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"One Man, One Vote" Part 1: Electoral Justice in the U.S. Electoral College: Banzhaf and Shapley/Shubik versus May*

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Abstract

This paper is dedicated to the measurement of (or lack of) electoral justice in the 2010 Electoral College using a methodology based on the expected influence of the vote of each citizen for three probability models. Our first contribution is to revisit and reproduce the results obtained by Owen (1975) for the 1960 and 1970 Electoral College. His work displays an intriguing coincidence between the conclusions drawn respectively from the Banzhaf and Shapley-Shubik's probability models. Both probability models conclude to a violation of electoral justice at the expense of small states. Our second contribution is to demonstrate that this conclusion is completely flipped upside-down when we use May's probability model: this model leads instead to a violation of electoral justice at the expense of large states. Besides unifying disparate approaches through a common measurement methodology, one main lesson of the paper is that the conclusions are sensitive to the probability models which are used and in particular to the type and magnitude of correlation between voters that they carry.

Classification JEL : D71, D72.

*We are very pleased to offer this paper as a contribution to this volume honoring Bill Gehrlein and Dominique Lepelley with whom we had the pleasure to cooperate in the recent years. Both have made important contributions to the evaluation of voting rules and electoral systems through probability models. Power measurement and two-tier electoral systems, the two topics of our paper, are parts of their general research agenda. We hope that they will find our paper respectful of the approach that they have been promoting in their work. Last, but not least, we would like to thank two anonymous referees who have attracted our attention on the imperfections and shortcomings contained in an earlier version and whose comments and suggestions have improved a lot the exposition of the ideas developed in our paper. Of course, they should not held responsible for any of the remaining insufficiencies. We gratefully acknowledge funding from the ANR under grant ANR-17-EURE-0010 (Investissements d'Avenir program).

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1 Introduction

While there are controversies about the appropriate precise definition of a perfect democratic electoral system, it is fair to say that a consensus exists among scholars and commentators on two properties that any such system should possess. These two properties, typically referred to as anonymity and neutrality in the jargon of social choice theory, are the two main pillars of any democratic electoral system. Anonymity calls for an equal treatment of voters[1] and neutrality calls for an equal treatment of candidates. If an electoral system is described in full mathematical generality as a mapping from the profiles of ballots into the set of candidates describing both the set of ballots available to each person and the selection of the winner for each profile of individual choices, anonymity implies first that all the persons have access to the same set of ballots and second that two profiles of votes which can deduced from each other through a permutation of names lead to the same electoral outcome. The second part of the definition amounts to an axiom of invariance to permutations of the names of the persons. Neutrality is an axiom of invariance to permutations of the names of the candidates. In this paper, we will focus exclusively on the anonymity property to which we will refer to alternatively as the "one person, one vote"[2]'s principle or electoral justice.[3]

Let us discuss briefly the first component of the anonymity axiom. For a meaningful discussion of that dimension, we need first the definition of a reference population listing all the persons who are considered to be part of the process on objective grounds. In practice, this set is a proper subset of the all population since for instance persons who are considered too young may not be listed in the reference population. Anonymity imposes as a necessary condition that all the persons in the reference population have access to the same set of ballots. The most extreme violation of that principle appears when some persons are totally excluded from the process (i.e. their set of ballots is empty) on the basis of one or several observable criteria like for instance gender, age, race, wealth or education. Most existing democracies went through a long period of time during which this basic version of anonymity has been violated. Even often, democracies have added temporarily an extra layer of departure from anonymity. Instead of having two classes of citizens: those (the non citizens) who do not vote at all and those (the voters) who do vote on equal grounds, the electoral system subdivides the second class into several classes defined by different ballot structures. Three examples from the European electoral history illustrate the second departure from anonymity: the law of double vote which

[1]It requires that the "votes" of any two persons should have the same influence.

[2]One man, one vote (or one person, one vote) is a slogan used by advocates of political equality through various electoral reforms such as universal suffrage, proportional representation, or the elimination of plural voting, malapportionment, or gerrymandering.

[3]For deep discussions of the notions of equity and justice, see Balinski (2005) and Young (1994).

was used from 1820 to 1830 in France,[4] the three-class franchise system[5] used from 1848 to 1918 in the Kingdom of Prussia and the "university constituency"[6] electoral system (university constituencies represent the members of one or more universities rather than residents of a geographical area) which has been used in the Parliament of Great Britain (from 1707 to 1800) and the United Kingdom Parliament, until 1950.[7]

Having access to the same ballots is necessary but not sufficient to obtain anonymity. A third and more subtle departure from anonymity arises when the ballots are the same but don't have the same influence on the final electoral outcome. The mathematical definition of anonymity is that for any given profile of ballots, the electoral outcome remains the same for all possible permutations of these ballots across voters. This condition is violated in two-tier electoral systems. A (single-seat) two-tier electoral system is a system where the population is partitioned into areas. In each area, the citizens elect a number of representatives who then meet in the upper-tier to elect the winner. In such system, the final outcome is sensitive to the geography of the votes i.e. to the distribution of the votes across the units composing the first tier. Two identical ballots will not have the same influence and the crucial question becomes: how to measure the differences across voters and the departure of the electoral system from perfect anonymity? In this paper, we will follow a popular approach pioneered by Banzhaf (1964) and Shapley-Shubik (1954) which consists in evaluating the power of a voter as the probability of this voter being pivotal where probability refers to a probability model where the profiles of preferences or utilities of the voters are the elementary events of the state space.[8] *Precisely, our methodology to evaluate the degree of electoral justice will consist in the computation of the*

[4]For an analysis, see Le Breton and Lepelley (2014), Newman (1974) and Spitzer (1983).

[5]This indirect election system (In German: Dreiklassenwahlrecht) has also been used for shorter intervals in other German states. Voters were grouped into three classes such that those who paid most tax formed the first class, those who paid least formed the third, and the aggregate tax revenue of each class was equal. Voters in each class separately elected one third of the electors (Wahlmänner) who in turn voted for the representatives. in Prussia from 1849 to 1909 and the law of sieges(called the law of double vote) created among the voters (only old enough males paying a critical amount of taxes were voters). For more on this electoral law, see Droz (1963) and Schilfert (1963).

[6]University constituencies originated in Scotland, where the representatives of the ancient universities of Scotland sat in the unicameral Estates of Parliament.[1] When James VI inherited the English throne in 1603, the system was adopted by the Parliament of England. It was also used in the Parliament of Ireland, in the Kingdom of Ireland, from 1613 to 1800, and in the Irish Free State from 1922 to 1936. It is still used in elections to Irish "senate". For more on this electoral law, see Beloff (1952).

[7]These may or may not involve plural voting, in which voters are eligible to vote in or as part of this entity and their home area's geographical constituency.

[8]We refer the readers to Felsenthal and Machover (1998) and Laruelle and Valenciano (2011) for overviews of the theory and its main applications. An alternative measurement approach could be based upon utilities. From the Penrose's formula (see for instance Felsenthal and Machover, 1998), under IC, utility is an affine function of power. This simple relationship ceases to hold true for other probabilistic models (See e.g. Laruelle and Valenciano, 2011 and Le Breton and Van Der Straeten, 2015). We have not explored the conclusions in terms of electoral justice drawn from utilities.

values of these power indices (there is one value per class of voters) and then the ratios of the numbers with respect to the smallest one. An alternative and equivalent way to present the same information would be to compute the relative shares of power. In this paper, we don't attempt to end up with a one dimensional uncontroversial measure of electoral inequity as we will consider several probability models. This concern together with some related statistical developments is the main topic addressed in our companion paper (De Mouzon, Laurent and Le Breton, 2019).

This paper will focus on an extremely popular and important two-tier electoral mechanism namely the U.S. Electoral College which is the electoral mechanism used by the United States of America to elect their president. In his pioneering and must read paper on the electoral college, Miller (2012) offers a very clear presentation of the issue of unequal representation in the context of this specific electoral institution. He writes:

> "Does the transformed Electoral College system give voters in different states unequal voting power? If so, are voters in large or small states favored and by how much? With respect to this question, directly contradictory claims are commonly expressed as a result of the failure by commentators to make two related distinctions: the theoretical distinction between 'voting weight' and 'voting power', and the practical distinction between how electoral votes are apportioned among the states (which determines their voting weights), and how electoral votes are cast by states (which influences their voting power).
>
> Those claiming that the Electoral College system favors voters in small states point to the advantage small states have with respect to the apportionment of electoral votes. States have electoral votes equal to their total representation in Congress. Since every state is guaranteed at least one seat in House and has two Senators, every state is entitled to at least three electors regardless of population. Approximate proportionality to population takes effect only beyond this three-electoral-vote floor, and this creates a substantial small-state advantage in the apportionment of electoral votes.
>
> However, other commentators (starting with like Luther Martin) emphasize that voting power is not proportional to voting weight (e.g., electoral voters), for two reasons. First, the voting power of a state depends not only on its share of electoral votes but on how the remaining electoral votes are distributed among the other states. Second, the voting power of a state depends on whether it casts its electoral votes as a bloc for a single candidate or splits them among two or more candidates, as well as how other states cast their votes. Intuition seems to tell us that the fact

that elector slates are elected on a general ticket and therefore cast as bloc produces a large-state advantage—but intuition doesn't tell us how big this advantage may be. Moreover, we saw earlier that this intuition is only weakly supported in the state voting power calculations. The large-state advantage in the 51-state weighted voting game resulting from winner-take-all is not great enough to counterbalance the small-state advantage with respect to apportionment except in the case of the megastate of California, so those claiming a (modest) small-state advantage may appear to be correct. However, the top-tier 51-state weighted voting game entailed by the transformed Electoral College is a chimera, and the picture changes dramatically when we consider the more realistic 130-million-voter two-tier popular election".

The literature on the qualities and weaknesses of the Electoral College is vast. We will here focus our attention on two questions: How to compare the voters from the different states in the Electoral College ? Is there an advantage to small states or large states? To address, these questions we will follow the vast area of research based on the use several distinct a-priori probability models on top of which the two most popular ones: the Banzhaf/IC probability model (Banzhaf, 1965) and the Shapley-Shubik/IAC probability model (Shapley and Shubik, 1954). These a-priori models have been criticized on several grounds among which the lack of empirical support in favor of these models.[9] We think that theoretical and empirical probability models serve different purposes. It depends whether[10] the emphasis of the analysis is either positive i.e. on predicting the power of the citizens on the basis of the current electoral data or normative i.e. on evaluating on a priori neutral grounds the current electoral system and its potential contenders.

In this paper, we revisit and complement the pioneering work of Owen (1975) who writes: "Discussion has frequently centered on the excessive power which this system seems to give to one group or another (the large states, the small states, organized minorities within one or another of the kinds of states, etc.), though there is also frequent disagreement about the identity of these favored groups". Owen computes for both the 1960 and 1970 apportionments and census, the $51-$dimensional vector of power indices of U.S. citizens (as a function of the U.S. State where they vote) for the two most popular probability models to which we have already alluded: the Banzhaf/IC model and the Shapley-Shubik/IAC probability model. From these calculations, he derives both for 1960 and 1970, the $50-$dimensional ratios of the power of the

[9] See Gelman, Katz, Bafumi (2004), Gelman, Katz, King (2002), Gelman, Katz, J.N. Tuerlinckx (2002), Gelman, King, Boscardin (1998). See also the empirical analysis of pivotality conducted by Mulligan and Hunter (2003).

[10] We refer to Miller (2009) and De Mouzon, Laurent, Le Breton and Lepelley (2019 b) for a defense of the a priori approach.

citizen of any given state by the smallest power (which corresponds to the District of Columbia). In 1960, the highest ratio is obtained for New York with a value equal to 3.312 for IC and 3.287 for IAC followed by California with a value of 3.162 for IC and 3.143 for IAC. In 1970, the situation is reversed with California on top with a value of 3.177 for IC and 3.166 for IAC followed by New York with a value of 3.004 for IC and 2.976 for IAC. Between 1960 and 1970, California gained 5 electoral votes (from 40 to 45) while New York lost 2 electoral votes (from 43 to 41). More precisely, Owen obtains for the two probability models a complete numerical ranking of the states according to these ratios. Two conclusions emerge from his work:

- For the two models and the two periods, citizens from large states have more influence on the electoral process than citizens from small states (around three times more for citizens in California and New York State).
- The numerical rankings (and *de facto* their ordinal implications) attached to IC and IAC are almost the same.

The first conclusion has been widely commented and criticized by many authors. This conclusion seems at odds with the conventional wisdom asserting that the conclusion should be opposite since the small states are endowed with at least 3 electoral votes irrespective of their populations. This conclusion is shared by political scientists. For instance, by calculating an *advantage ratio* for each state by simply dividing its share of the total electoral vote by its share of the national population, Shugart (2004) obtains that "these ratios range from 0.85 for California and Texas to 3.18 for Wyoming. In other words, California's weight in electing a president is only 85% of its contribution to the national population, while Wyoming's is more than three times as great as its population". On a figure, he plots each state's advantage ratio against its population and claims that "it shows very clearly how the smallest states are significantly overrepresented... There are 13 states with an advantage ratio greater than 1.5...".[11]

Our paper revisits the two conclusions and elucidates the difference between Owen's conclusions that large states are overrepresented and traditional views about over representation of small states.

First, to the best of our knowledge, the second result in Owen's paper has not received any attention. This result, that we suggest to call *Owen's coincidence result*, is first mathematically intriguing. How could it be the case that two models which are very different[12] lead both to

[11] In the same vein, see also Durran (2017).

[12] Among the differences, note in particular that, as demonstrated by De Mouzon, Laurent, Le Breton and Lepelley (2019a), the probability of an election inversion (that is an electoral college winner different from the popular winner) in the Electoral College tends to 0 with the population size for the IAC model while the limit is positive for the IC model.

over representation of large states. The IC model postulates complete independence among voters while the IAC model displays correlation among voters within and across states. This coincidence is not obvious at all and is not empathized as such in Owen's work who derives his results through ingenious and sophisticated numerical approximation arguments. Our paper will revisit this coincidence as without any mathematical general result, we could indeed speculate that it may just well be the case that this coincidence is specific to the 1960 and 1970 data. Our first main result is that this coincidence extends to more recent data as well.

Besides the mathematical curiosity, this coincidence also obliges to have a different view about the so called Banzhaf's fallacy (Margolis (1983)). Conclusions derived from Banzhaf are often disregarded as the IC model is very special. It is very special indeed and it is fair to say that the square root law often attached to it should be considered with caution. With the IAC model, the order of magnitude of the probability of influence for any voter in any state is $\frac{1}{n}$ instead of $\frac{1}{\sqrt{n}}$. But the surprise is that when we compute the ratios for any pair of different states, they are the same for the two models. *This means that the conclusions in terms of electoral justice does not depend exclusively upon the choice of* IC.

The second contribution of the paper is to point out that the statement that small states are over represented can be obtained as the result of the methodology adopted here for a third probability model which has been invented first by May (1949) and used by several authors and that we will call the May's probability model.[13] This probability model is identical to IAC within states and IC across states i.e. correlations exist between voters from the same state and are absent between voters from different states. The ranking of the states according to influence is now almost a complete reversal of the ranking attached to the IC and IAC models. Further the largest ratio is now about three times the smallest one. The mathematical side of this result is easy but it is quite surprising to observe that the ordering of the states depends very much on whether the preferences of the voters are correlated or not across states. *With no correlation across states, electoral justice is against small states while with enough correlation electoral justice is against large states.* There is likely a critical level of inter state correlation separating the two conclusions!

The paper is organized as follows. In section 2, we present the main notations and definitions. Then in section 3, we present the main results of the paper. They are all based on simulated elections based upon the 2010 apportionment and census for the three a priori probability models which are considered. Our two main results on electoral justice (identical over representation of large states for both Banzhaf and Shapley-Shubik and over representation of small states for May) are presented in that section. In the appendix we do two things. First,

[13] It is sometimes called the IAC* probability model (Le Breton, Lepelley and Smaoui (2016)).

we discuss the issue of their asymptotic coincidence in the case of discrete versions of the May and Shapley-Shubik models and we sketch an explanation of their coincidence/difference in the case of the continuous version. Second, we examine the validity of Penrose's approximation in the second tier of the electoral college by comparing the exact ratios of power indices of the states with the ratio of weights.

2 Notations and Definitions

The purpose of this section is twofold. First, we introduce the main notations and definitions with a special emphasis on the notion of two-tier weighted majority mechanism. Second, we present the measure of influence of a voter which is used in this paper and the three main probability models that are considered to conduct the computations.

2.1 Two-Tier Weighted Majority Mechanisms

We consider a society N of n voters which must chose among two alternatives:[14] D versus R. Each member i of N is described by his/her preference P_i. There are two possible preferences: D or R. We assume that N is partitioned into K states: $N = \cup_{1 \leq k \leq K} N^k$. The n^k voters of state k are endowed with w^k electoral votes. The electoral outcome $F(P) \in \{D, R\}$ attached to the profile of preferences $P = (P_1, \ldots, P_n) \in \{D, R\}^n$ is determined by the following[15] two-tier weighted majority mechanism.[16] Let $F^k(P^k)$ be the majority winner in state k i.e.[17]

$$F^k(P^k) = \begin{cases} D \text{ if } \left|\left\{i \in N^k : DP_iR\right\}\right| \geq \left|\left\{i \in N : RP_iD\right\}\right| \\ R \text{ if } \left|\left\{i \in N^k : DP_iR\right\}\right| < \left|\left\{i \in N : RP_iD\right\}\right| \end{cases}$$

Then:

[14] In our simplified setting, like Owen (1975), we neglect the spoiler effects due to the existence of candidates in addition to the two main ones.

[15] A general electoral mechanism F is defined as a monotonic mapping from $\{0,1\}^n$ into $\{0,1\}$ where $D \equiv 1$ and $R \equiv 0$.

[16] Alternatively and equivalently, any electoral mechanism F can be described in terms of winning coalitions. A coalition of voters $S \subseteq N$ is winning, denoted $S \in \mathcal{W}$, iff $F(P) = D$ whenever $P_i = D$ for all $i \in S$. It is straightforward to check that the family $\mathcal{W}$ of winning coalitions is monotonic with respect to inclusion. The pair $(N, \mathcal{W})$ is called a *simple game* (Owen, 2001). Among those, weighted majority games are central. A weighted majority game on N is described by a vector of weights $w = \left(w^1, \ldots, w^n\right)$ and a quota q: $S \subseteq N$ is winning, denoted $S \in \mathcal{W}(q, w)$, iff $\sum_{i \in S} w^i \geq q$. When $w = (1, \ldots, 1)$ and $q = \frac{n}{2}$, we obtain the ordinary majority game.

[17] In this definition, in both tiers, ties are broken in favor of D. The details of the tie breaking rule do not impact our results. In fact, our simulations are conducted under the assumption that in both tiers, ties are broken through a fair random choice between D and R. We will offer further comments on that, later in the paper.

$$F(P) = \begin{cases} D \text{ if } \sum_{1 \le k \le K: F^k(P^k) = D} w^k \ge \sum_{1 \le k \le K: F^k(P^k) = R} w^k \\ R \text{ if } \sum_{1 \le k \le K: F^k(P^k) = D} w^k < \sum_{1 \le k \le K: F^k(P^k) = R} w^k \end{cases}$$

Candidate D or R wins the election if the total number of electoral votes attached to the states where he/she wins a majority of the votes is larger than the total number of electoral votes attached to the states where his/her opponent wins a majority of the votes.

The simple game attached to a two-tier weighted majority mechanism is called a *compound simple game*[18] (Owen, 2001, Shapley, 1962). From that perspective, the ingredients of the two-tier electoral mechanism consist of $K+1$ simple games:

- K ordinary majority games (N^k, W^k_{maj}) $k = 1, \ldots, K$: in each state, the allocation of the totality of the w^k electoral votes of the state is decided by ordinary majority voting within the state.

- The weighted majority game $(\{1, \ldots, K\}, \mathcal{W}(q, w))$ where $w = \left(w^1, \ldots, w^K\right)$ and $q = \frac{\sum_{1 \le k \le K} w^k}{2}$: in the second tier, the representatives of each state (voting as a block) elect the president through majority voting.

This mechanism can receive two interpretations. Either, it describes the election of a president through an electoral college. Or it describes the election of a parliamentary house through a plurality formula.

In the presidential interpretation, the states represent the (geographical) states in the Federal Union. The majority winner in state k wins all[19] the electoral votes in state k. The upper-tier, called the electoral college, is composed of $\sum_{1 \le k \le K} w^k$ electors who are either on the D side or on the R side. It is assumed that they elect the president through an ordinary majority vote. In the parliamentary interpretation, the states represent the electoral districts of the country and w^k is the district magnitude of district k. If all the seats of district k go to the majority winner[20] in district k, then $F(P)$ denotes the majority "color" of the parliament while $\sum_{1 \le k \le K: F^k(P^k) = D} w^k$ and $\sum_{1 \le k \le K: F^k(P^k) = R} w^k$ denote the number of seats won respectively by

[18] The notion of composition is quite general and can be applied to very abstract simple games.

[19] This is the "winner takes all" feature of the mechanism. In our paper, we ignore the fact that for Maine and Nebraska "winner takes all" does not fully apply. Strictly speaking, congressional districts should be treated as additional states for the purpose of the modeling. We conjecture that our results are not significantly impacted by this simplification.

[20] In the real world electoral systems which are used to elect the representatives, when the district magnitude is larger than 1, it is often the case that the "winner-takes-all" principle is replaced by a proportional principle. In such a case, the formal description of the electoral mechanism differs from the one considered here. For a general approach, when the district magnitude is equal to 2, the reader is referred to Le Breton, Lepelley, Merlin and Sauger (2017).

D and R. In the main real-world applications of this second interpretation (U.S.; U.K.,...), the district magnitude of all the districts is equal to 1.

In this paper, we will focus on the first interpretation. The above formal definition calls for a comment as, strictly speaking, it deviates at the margin from the real one. Indeed, since we cannot exclude a priori the cases where there is a tie either within a state (this may happen if n^k is even) or more seriously within the electoral college (this may happen if $\sum_{1 \leq k \leq K : F^k(P^k)=D} w^k = \sum_{1 \leq k \leq K : F^k(P^k)=R} w^k$), we need to define how these ties are broken. In this event, to make our presentation simple, we have broken deterministically the tie in favor of the candidate D. Instead, we could have decided to break the tie in favor of the candidate R or to use a probabilistic device like flipping a fair coin. In our simulations, to make the rule as neutral as possible, we opted for the random draw but in this general presentation, we decided not to do so as this calls for some cumbersome notational adjustments in the definition of pivotality that we wanted to avoid.

For the unbiased probabilistic models on preferences that will be introduced in the next section, all three tie-breaking rules lead to the same computations and conclusions. The U.S. electoral college uses a different rule in the case of such a contingent presidential election. The Twelfth Amendment requires the House of Representatives to go into session immediately to vote for a president if no candidate for president receives a majority of the electoral votes. In this event, the House of Representatives is limited to choosing from among the three candidates who received the most electoral votes for president. Each state delegation votes "en bloc": each delegation having a single vote; the District of Columbia does not receive a vote. A candidate must receive an absolute majority of state delegation votes (i.e., at present, a minimum of 26 votes) in order for that candidate to become the president-elected. The House continues balloting until it elects a president.[21] Like Owen (1975) we depart from the "real" tie-breaking rule. In section 4, we speculate that the results that we will obtain for the three equivalent and simple theoretical breaking rules defined above are identical to those that would be obtained for the Twelfth Amendment rule.

2.2 Probability Models

To evaluate the power and utilities of voters and the properties of a voting mechanism F, we introduce a probability model π on the set of profiles $\{D, R\}^n$: $\pi(P)$ denotes the probability (frequency, $\ldots$) of profile P. Let us examine the situation from the perspective of voter i. To evaluate how often i is influential, we consider the frequency of profiles P such that $F(D, P_{-i}) \neq$

[21] The House of Representatives has chosen the president only once in 1825 under the Twelfth Amendment. Senate is involved along similar principles in the election of the vice-president.

$F(R, P_{-i})$ or equivalently the frequency of coalitions S, such that $S \in \mathcal{W}$ and $S \setminus \{i\} \notin \mathcal{W}$ or $S \notin \mathcal{W}$ and $S \cup \{i\} \in \mathcal{W}$. In such situation , we say that voter i is *pivotal*. The probability $Piv_i(i, F, \pi, n)$[22] of such an event is:

$$\sum_{T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} \pi(T) + \sum_{T \in \mathcal{W} \text{ and } T \setminus \{i\} \notin \mathcal{W}} \pi(T) = \sum_{T \subseteq N \setminus \{i\} : T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} \pi_{-i}(T)$$

where π_{-i} denotes the (marginal) probability induced by π on the product subspace $\{D, R\}^{N \setminus \{i\}}$. This formula[23] makes clear that the evaluation depends upon the probability π which is considered. Two popular specifications have attracted most of the attention and dominate the literature.

The first (known under the heading *Impartial Culture (IC))* leads to the Banzhaf's index.[24] It corresponds to the setting where all the preferences P_i proceed from independent Bernoulli draws with parameter $\frac{1}{2}$. In this case, for all $T \subseteq N \setminus \{i\} : \pi_{-i}(T) = \frac{1}{2^{n-1}}$. The Banzhaf power $B(i, F, n)$ of voter i is equal to:

$$\frac{\eta_i(\mathcal{W})}{2^{n-1}}$$

where $\eta_i(\mathcal{W})$ denotes the number of coalitions $T \subseteq N \setminus \{i\}$ such that $T \notin \mathcal{W}$ and $T \cup \{i\} \in \mathcal{W}$ (in the literature, any such coalition T is referred to as a "swing" for voter i).

The second model (known under the heading *Impartial Anonymous Culture (IAC) Assumption)* leads to the Shapley-Shubik's index.[25] It is defined as follows. Conditionally to a draw of the parameter p in the interval $[0, 1]$, according to the uniform distribution,[26] the preferences P_i

[22] $Piv(i, F, \pi, n)$ contains a little abuse in notation since π and n cannot be separated as π is defined on $\{D, R\}^n$. $Piv(i, F, \pi, n)$ could also be denoted $Piv(i, \mathcal{W}, \pi, n)$ and it is often called the power of voter i for the voting rule F/W according to the probability model π. When the reference to F/W will be clear, we will drop it from the notation.

[23] This definition needs to be adjusted when the voting mechanism when ties are not broken deterministically. Let us denote by T (T for ties) the set of profiles $P \in \{D, R\}^n$ such that D is elected with probability $0 < \chi(P) < 1$. Assuming that a tie is broken as soon as a single voter changes her mind, then the probability of pivotality is the probability over subprofiles P_{-i} of having $F(P_{-i}, D) \neq F(P_{-i}, R)$. When both outcomes are deterministic, then this happens only when $F(P_{-i}, D) = D$ and $F(P_{-i}, R) = R$. But with ties this may also happen when : $F(P_{-i}, D) = T$ and $F(P_{-i}, R) = R$ or when $F(P_{-i}, R) = T$ and $F(P_{-i}, D) = D$. In the last two cases, the probability of having different outcomes is not equal to 1 anymore but to $\chi(P)$.

[24] Here, we have only two candidates. The wording IC is used more generally to define the situation of independent and identically draws of preferences over an arbitrary number of candidates. Here, we use the terms Banzhaf and IC equivalently.

[25] It can be proved that the Shapley-Shubik model amounts drawing uniformly the number of voters who prefer D to R. It can also be showed that for the IAC model the preferences display some correlation. Here we have only two feasible preferences. For an arbitrary number of candidates, the wording IAC is used more generally to define the situation where the draws of the vectors describing the numbers of preferences of each type are uniform. Here, we use the terms Shapley-Shubik and IAC equivalently.

[26] If we take an arbitrary absolutely continuous distribution, we obtain a generalized version of the Shapley-Shubik's probability model which has been analyzed by Chamberlain and Rothschild (1981) and Good and

proceed from independent Bernoulli draws with parameter p. In such a case, for all $T \subseteq N \setminus \{i\}$: $\pi_{-i}(T) = \int_0^1 p^t(1-p)^{n-1-t}dp$ where $t \equiv \#T$. The Shapley-Shubik power $Sh(i, F, n)$ of voter i is equal to:

$$\int_0^1 \left(\sum_{T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} p^t (1-p)^{n-1-t} \right) dp.$$

In addition to these two models, we consider a third one, called here IAC^*, which is intermediate between IC and IAC. It was first introduced by May (1949) in his analysis of election inversions and pivotality was studied recently by Le Breton, Lepelley and Smaoui (2016) when F is popular majority. This model is defined as follows.

Assume from now that N is partitioned into K states: $N = \cup_{1 \leq k \leq K} N^k$. Conditionally on K independent and identically distributed, draws $p_1, \ldots, p_K$ in the interval $[0,1]$, according to the uniform distribution, the preferences in group N_k proceed from independent Bernoulli draws with parameter p_k. In such a case, for all $T \subseteq N \setminus \{i\}$ such that i belongs to state $k(i)$:

$$\pi_{-i}(T) = \left(\prod_{1 \leq k \leq K : k \neq k(i)} \int_0^1 p_k^{t^k} (1-p_k)^{n^k - t^k} dp_k \right) \times \int_0^1 p_{k(i)}^{t^{k(i)}} (1-p_{k(i)})^{n^{k(i)}-1-t^{k(i)}} dp_{k(i)},$$

where $t^k \equiv \left| N^k \cap T \right|$ for all $k = 1, \ldots, K$. The May power $M(i, F, n)$ of voter i is then equal to:

$$\sum_{T \notin \mathcal{W} \text{ and } T \cup \{i\} \in \mathcal{W}} \left(\prod_{1 \leq k \leq K : k \neq k(i)} \int_0^1 p_k^{t^k} (1-p_k)^{n^k - t^k} dp_k \right) \times \int_0^1 p_{k(i)}^{t^{k(i)}} (1-p_{k(i)})^{n^{k(i)}-1-t^{k(i)}} dp_{k(i)}.$$

In the case where F is the direct/popular (i.e. one tier) majority mechanism, all the voters have the same influence. In such case, we can drop the reference to i. When n, the number of voters, is large, it is well known that the Banzhaf power of a voter is[27] approximatively equal to $\sqrt{\frac{2}{\pi n}}$ while the Shapley-Shubik index of a voter is equal to $\frac{1}{n}$. The combinatorics of the May's index are more involved and explored in Le Breton, Lepelley and Smaoui (2016).

In the case where F is the two-tier weighted majority mechanism considered in this paper, note first that we cannot drop the reference to i anymore but all voters from the same state will have the same power as long as the probability model displays symmetry accross players.

Mayer (1975).

[27]If n is odd, then for all i, $B_i = \binom{n-1}{\frac{n-1}{2}}/2^{n-1}$. If n is even, $B_i = \left[\binom{n-1}{\frac{n}{2}}/2^{n-1} + \binom{n-1}{\frac{n-2}{2}}/2^{n-1}\right] \times \frac{1}{2}$. The assertion follows from Stirling's formula.

As this is the case for the Banzhaf, Shapley-Shubik and May probability models, we will have to compute K different values for these three models.

From now on, we will focus on the two-tier electoral mechanism F defined in section 2.1 and we drop the reference to F in the coming computations of pivotality.[28] For any $i \in N$, we will denote by $Piv\,(i, \pi, n)$ the probability that voter i is pivotal according to the probability model π. We denote by $\underline{Piv}(i, \pi, n)$ the probability that i is pivotal according to π in his/her state $k(i)$.[29] When k is a representative in the upper-tier, we will denote by $\overline{Piv}(k, \pi)$ the probability that k is pivotal in the upper tier. Note that when we examine *per se* the upper-tier, we do not need the full knowledge of π but the probability induced by π on the set of representative preference profiles $\{D, R\}^K$. Let us illustrate that point for IC, IAC^* and IAC when $K = 2$. When $\pi = IC$, the probability that both representatives vote D i.e. the probability that there is a majority of representatives voting D in both states is equal to $\frac{1}{4}$. In such a case, the probability induced by π on the upper-tier is simply IC on the set of representatives $\{1, 2\}$. By the same token, we obtain that when $\pi = IAC^*$, the probability induced by π on the upper-tier is simply IC on the set of representatives $\{1, 2\}$. In contrast, when π is IAC, things get far more subtle. Under the assumption that the two states are equipopulated with $n^1 = n^2 \equiv m$ (m odd), the probability that both states vote democrat is equal to:

$$\sum_{k=\frac{m+1}{2}}^{m} \sum_{r=\frac{m+1}{2}}^{m} \frac{m!}{k!(m-k)!} \times \frac{m!}{r!(m-r)!} \times \frac{(k+r)!(2m-k-r)!}{(2m+1)!}.$$

We do not know any closed form. When $m = 11$ and therefore $n = 22$, this probability is equal to 0.42107 which is much larger than the value 0.25 obtained for IC and IAC^*. Note that the probability induced by IAC on $\{1, 2\}$ is not IAC on $\{1, 2\}$. Indeed, if we consider IAC on $\{1, 2\}$, the probability of the profile (D, D) is $\int_0^1 p^2 dp = \frac{1}{3} = 0.3333...$

Let us consider the computation of $Piv\,(i, \pi, n)$ for $\pi = IC$, IAC and IAC^*. From the description of F as a compound simple, it is straightforward that for a voter to be influential/pivotal, we need the combination of two events: the voter must be pivotal in his state and the representatives of his state must be themselves pivotal in the electoral college. In general unfortunately the two events are not independent. If the two events are independent for some probability model π, then the computation of the pivotality P_i of voter i proceeds from a simple multiplicative formula:

[28]This means also that we will not explicitly refer to the division $n^1, ..., n^K$ of the n voters into the K states and to the electoral votes $w^1, ..., w^K$ of the states.

[29]Truly only the restrictions of F and π on the subset $N^{k(i)}$ matters. Since the restriction of F onto $N^{k(i)}$ is the ordinary majority mechanism with $n^{k(i)}$ voters, the computation of $\underline{Piv}(i, \pi, n)$ amounts to the computation of the pivotality according to π for the ordinary majority mechanism.

$$Piv\left(i,\pi,n\right)=\underline{Piv}\left(i,\pi,n^{k(i)}\right)\times\overline{Piv}\left(k(i),\pi\right),$$

where $\underline{Piv}\left(i,\pi,n^{k(i)}\right)$ denotes the pivotality power of voter i in his state $k(i)$ and $\overline{Piv}\left(k(i),\pi\right)$ denotes the pivotality power of the representative(s) of state $k(i)$ in the second tier. For the Banzhaf and May probability models, the two events are independent. Further given the neutrality of these two probability models between the candidates and the neutrality of the ordinary majority mechanism between the two candidates, the pivotality power of representative k in the second tier is simply his Banzhaf power in the second tier. Therefore, when the number of voters in each state is large:

$$B(i,n)=\underline{B}(i,n^{k(i)})\times\overline{B}(k(i))\simeq\sqrt{\frac{2}{\pi n^{k(i)}}}\overline{B}(k(i))$$

and

$$M(i,n)=\underline{Sh}(i,n^{k(i)})\times\overline{B}(k(i))=\frac{1}{n^{k(i)}}\overline{B}(k(i)).$$

In such case, we are left with the computation of $\overline{B}(k)$ for $k=1,\ldots,K$. This can be done in several ways. Either by using existing software which works well as long as K is not too large. Another road is to use (if possible) Penrose's theorem which asserts that under some conditions, the Banzhaf power of player k in the weighted majority game $(\{1,\ldots,K\},\mathcal{W}(q,w))$ is proportional to w^k. The exact computation of these values as well as the validity of the Penrose's approximation are presented and discussed in appendix 3. Under the presumption that the Penrose's approximation is valid, we obtain for all $i,j\in N$:

$$\frac{B(i,n)}{B(j,n)}=\frac{\sqrt{n^{k(j)}}}{\sqrt{n^{k(i)}}}\times\frac{w^{k(i)}}{w^{k(j)}}\tag{1}$$

and

$$\frac{M(i,n)}{M(j,n)}=\frac{n^{k(j)}}{n^{k(i)}}\times\frac{w^{k(i)}}{w^{k(j)}}\tag{2}$$

Unfortunately, we cannot proceed similarly for the Shapley-Shubik model. Remember that the preferences of the voters within and across states are correlated. This means that for that probability model, we cannot in principle separate the two computations on pivotality.[30] Owen (1975, 2001) presents developments on how to calculate the Shapley value of a compound simple game.[31] There is no easy way to proceed and Owen states some results on the relationships of

[30]When $K=2$, and $n^1=n^2\equiv m$, the probability that any player is pivotal for IAC is equal to $\frac{(m-1)!}{(\frac{m-1}{2})!(\frac{m-1}{2})!}\times\sum_{r=\frac{m+1}{2}}^{m}\frac{m!}{r!(m-r)!}\times\frac{(\frac{m-1}{2}+r)!(2m-\frac{m-1}{2}-r)!}{(2m+1)!}$, while it equals to $\frac{1}{m}\times\frac{1}{2}$ for IAC^* and to $\frac{\binom{m-1}{\frac{m-1}{2}}}{2^{m-1}}\times\frac{1}{2}$ for IC. When $m=11$, we obtain the values 0.019, 0.05 and 0.123.

[31]There is the place to remind to the reader that Sh_i is also the Shapley value of the TU simple game $(N,V_{\mathcal{W}})$ where $V_{\mathcal{W}}(S)=1$ iff $S\in\mathcal{W}$ and 0 otherwise.

the multilinear extension of a compound simple game to the multilinear extensions of the simple games used in the composition. He uses these results to conduct his numerical computations and to obtain estimates of

$$\frac{Sh(i,n)}{Sh(j,n)}$$

for all $i, j \in N$.

3 Electoral Justice in the Electoral College

The main purpose of this section is to present our computational results on electoral justice in the Electoral College for the three probability models that have been defined. The codes of our computer program are available at `http://www.thibault.laurent.free.fr/code/DL_issue/`. In the first section, we present the 2010 apportionment and census data which is used in our analysis. Then, in three distinct subsections, we present and comment separately our results for Banzhaf, Shapley-Shubik and May. All these computations are derived through a simulator that works as follows:

Algorithm 1: Main steps of our algorithm

Initialization of program constants

K = number of states (51)

B = number of simulations (10^{12})

$\forall k \in \{1, \ldots, K\}, Seats_k$ denotes the number of votes of State k for the presidential election and n_k denotes the number of voters in State k.

Computation

for *Model in {IC, IAC, IAC*}* **do**

  Initialize the number of pivotal voters (at the presidential level): $\forall k \in \{1, \ldots, K\}, Piv_k := 0$.

  for *b in 1 to B* **do**

    Initialize the total number of seats for D or R: $Seats_D := 0$ and $Seat_R := 0$.

    for *k in 1 to K* **do**

      Simulate the choice of each voter in State k between D and R, following the chosen Model distribution (IC, IAC or IAC*)

      Compute Piv_k^{State}, the number of pivotal voters in State k, considering only State k choice.[a]

      Compute State k choice for the presidential election: $C_k \in \{D, R\}$.

      Update the total number of seats for D or R: $\forall P \in \{D, R\}$, if $C_k = P$, then $Seats_P := Seats_P + Seats_k$.

    for *k in 1 to K* **do**

      If State k has pivotal voters ($Piv_k^{State} > 0$) and Seats of State k are pivotal,[b] then update the number of Pivotal voters: $Piv_k := Piv_k + Piv_k^{State}$.

  Compute the estimated probability of a voter to be pivotal at the presidential election:

  $\forall k \in \{1, \ldots, K\}$, the probability for a voter in State k to be pivotal at the presidential election is: $\frac{Piv_k}{B * n_k}$.

[a] This number is often 0. Considering, on the one hand, a State k with an odd number of voters, n_k, there are either no pivotal voters or $\frac{n_k+1}{2}$ (when there is almost a tie). Considering, on the other hand, a State k with an even number of voters, either there is a tie and half of the voters are pivotal or there is almost a tie and $\frac{n_k+2}{2}$ voters are pivotal in half of the cases. In all other cases, there are no pivotal voters.

[b] Seats of State k are pivotal if $Seats_{C_k} - Seats_k <= Seats_{-C_k} + Seats_k$, $-C_k$ denoting the non chosen party by State k. In presence of a tie ($Seats_{C_k} = Seats_{-C_k}$ or $Seats_{C_k} - Seats_k = Seats_{-C_k} + Seats_k$), only half of the cases are pivotal.

It is interesting to point out that in our algorithm, instead of counting, for a fixed i in any given state k, the number of profiles for which a change in the vote of i changes with some positive probability the electoral outcome, we count for each profile the total number of individuals who are pivotal. When the voting rule is anonymous, there is an obvious link between the two. Consider for instance the majority game with n odd and define $X(P)$ to be the random variable counting the number of pivotal voters at P and $Y(P)$ ($Y_i(P)$) the Bernoulli random variable equal to 1 if there is a pivotal voter (if i is a pivotal voter) at P.

Then $X(P) \neq 0$ iff the difference between the number of voters voting D and the number of voters voting R is either 1 or -1. In both cases, the number of pivotal voters is equal to $\frac{n+1}{2}$. The expectation of X is equal to:

$$2 \times \frac{\binom{\frac{n-1}{2}}{n}}{2^n} \times \frac{n+1}{2} = \frac{\binom{\frac{n-1}{2}}{n-1}}{2^{n-1}} n.$$

If we divide by n we obtain $\frac{\binom{\frac{n-1}{2}}{n-1}}{2^{n-1}}$. Note however that the probability that $Y(P) = 1$ is equal to:

$$2 \times \frac{\binom{\frac{n-1}{2}}{n}}{2^n} = 2\frac{n}{n+1}\frac{\binom{\frac{n-1}{2}}{n-1}}{2^{n-1}}$$

i.e. about 2 times the probability that $Y_i(P) \neq 0$. Therefore the event $Y(P) \neq 0$ is two times more frequent than the event $Y_i(P) \neq 0$. This implies via Bienaymé-Tchebychev that we can obtain estimates with the same confidence while dividing by two the number of simulations.

When n is even we proceed similarly. $X(P) \neq 0$ iff the difference between the number of voters voting D and the number of voters voting R is either 0, 2 or -2. In the first case, the number of pivotal voters is equal to n but is counted $\frac{n}{2}$ as the change occurs only with probability $\frac{1}{2}$. In the second case, the number of pivotal voters is equal to $\frac{n+2}{2}$ but is counted $\frac{n+2}{4}$ as the change occurs only with probability $\frac{1}{2}$. Therefore, the expectation of X is equal to:

$$\frac{n}{2} \times \frac{\binom{\frac{n}{2}}{n}}{2^n} + 2 \times \frac{\binom{\frac{n-2}{2}}{n}}{2^n} \times \frac{n+2}{4} = \frac{\binom{\frac{n-2}{2}}{n-1}}{2^{n-1}}\left[\frac{n}{2} + \frac{n}{2}\right] = \frac{\binom{\frac{n-2}{2}}{n-1}}{2^{n-1}} n.$$

If we divide by n, we obtain as before $\frac{\binom{\frac{n-2}{2}}{n-1}}{2^{n-1}}$. Finally, note also that the probability that $Y(P) = 1$ is equal to:

$$\frac{\binom{\frac{n}{2}}{n}}{2^n} + 2 \times \frac{\binom{\frac{n-2}{2}}{n}}{2^n} = \frac{\binom{\frac{n-2}{2}}{n-1}}{2^{n-1}}\left[2 + \frac{2n}{n+2}\right]$$

i.e. again about 2 times the probability that $Y_i(P) \neq 0$.

3.1 The 2010 U.S. Electoral College and Population Data

Table 1 presents the number of voters and seats[32] which have been used in our simulator. It corresponds to the 2010 population census and 2012 electoral college (which holds also for 2016 and 2020).

[32]The number of electoral votes (called hereafter 'seats') of a state is the sum of its number of representatives and number of senators (which is 2 for all states). The District of Columbia is allocated 3 seats.

state	pop_2010	college_2012
Alabama	4802982	9
Alaska	721523	3
Arizona	6412700	11
Arkansas	2926229	6
California	37341989	55
Colorado	5044930	9
Connecticut	3581628	7
Delaware	900877	3
District of Columbia	601766	3
Florida	18900773	29
Georgia	9727566	16
Hawaii	1366862	4
Idaho	1573499	4
Illinois	12864380	20
Indiana	6501582	11
Iowa	3053787	6
Kansas	2863813	6
Kentucky	4350606	8
Louisiana	4553962	8
Maine	1333074	4
Maryland	5789929	10
Massachusetts	6559644	11
Michigan	9911626	16
Minnesota	5314879	10
Mississippi	2978240	6
Missouri	6011478	10
Montana	994416	3
Nebraska	1831825	5
Nevada	2709432	6
New Hampshire	1321445	4
New Jersey	8807501	14
New Mexico	2067273	5
New York	19421055	29
North Carolina	9565781	15
North Dakota	675905	3
Ohio	11568495	18
Oklahoma	3764882	7
Oregon	3848606	7
Pennsylvania	12734905	20
Rhode Island	1055247	4
South Carolina	4645975	9
South Dakota	819761	3

state	pop_2010	college_2012
Tennessee	6375431	11
Texas	25268418	38
Utah	2770765	6
Vermont	630337	3
Virginia	8037736	13
Washington	6753369	12
West Virginia	1859815	5
Wisconsin	5698230	10
Wyoming	568300	3

Table 1: U.S. Electoral college and population data per State (Source: `http://www.thegreenpapers.com/Census10/HouseAndElectors.phtml`)

Figure 1 shows that number of representatives are allocated proportionally to the population of the State (census 2010).[33] The exact distribution is derived from the Huntington-Hill method. The average is around 1.4 representative per million of inhabitants (see red dashed line). Due to integer rounding effects, the actual number of representatives per million of inhabitants varies from one state to another between 1.0 (Montana) and 1.9 (Rhode Island). Due to the distribution rule, the variability is higher among states with small number of representatives.

Although the distribution seems as fair as possible among states, it is clear that some voters have more representatives than others. Hence a voter in Rhode Island has almost twice as many representatives than a voter in Montana.

Moreover, the number of seats in the presidential election is the number of representatives added to the two senator votes. Hence, the distribution of seats per inhabitants is even more distorted, as shown on Figure 2. The red dashed line represents the average of 1.7 seats per million of inhabitants in the USA. Depending on the State, this number goes from 1.5 (California) to 5.3 (Wyoming). Hence, a voter from Wyoming seems to have around 3.6 more representation than a voter from California. Again, the variability in Figure 2 is higher for small number of seats. Yet, it seems that the average number of seats per inhabitants is almost always decreasing in population!

In the end, the question is whether this distortion biases the outcome of the presidential election or if it corrects another distortion as a big state might be more often pivotal than a small one.

The aim of our simulator is exactly to study this question in the case of different standard probability models.

[33] To be consistent, we have assumed that District of Columbia has 1 representative.

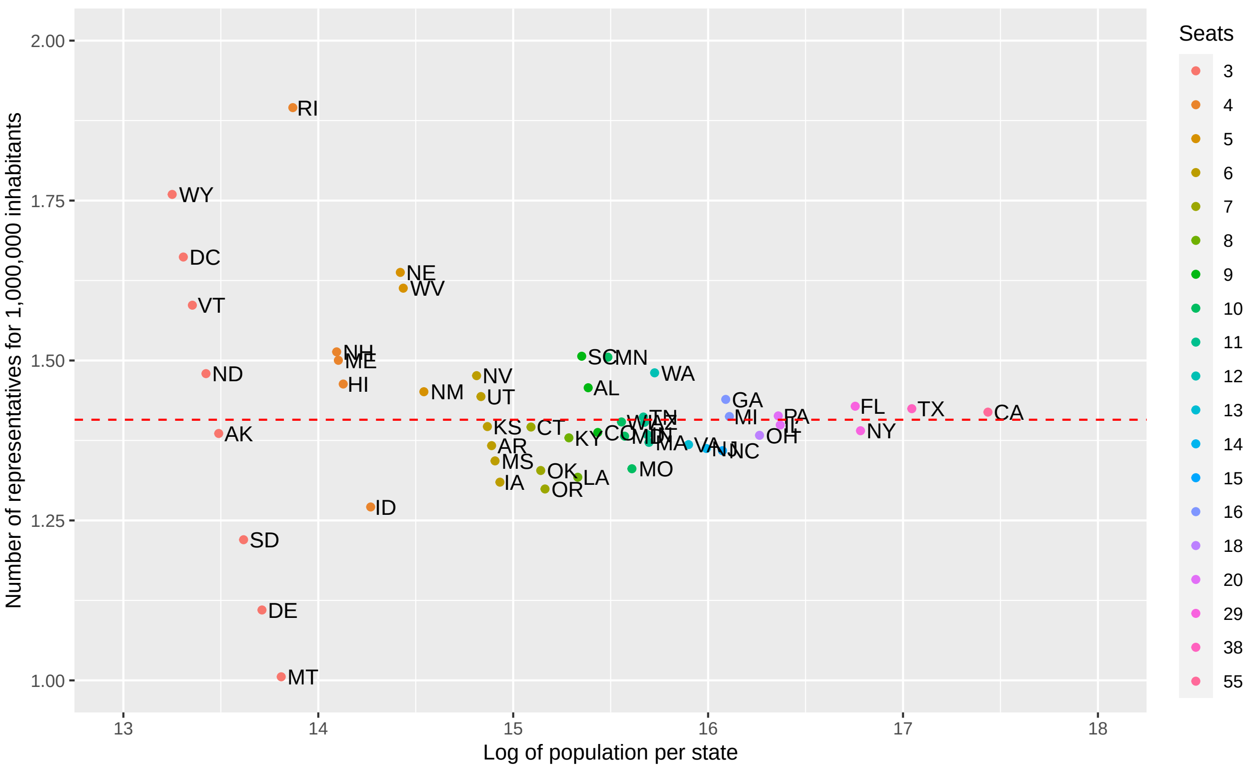

Figure 1: Electoral representatives per inhabitant ratio in each state in years 2012, 2016 and 2020

3.2 Electoral Justice with respect to Banzhaf

In this section, the simulations have been made by keeping the exact population per state. We have done 10^{12} simulations and the computation time was around 5 days, using 40 cores on a server of 58 logical cores at 3.07 GHz.

As shown on Table 2, the obtained probabilities to be pivotal are between 1.810×10^{-5} (Montana) and 6.153×10^{-5} (California). According to Bienaymé-Tchebychev, those results are significant and accurate ($\pm 3.5 \times 10^{-8}$) at a confidence level better than 95%.

Unsurprisingly, the results derived from our simulations are consistent with the theoretical ones (available for Banzhaf). Note that the maximal difference between the two values is less than $+/-10^{-8}$. So the significance and accuracy of our simulations are even better than what could be guaranteed according to Bienaymé-Tchebychev.

From Bienaymé-Tchebychev we know that the ranking of States according to the probability for a voter to be pivotal is significant and accurate at a confidence level better than 95%, except for four groups of States:

- pivotality around 2.09×10^{-5} for New Mexico < Mississippi < New Hampshire

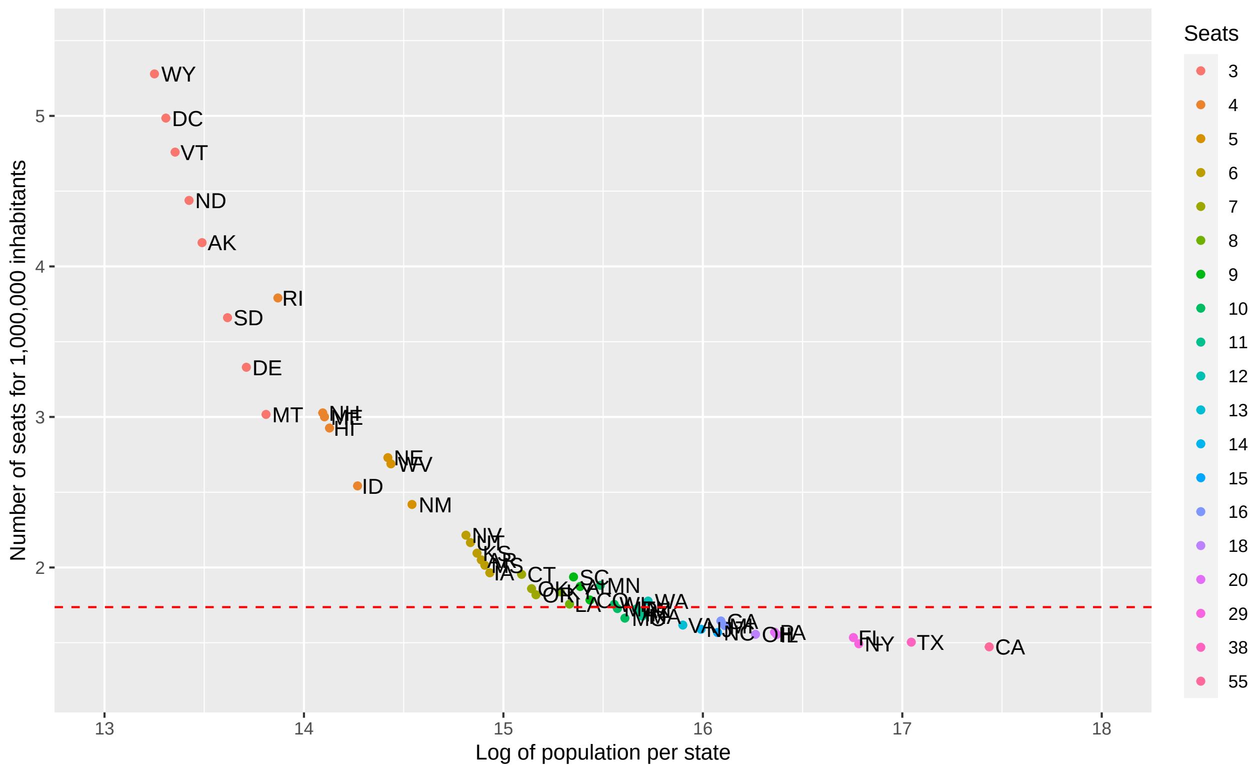

Figure 2: Electoral seats per inhabitant ratio in each state in years 2012, 2016 and 2020

- pivotality around 2.17×10^{-5} for Utah < Oklahoma (the confidence level in this ranking is of 83.2%)
- pivotality around 2.19×10^{-5} for Nevada < North Dakota
- pivotality around 2.22×10^{-5} for Nebraska < Connecticut (the confidence level in this ranking is of 93.5%)

Yet, we know that the ranking obtained trough the simulations matches perfectly the theoretical one, even in those four groups. Hence, the ranking presented in Figure 3 is not debatable.

State	Theoretical	Simulation results	
	value	pivot_IC	ratio_IC
Alabama	2.476e-05	2.475e-05	1.368
Alaska	2.125e-05	2.124e-05	1.174
Arizona	2.622e-05	2.622e-05	1.449
Arkansas	2.112e-05	2.113e-05	1.167
California	6.152e-05	6.153e-05	3.399
Colorado	2.416e-05	2.415e-05	1.334
Connecticut	2.228e-05	2.228e-05	1.231

State	Theoretical value	Simulation results pivot_IC	ratio_IC
Delaware	1.902e-05	1.901e-05	1.050
District of Columbia	2.327e-05	2.327e-05	1.286
Florida	4.111e-05	4.111e-05	2.272
Georgia	3.108e-05	3.107e-05	1.717
Hawaii	2.059e-05	2.059e-05	1.138
Idaho	1.919e-05	1.919e-05	1.060
Illinois	3.392e-05	3.392e-05	1.874
Indiana	2.604e-05	2.603e-05	1.438
Iowa	2.067e-05	2.067e-05	1.142
Kansas	2.135e-05	2.135e-05	1.180
Kentucky	2.311e-05	2.311e-05	1.277
Louisiana	2.259e-05	2.259e-05	1.248
Maine	2.085e-05	2.085e-05	1.152
Maryland	2.507e-05	2.506e-05	1.385
Massachusetts	2.592e-05	2.592e-05	1.432
Michigan	3.079e-05	3.079e-05	1.701
Minnesota	2.616e-05	2.616e-05	1.445
Mississippi	2.093e-05	2.094e-05	1.157
Missouri	2.460e-05	2.460e-05	1.359
Montana	1.810e-05	1.810e-05	1.000
Nebraska	2.224e-05	2.224e-05	1.229
Nevada	2.195e-05	2.195e-05	1.212
New Hampshire	2.094e-05	2.093e-05	1.157
New Jersey	2.853e-05	2.853e-05	1.576
New Mexico	2.093e-05	2.093e-05	1.156
New York	4.055e-05	4.055e-05	2.240
North Carolina	2.935e-05	2.935e-05	1.622
North Dakota	2.195e-05	2.196e-05	1.213
Ohio	3.212e-05	3.212e-05	1.775
Oklahoma	2.173e-05	2.173e-05	1.201
Oregon	2.149e-05	2.149e-05	1.187
Pennsylvania	3.409e-05	3.408e-05	1.883
Rhode Island	2.343e-05	2.343e-05	1.295
South Carolina	2.517e-05	2.517e-05	1.391
South Dakota	1.994e-05	1.994e-05	1.102
Tennessee	2.629e-05	2.629e-05	1.453
Texas	4.744e-05	4.744e-05	2.621
Utah	2.170e-05	2.171e-05	1.199
Vermont	2.273e-05	2.273e-05	1.256
Virginia	2.771e-05	2.771e-05	1.531
Washington	2.789e-05	2.789e-05	1.541

State	Theoretical value	Simulation results pivot_IC	ratio_IC
West Virginia	2.207e-05	2.206e-05	1.219
Wisconsin	2.527e-05	2.527e-05	1.396
Wyoming	2.394e-05	2.394e-05	1.323

Table 2: Probability for a voter to be pivotal at the presidential election in each State, with respect to Banzhaf model

Figure 3 presents for each State the ratio of pivotality ordered from the maximum to the minimum. Colors correspond to the number of electoral seats in the states. It seems that the ratio of pivotality is higher for states with larger number of seats.

Hence, in the case of Banzhaf's vote distribution model, the distortion of seats in favor of small populated states does not compensate the electoral advantage of a voter living in a high populated state. This is very clear on Figure 4.[34] For instance, a voter from California has more than two and a half more chances to be pivotal than a voter from Wyoming, although the one from Wyoming accounts for more than three and a half more seats than the one from California.

Of course, for a given number of seats, the order between states seen on Figure 2 still holds on Figure 3. For instance, for 3 seats, Wyoming is better off than District of Columbia, then Vermont, North Dakota, Alaska, South Dakota, Delaware and finally Montana (where a voter has minimal power, in the case of Banzhaf, in all the USA, for the presidential election). But the comparison does not hold between states with different number of seats. For instance, Rhode Island is in between Alaska and South Dakota on Figure 2, but much higher on Figure 3, where it is in between Wyoming and District of Columbia.

In order to better understand the mechanisms at stake, Figure 5 decomposes the pivotality part due to being a pivotal voter in his state (middle figure) and the part due to pivotality in the second tier (bottom figure). In Banzhaf's case, the total pivotality (top figure) is computed as the product of the two parts. For instance, for California (the sole pink dot corresponding to the state with 55 electoral votes): $(1.3 \times 10^{-4}) \times 0.47 \approx 6.15 \times 10^{-5}$. It is obvious that the second part plays the biggest role in the probability of being pivotal. Indeed, the second part is increasing with respect to the size of the population as well as the probability of being pivotal,

[34] We have also drawn figures with the same y-axis but with the number of electoral votes on the x-axis. These three figures should be compared to the one derived by Gelman, Silver and Edlin (2012) for an econometric model of elections.

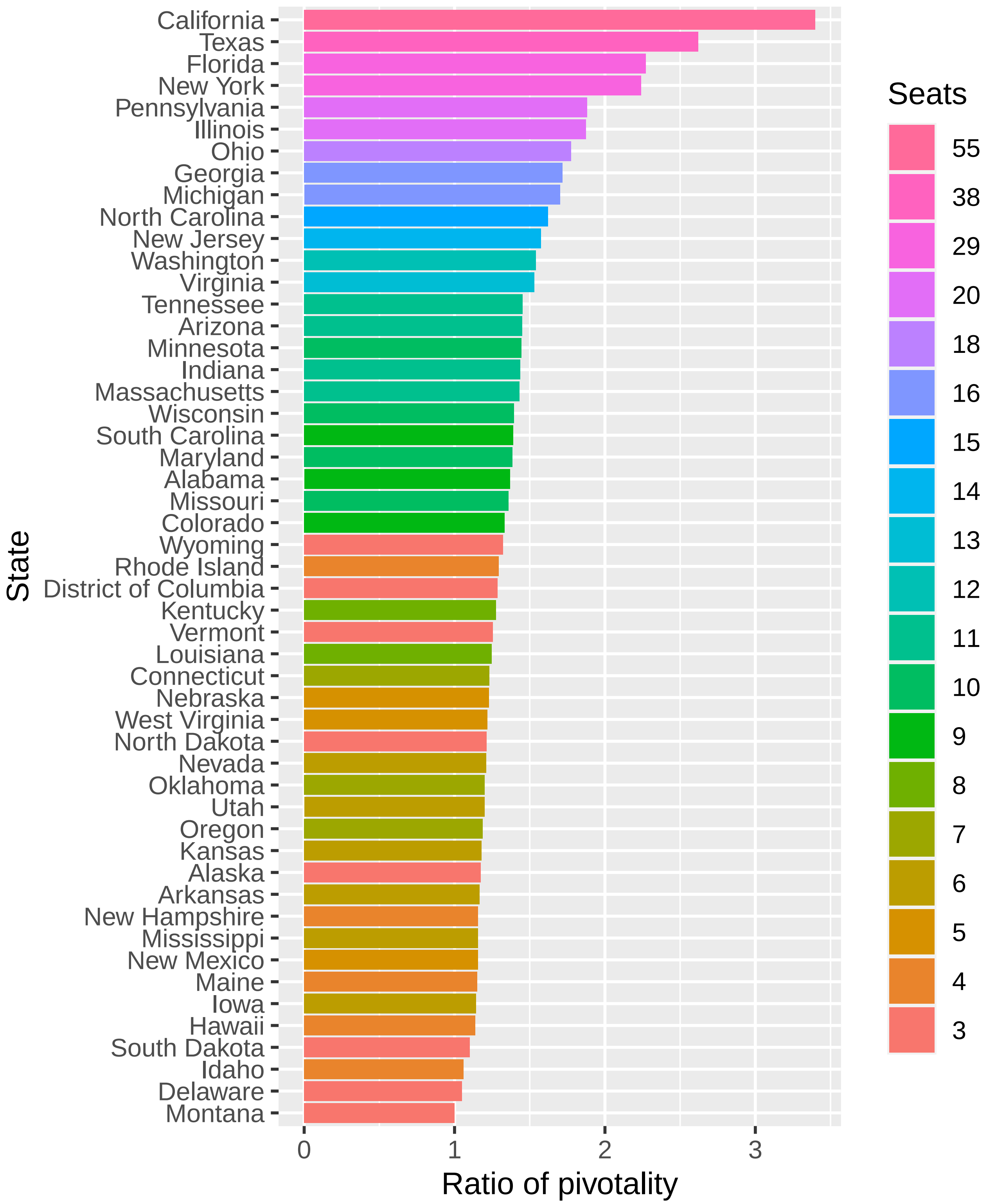

Figure 3: Pivotality ratio by state and number of seats in the case of Banzhaf, ordered by decreasing pivotality ratio

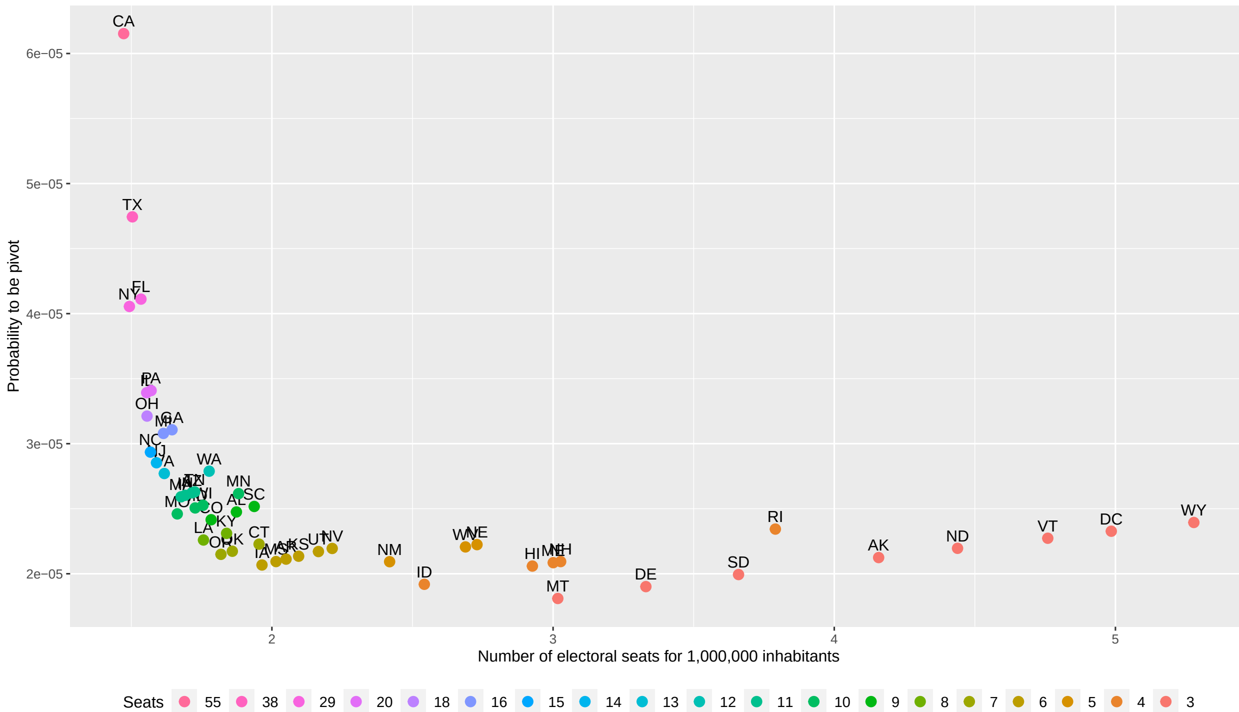

Figure 4: Pivotality ratio and number of seats per 1,000,000 inhabitants in the case of Banzhaf, depending on the State

whereas the first part is decreasing proportionally to $1/\sqrt{n^{k(i)}}$. Besides, for the states which have an equal number of seats, the second part is constant; in that configuration, this is the first part which differentiates the probability of being pivotal, and the states with a lower size of population are advantaged. This can be seen in the top figure by the linear shapes which appear by group of states with the same number of seats. Finally, we have plotted in pink (resp. blue) dashed line the average mean for a voter to be pivotal in the electoral college (resp. the popular vote) case. Most states are below the two lines which confirms in the case of Banzhaf model an inequality between citizens belonging to small states and citizens belonging to large states. Only the two biggest states (California and Texas) are above the blue line and two more additional states (New-York and Florida) are above the pink line.

3.3 Electoral Justice with respect to Shapley-Shubik

In this section, the simulations have been made by keeping the exact population per state. Again, we have done 10^{12} simulations and the computation time was around 5 days, using 40 cores on a server of 58 logical cores at 3.07 GHz.

As shown on Table 3, the obtained probabilities to be pivotal are between 1.73×10^{-9} (Montana) and 5.72×10^{-9} (California). According to Bienaymé-Tchebychev, those results are

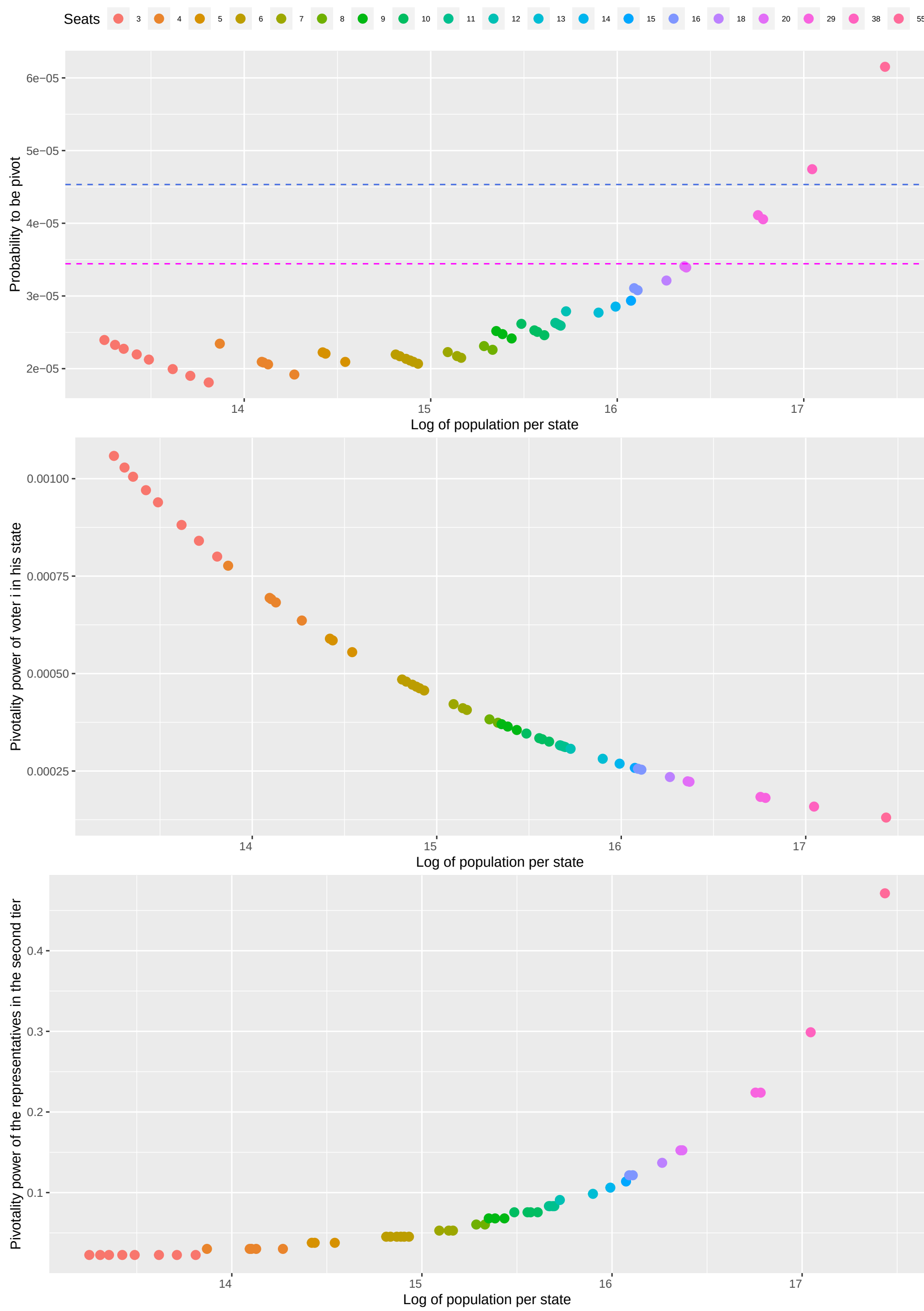

Figure 5: Pivotality probabilities overall or inside the state or among states by state and number of seats in the case of Banzhaf

significant and accurate ($\pm 2 \times 10^{-10}$) at a confidence level better than 85%. As the probabilities to be pivotal are much smaller in the Shapley-Shubik case than in the Banzhaf case, with the same setting the results are not as precise as in the Banzhaf case.

If we had wanted a precision of the same order ($\pm 3.5 \times 10^{-12}$ at a confidence level better than 95%) we should have done 10^{16} simulations. But this would take more than a century to obtain the results!

Yet, as the actual precision in the Banzhaf case was proven (from theoretical values) much better than what was guaranteed from Bienaymé-Tchebychev, it might also be the case in the Shapley-Shubik case. So we decided to use the results, even though they certainly are less precise.

The same problem occurs for the ranking of States according to the probability for a voter to be pivotal. In fact, the ranking of the top four states is known with confidence level better than 95% (according to Bienaymé-Tchebychev), except for third and fourth which could be reversed. But for the other states the ranking is known only up to +/-15 positions in average (more precise for top States and less precise for bottom States) if we want to maintain a 95% confidence level (but it is still +/-10 positions with a lower confidence level of 85%).

Hence, as the results for Banzhaf case were better than the minimum guaranteed, it is likely that the situation of Shapley-Shubik case is also better than the minimum guaranteed, but the ranking presented in Figure 6 can be debatable.

state	pivot_IAC	ratio_IAC
Alabama	2.311e-09	1.336
Alaska	1.905e-09	1.101
Arizona	2.465e-09	1.425
Arkansas	1.941e-09	1.122
California	5.721e-09	3.307
Colorado	2.263e-09	1.308
Connecticut	2.098e-09	1.213
Delaware	1.823e-09	1.054
District of Columbia	2.169e-09	1.253
Florida	3.873e-09	2.239
Georgia	2.990e-09	1.728
Hawaii	1.929e-09	1.115
Idaho	1.842e-09	1.065
Illinois	3.182e-09	1.839
Indiana	2.469e-09	1.427
Iowa	1.942e-09	1.123
Kansas	2.055e-09	1.188
Kentucky	2.178e-09	1.259
Louisiana	2.066e-09	1.194
Maine	1.955e-09	1.130
Maryland	2.377e-09	1.374

state	pivot_IAC	ratio_IAC
Massachusetts	2.448e-09	1.415
Michigan	2.851e-09	1.648
Minnesota	2.490e-09	1.439
Mississippi	1.951e-09	1.128
Missouri	2.306e-09	1.333
Montana	1.730e-09	1.000
Nebraska	2.061e-09	1.191
Nevada	2.091e-09	1.209
New Hampshire	1.955e-09	1.130
New Jersey	2.601e-09	1.503
New Mexico	2.036e-09	1.177
New York	3.882e-09	2.244
North Carolina	2.694e-09	1.557
North Dakota	2.061e-09	1.191
Ohio	3.029e-09	1.751
Oklahoma	2.050e-09	1.185
Oregon	2.072e-09	1.197
Pennsylvania	3.227e-09	1.865
Rhode Island	2.250e-09	1.301
South Carolina	2.391e-09	1.382
South Dakota	1.896e-09	1.096
Tennessee	2.530e-09	1.463
Texas	4.491e-09	2.596
Utah	2.071e-09	1.197
Vermont	2.161e-09	1.249
Virginia	2.565e-09	1.482
Washington	2.568e-09	1.484
West Virginia	2.081e-09	1.203
Wisconsin	2.389e-09	1.381
Wyoming	2.226e-09	1.287

Table 3: Probability for a voter to be pivot at the presidential election in each State, with respect to Shapley-Shubik model

Figure 6 presents for each State the ratio of pivotality ordered from the maximum to the minimum. Note that, as in the Banzhaf setting, Califorina and Montana are found in the top and bottom positions, with the same ratio (around 3.4). More generally, it seems that the Shapley-Shubik and Banzhaf ratios of pivotality behave similarly: Figures 3 and 6 are almost indistinguishable. The biggest states are advantaged over others.

In order to highlight the small differences between the two figures, we have represented in Figure 10 the ranking of the states according to each probability model. It appears that the ranks are indeed very similar (particularly for the biggest states). The small differences concern essentially the states which have less than 10 seats. These states have very close values of probability of being pivotal and the differences of ranking could be attributed to the precision of our simulations, as explained in introduction of this section.

Indeed, we have seen, in the case of Shapley-Shubik, that the top four States were known (confidence level of 95%) except that it was not sure between the third and fourth which actually came first, whereas in the case of Banzhaf, the order was known for sure. Figure 10 shows that both models rank identically the top four States, except that third and fourth are in reverse order, which could absolutely be due to the lack of precision between which State is third or fourth in the case of Shapley-Shubik. So this 'inversion'-like shape does not necessarily mean that both models disagree. With more simulations, Shapley-Shubik's ranking could be the same as Banzhaf's.

The same applies for all other 'inversion'-like shapes. For instance, the difference in ranking between the two models is at most of 7 positions (Alaska), which is far below the 17 that would still be non significant (confidence level of 85% or even 25 positions with confidence level of 95%).

The fact that the two models give similar results is also confirmed in Figure 8 which represents the probability for being pivotal with respect to the number of seats per 100,000 inhabitants: This figure is almost indistinguishable from the equivalent Figure 4 for the Banzhaf setting. It clearly appears that the biggest states have the highest probabilities for being pivotal. It is also interesting to notice that for the states with less than 9 seats, the differences of pivotality are very weak. Still, for the states which have the same number of seats, the size of population has a negative effect on the probability of being pivotal, as it can be seen for the lowest states with 3 seats. Thus, the smaller state of the USA (Wyoming) in term of population has a probability of being pivotal nearly equivalent to the states which have around 10 seats.

To summarize the findings of this section, we believe that Banzhaf and Shapley-Shubik cases lead to identical ratio differences in the probability for a voter of being pivotal in a given State. The mathematical proof of this is beyond the scope of this paper and the number of simulations (leading to $5 + 5 = 10$ days of computations) does not give empirical certainty of this[35], but

[35] To achieve this certainty, we have seen that more than a century of computations would be necessary. Another idea, which was not fully implemented here, would be to test the hypothesis with a lower number of voters: for example, dividing State population by 1000. The expected probabilities should be around 10^{-6} instead of 10^{-9}, so we could obtain accurate and more precise results with the same number of simulations (10^{12}, and so again in 5 days). Of course, for the sake of comparison, we should do the same with the Banzhaf case. In a former, less efficient version of our simulator, we tested for Shapley-Schubik a population divide factor of 5683, leading to States with population in the interval $(100; 6571)$. But we were able to perform only 10^8 simulations, at that time, and so ended with the same type of conclusion as here. The obtained probabilities to be pivotal

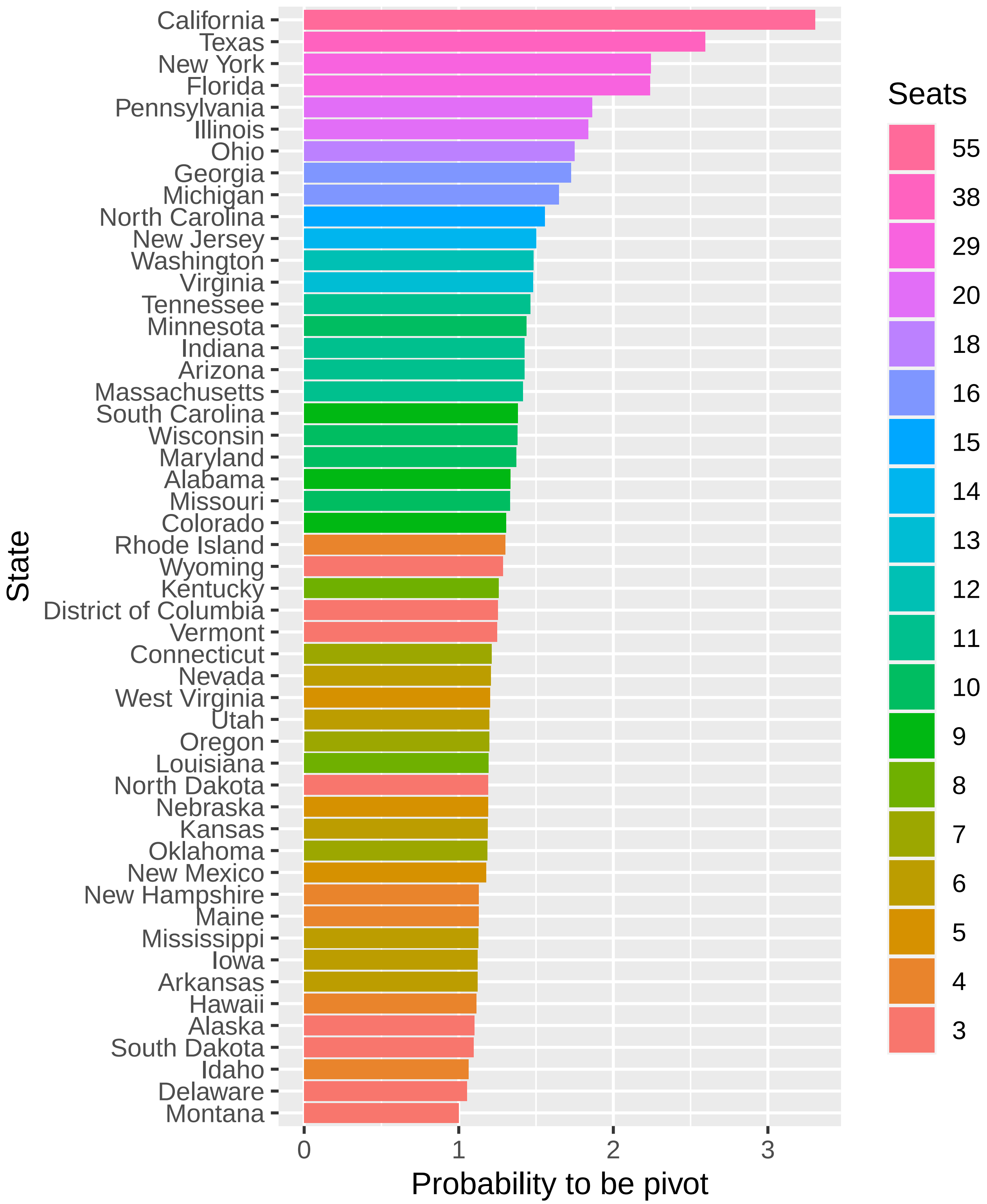

Figure 6: Pivotality ratio by state and number of seats in the case of Shapley-Shubik, ordered by decreasing pivotality ratio

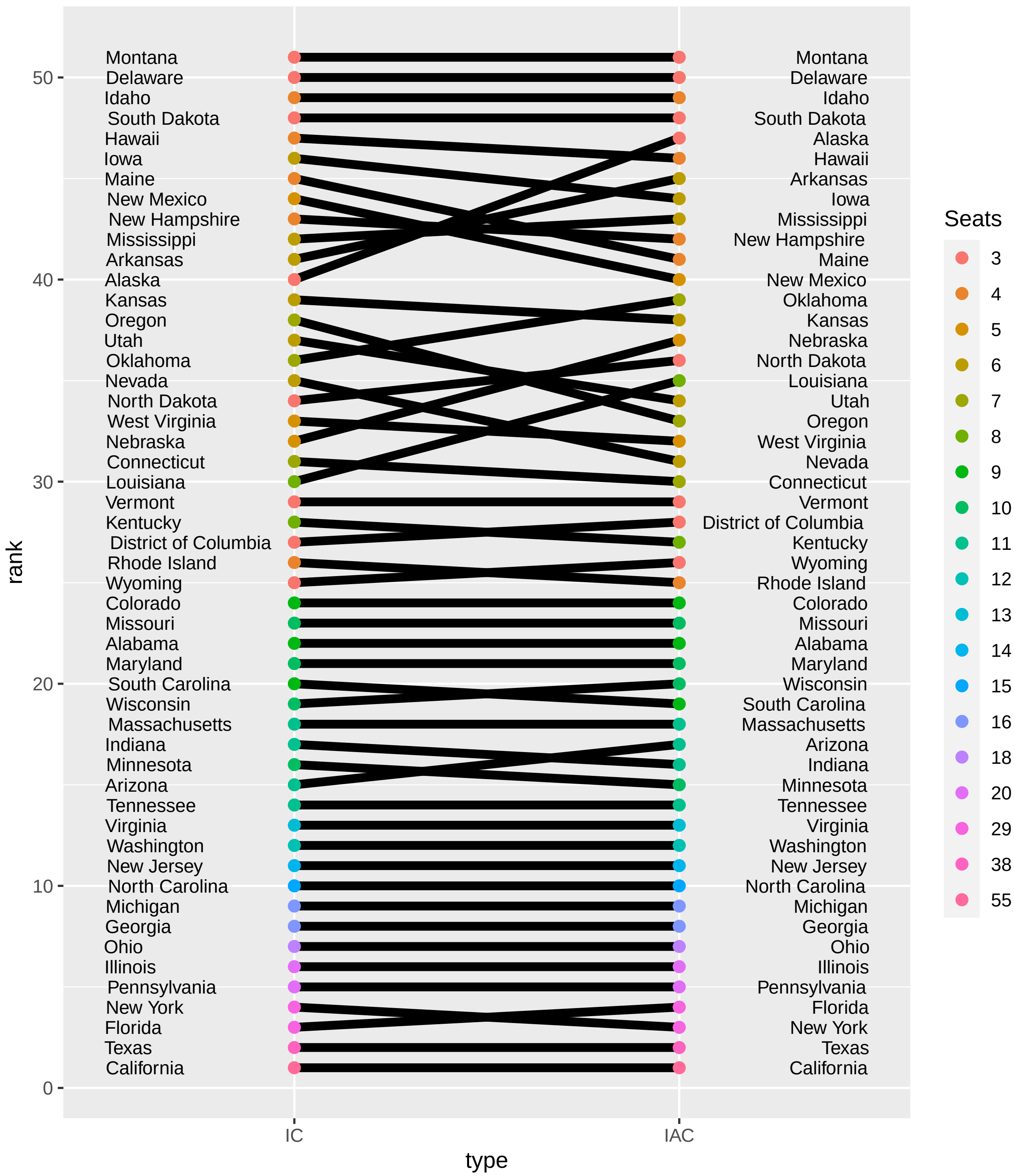

Figure 7: Comparison of the ranking between Banzhaf and Shapley-Shubik

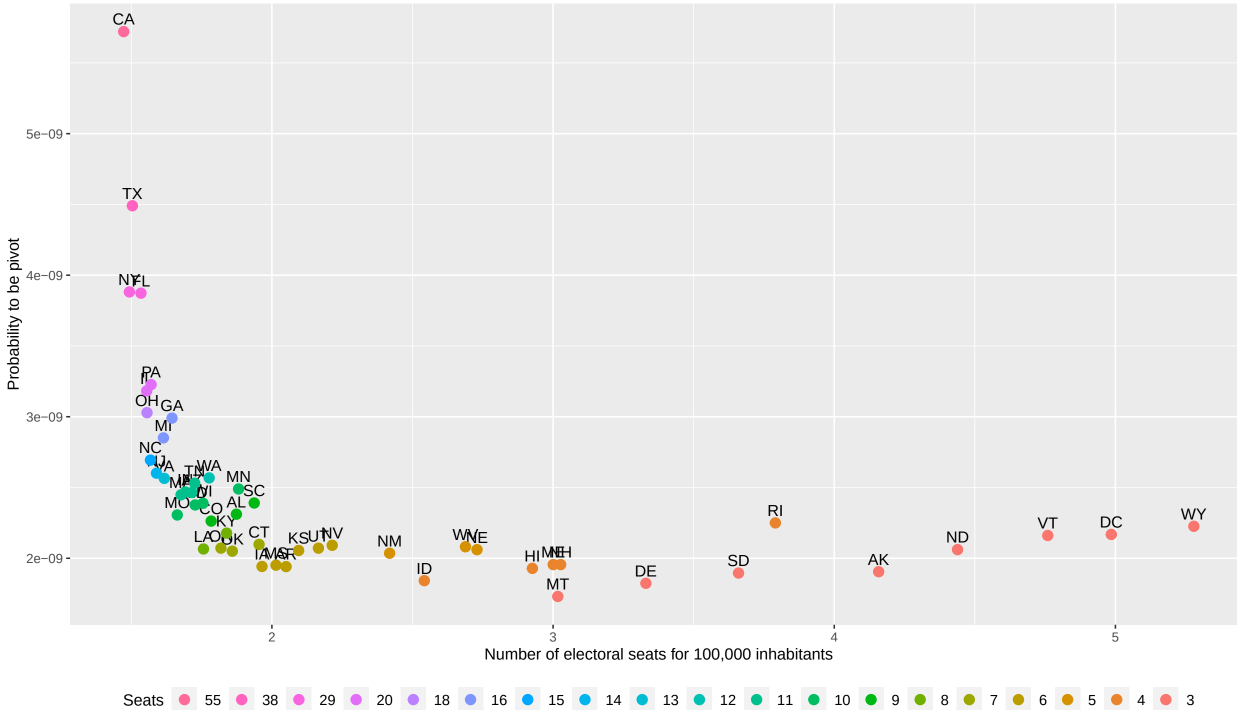

Figure 8: Pivotality ratio and number of seats per 100,000 inhabitants in the case of Shapley-Shubik, depending on the State

it also does prove that our belief is not ruled out. Hence, although it might seem puzzling, as the two models behave very differently in many other ways, we still believe that they lead to identical ratio differences in the probability for a voter of being pivotal in a given State[36].

3.4 Electoral Justice with respect to May

In this section, the simulations have been made by keeping the exact population per state. Once again, we have done 10^{12} simulations and the computation time was around 5 days, using 40 cores on a server of 58 logical cores at 3.07 GHz.

As shown on Table 4, the obtained probabilities to be pivotal are between 1.158×10^{-8} (New-York) and 3.977×10^{-8} (Wyoming). According to Bienaymé-Tchebychev, those results are significant and accurate $(5 \times \pm 10^{-10})$ at a confidence level better than 85%. As the probabilities to be pivotal are much smaller in the May case than in the Banzhaf case, but higher than in the Shapley-Schubik case, with the same setting the results precision will also be in-between those of the two previous cases.

were between 3.284×10^{-5} (California) and 9.680×10^{-6} (Montana). According to Bienaymé-Tchebychev, those results were significant and accurate $(\pm 3 \times 10^{-6})$ at a confidence level better than 95%.

[36] We call the attention of the reader on the fact that appendices 1 and 2 shed some light on these questions.

If we had wanted a precision of the same order of Banzhaf case ($\pm 3.5 \times 10^{-11}$ at a confidence level better than 95%) we should have done close to 10^{15} simulations. But this would take more than a decade to obtain the results!

Yet, as the actual precision in the Banzhaf case was proven (from theoretical values) much better than what was guaranteed from Bienaymé-Tchebychev, it might also be the case in the May case.

Indeed, and unsurprisingly, the results derived from our simulations are consistent with the theoretical ones (available for May). Note that the maximal difference between the two values is less than $\pm 2.4 \times 10^{-10}$. Hence, the observed accuracy is twice better as the guaranteed one and for all of the 51 States.

The same question arises for the ranking of States according to the probability for a voter to be pivotal. In fact, the ranking of the top 16 states is known with confidence level better than 95% (according to Bienaymé-Tchebychev), except for a few group of States for which there are still some uncertainties. For instance, between States 15 and 16, the order could be reversed (but not if we allow for a 90.9% confidence level). Same situation with States 2 and 3 (but not if we allow for a 93.3% confidence level). But then there are still 3 groups of States (in the top 16) for which the inside-group ranking is debatable (even at confidence level of 80%:

- pivotality around 2.0×10^{-8} for States 13 and 14
- pivotality around 2.2×10^{-8} for States 9 to 12
- pivotality around 2.8×10^{-8} for States 6 and 7

Yet, we know that the ranking obtained trough the simulations matches almost perfectly the theoretical one, even in those three groups, except for States 9 and 10: New Hampshire should be ranked just before Montana (and not the other way around, as found by the simulations, but the difference found between the two States is very small: around 3×10^{-11} only).

As for the other states (ranked 17 or below), the ranking, according to Bienaymé-Tchebychev, is known only up to +/-7 positions in average (more precise for top States and less precise for bottom States) if we want to maintain a 95% confidence level (but it is still +/-4 positions with a lower confidence level of 85%). Yet, the simulated ranking is much closer to the theoretical one. In fact, there are only 5 groups of States where the inside-group ranking would be different with more simulations:

- pivotality around 1.19×10^{-8} for States 45 to 49: the real order should be Pennsylvania (45), North Carolina, Illinois, Florida and Ohio (49)

- pivotality around 1.26×10^{-8} for States 38 to 40: the real order should be Massachusetts (38), Missouri and California (40)
- pivotality around 1.29×10^{-8} for States 35 and 36: the real order should be Tennessee (35) and Arizona (36)
- pivotality around 1.32×10^{-8} for States 32 and 33: the real order should be Louisiana (32) and Wisconsin (33)
- pivotality around 1.33×10^{-8} for States 30 to 31: the real order should be Colorado (30) and Washington (31)

Each time, the difference to obtain the correct order is quite small (between 5×10^{-12} only and 1.5×10^{-10}).

Hence, the ranking presented in Figure 3 is not perfect, but not so bad either.

State	Theoretical value	Simulation results pivot_May	ratio_May
Alabama	1.416e-08	1.415e-08	1.222
Alaska	3.135e-08	3.116e-08	2.691
Arizona	1.297e-08	1.304e-08	1.126
Arkansas	1.547e-08	1.541e-08	1.331
California	1.262e-08	1.271e-08	1.098
Colorado	1.348e-08	1.338e-08	1.155
Connecticut	1.475e-08	1.475e-08	1.273
Delaware	2.511e-08	2.530e-08	2.185
District of Columbia	3.759e-08	3.748e-08	3.236
Florida	1.185e-08	1.202e-08	1.038
Georgia	1.249e-08	1.245e-08	1.075
Hawaii	2.207e-08	2.210e-08	1.908
Idaho	1.917e-08	1.893e-08	1.635
Illinois	1.185e-08	1.187e-08	1.025
Indiana	1.280e-08	1.272e-08	1.099
Iowa	1.483e-08	1.477e-08	1.275
Kansas	1.581e-08	1.588e-08	1.372
Kentucky	1.389e-08	1.384e-08	1.195
Louisiana	1.327e-08	1.319e-08	1.139
Maine	2.263e-08	2.259e-08	1.951
Maryland	1.306e-08	1.307e-08	1.128
Massachusetts	1.268e-08	1.256e-08	1.085
Michigan	1.226e-08	1.227e-08	1.059
Minnesota	1.422e-08	1.421e-08	1.227
Mississippi	1.520e-08	1.531e-08	1.322

State	Theoretical value	Simulation results pivot_May	ratio_May
Missouri	1.257e-08	1.259e-08	1.087
Montana	2.275e-08	2.283e-08	1.971
Nebraska	2.059e-08	2.062e-08	1.781
Nevada	1.671e-08	1.671e-08	1.443
New Hampshire	2.283e-08	2.279e-08	1.968
New Jersey	1.205e-08	1.224e-08	1.057
New Mexico	1.825e-08	1.803e-08	1.557
New York	1.153e-08	1.158e-08	1.000
North Carolina	1.189e-08	1.201e-08	1.037
North Dakota	3.347e-08	3.342e-08	2.886
Ohio	1.184e-08	1.198e-08	1.034
Oklahoma	1.404e-08	1.406e-08	1.214
Oregon	1.373e-08	1.370e-08	1.183
Pennsylvania	1.197e-08	1.195e-08	1.032
Rhode Island	2.859e-08	2.837e-08	2.450
South Carolina	1.464e-08	1.463e-08	1.263
South Dakota	2.760e-08	2.764e-08	2.387
Tennessee	1.305e-08	1.296e-08	1.119
Texas	1.183e-08	1.185e-08	1.023
Utah	1.634e-08	1.633e-08	1.410
Vermont	3.589e-08	3.599e-08	3.108
Virginia	1.225e-08	1.226e-08	1.059
Washington	1.345e-08	1.339e-08	1.156
West Virginia	2.028e-08	2.039e-08	1.761
Wisconsin	1.327e-08	1.328e-08	1.146
Wyoming	3.981e-08	3.977e-08	3.434

Table 4: Probability for a voter to be pivotal at the presidential election in each State, with respect to May model

Figure 9 presents for each State the ratio of pivotality ordered from the maximum to the minimum. Interestingly, the results seems somewhat opposite of those derived in the Banzhaf and Shapley-Shubik settings. For instance, California flipped from upper position to one of the smallest ratios (1.1), while Wyoming (previously with a ratio of 1.3) is now on the top (with a ratio of 3.4). Notice also that the highest ratio in the three models is 3.4.

Hence, wih May's model, the states with the highest probability of being pivotal are those with the smallest number of seats. This is in line with the common popular wisdom concerning the current representation of states in the Electoral College.

Figure 10 shows the ranking difference between the states according to Shapley-Shubik and

May. It appears that the ranks have been drastically changed. The states with high (resp. low) probability of being pivotal in the Shapley-Shubik case are often those which have low (resp. high) probabilities in the May case.

Figure 11 decomposes the pivotality part due to being a pivotal voter in his state (middle figure) and the part due to pivotality in the second tier (bottom figure). In May's case, the total pivotality (top figure) is computed as the product of the two parts. The first part of the equation is decreasing proportionally to $1/n^{k(i)}$. It appears that this part has a higher effect on the pivotality compared to the Banzhaf case. This explains why the small states have such a high probability of being pivotal.

Another way to understand the phenomenon is to represent the probability of being pivotal with respect to the number of seats per 100,000 inhabitants, as in Figure 12. It appears that the trend is linear and increasing.

To summarize the findings of this section, we believe that May case can lead to almost opposite rankings to the two other cases, concerning the ratio differences in the probability for a voter of being pivotal in a given State. When States have the same number of seats, the order is maintained, but when they have a different number of seats, the order is reversed. The mathematical proof of this is beyond the scope of this paper and the number of simulations (leading to $5 + 5 = 10$ days of computations) does not give empirical certainty of this[37], but already give some insights of this almost reverse ranking. It is important to stress that not all models agree on the ranking. So conclusions on whether big or small States take most advantage of the current electoral system are not straightforward.

4 The Twelve Amendment

As already pointed out several times in section 2, our description of the Electoral College departs slightly from the real one. First, two states (Maine and Nebraska) do not allocate their electoral votes according to the "winner take all" rule but use instead the following rule : the two "senators" electoral votes go to the state winner while the congressional electoral votes go the congressional district winners. We don't think that this difference has a great impact on our analysis but we have not done any estimations of the differences. The second and seemingly more serious difference has to do with our treatment of ties. In our simulations, we have assumed that in case of a tie in either a state election and/or in the Electoral College[38],

[37] To achieve this certainty, we have seen that more than a decade of computations would be necessary. Another idea, which was not fully implemented here, would be to test the hypothesis with a lower number of voters, as suggested in footnote 35.

[38] According to Hayes (2012), there are $0.16976480564070 \times 10^{14}$ ways of arriving at a tie roughly 0.75 percent of 2^{51}, the total number of profiles in the upper-tier.

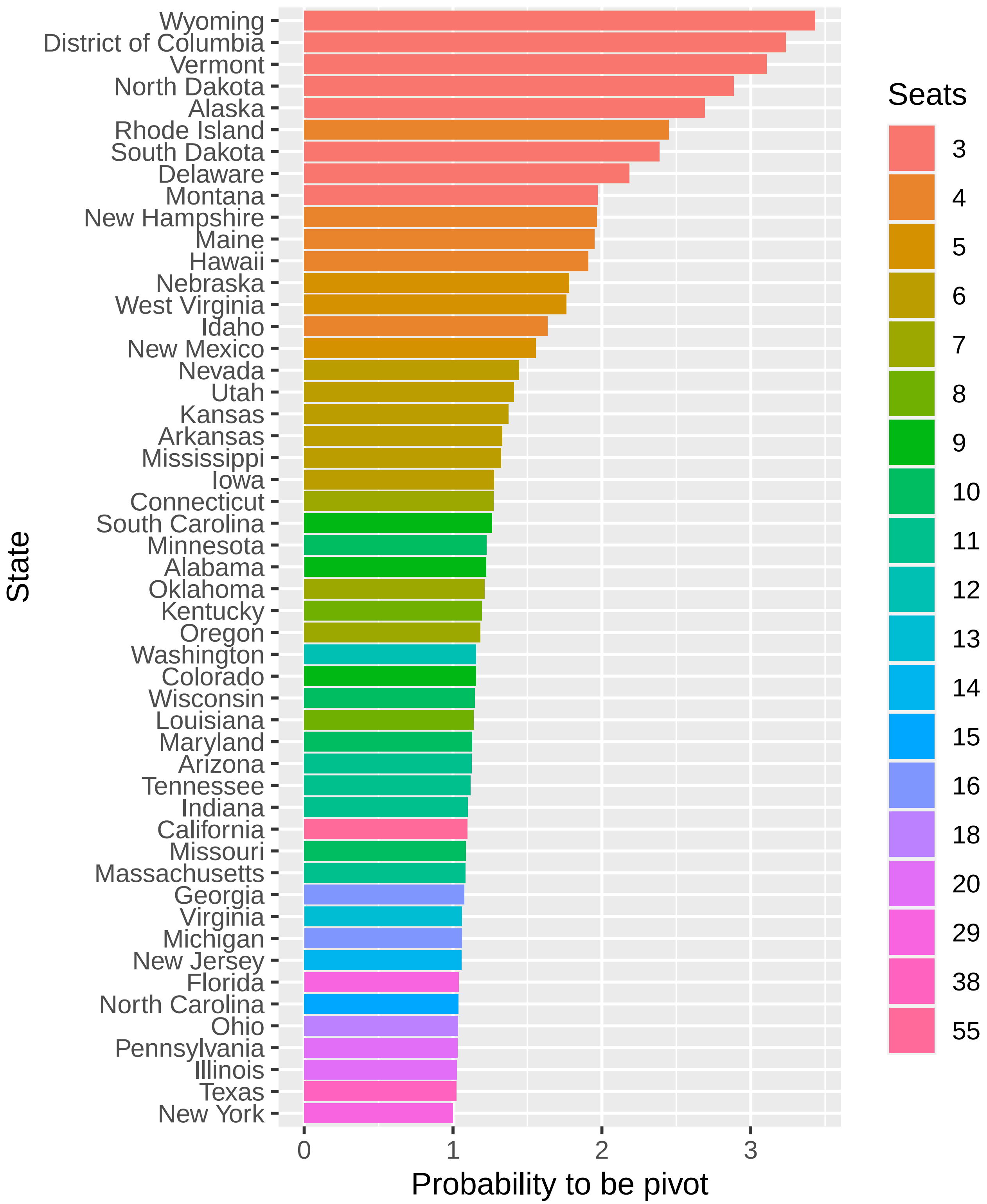

Figure 9: Pivotality ratio by state and number of seats in the case of May, ordered by decreasing pivotality ratio

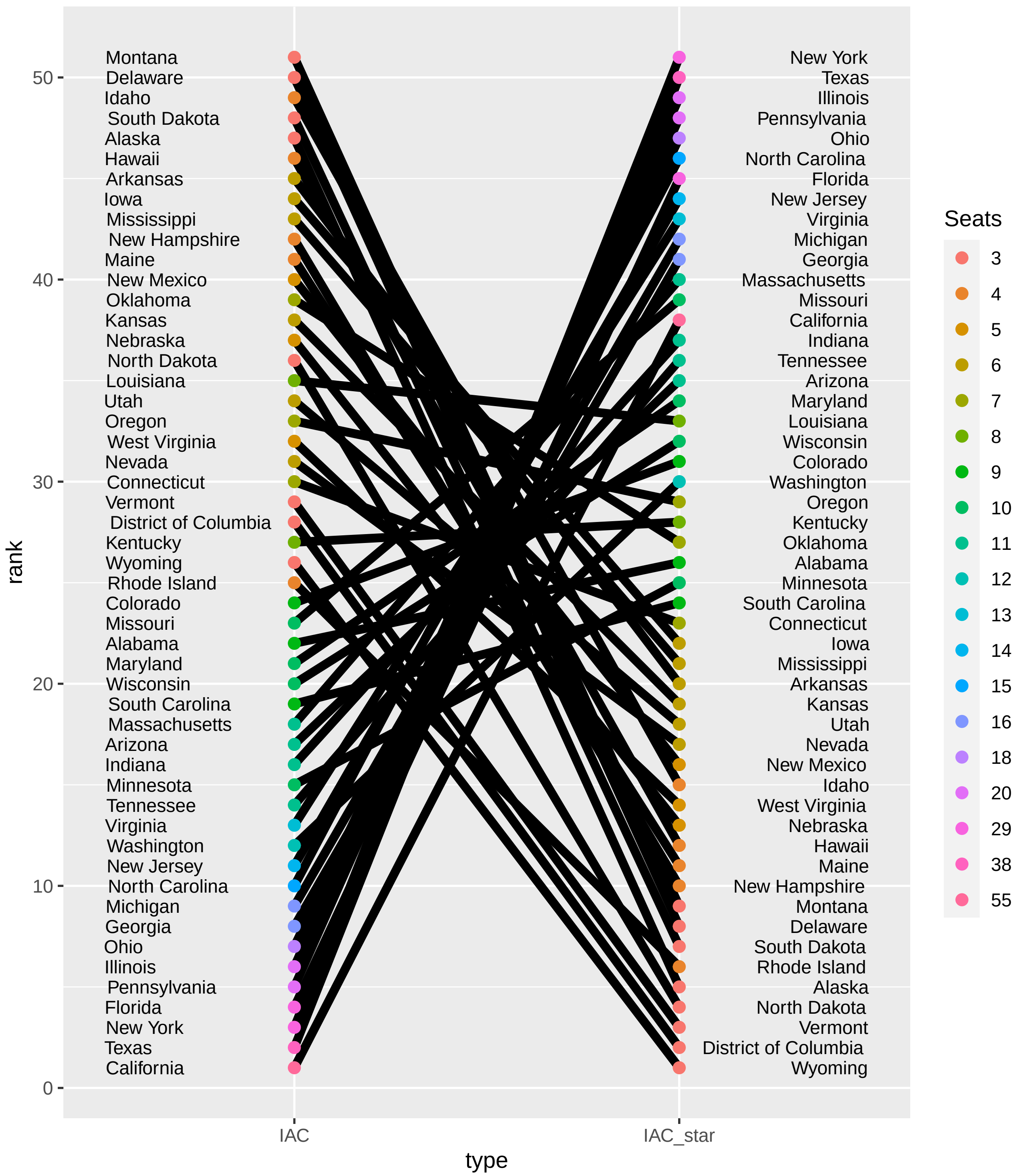

Figure 10: Comparison of the ranking between Shapley-Shubik and May

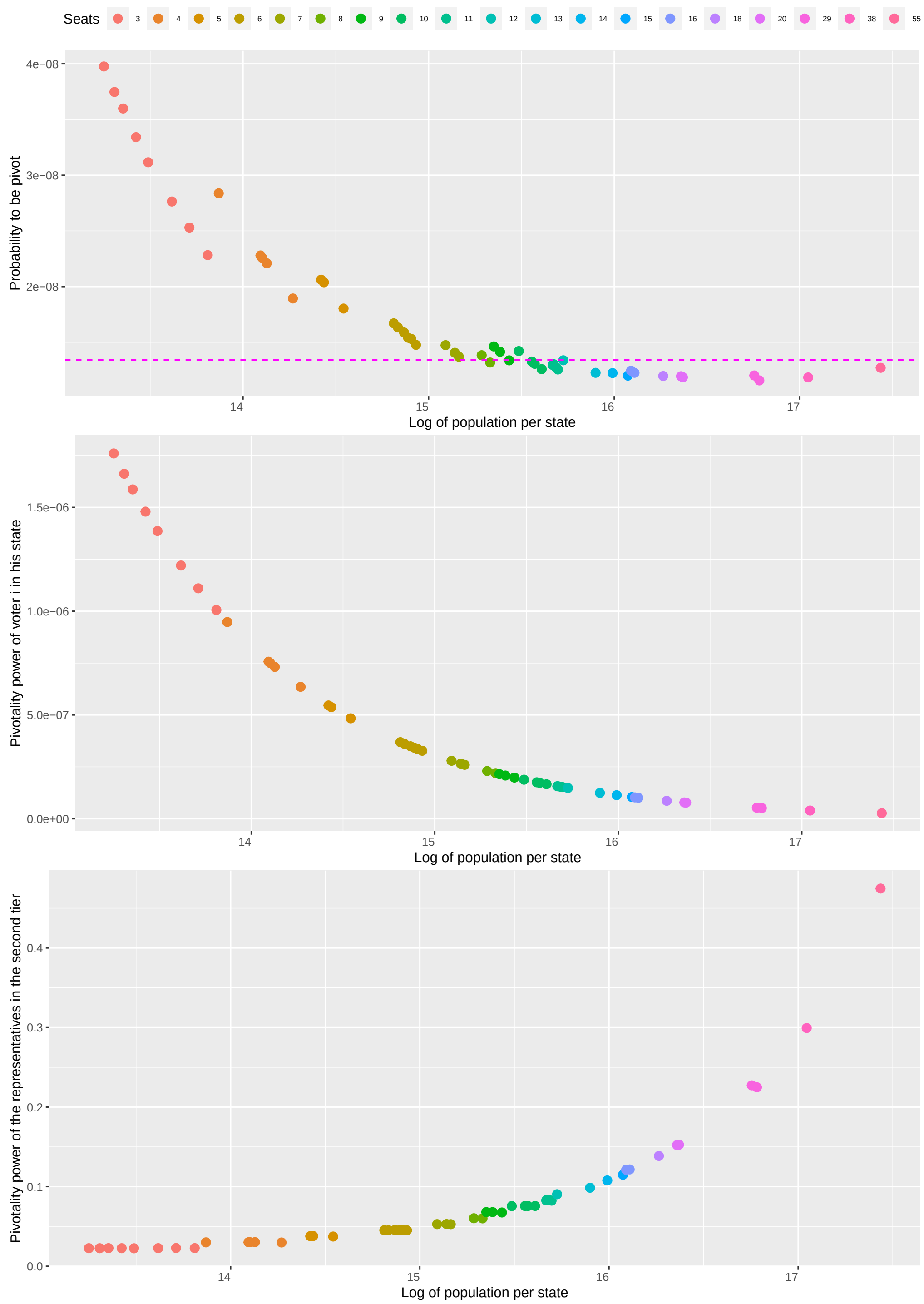

Figure 11: Pivotality probabilities overall or inside the State or among states by state and number of seats in the case of May

the winner was determined through the random draw of fair coin. In reality, a new simple game

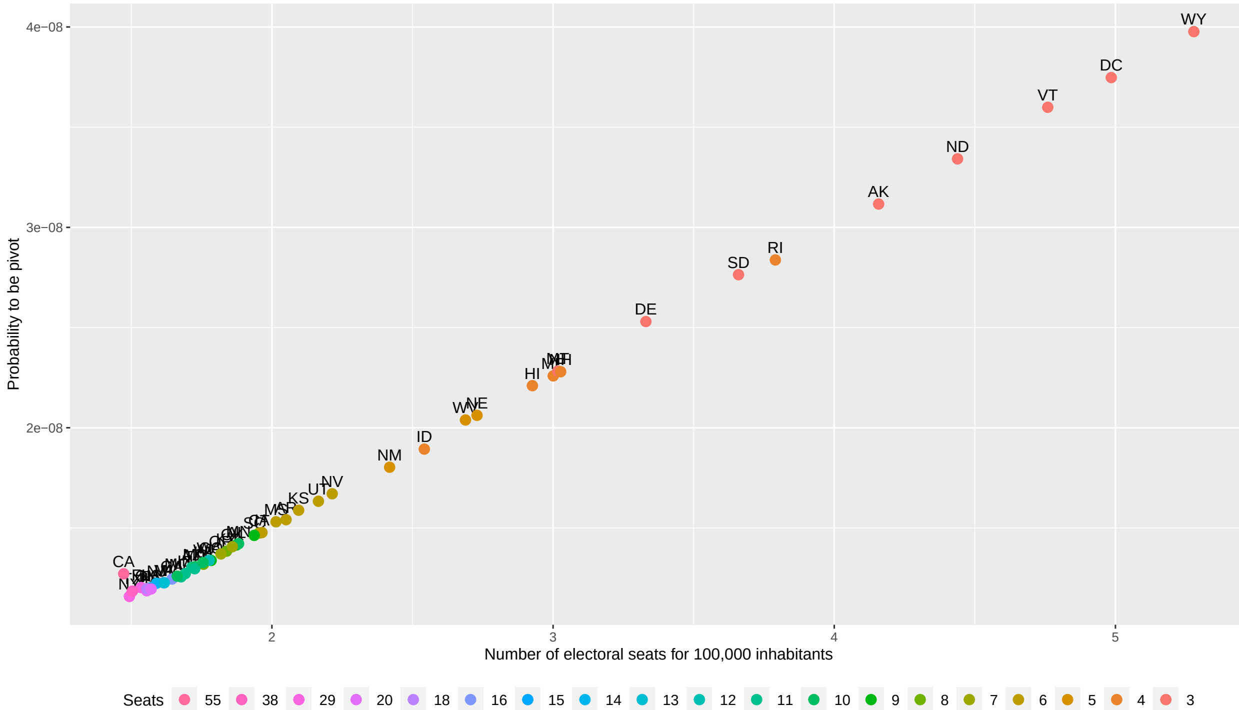

Figure 12: Pivotality ratio and number of seats per 100,000 inhabitants in the case of May, depending on the State

with a different set of players (the members of the house of representatives[39] at the time of the presidential election with each state represented by a single voter with only one vote) with preferences possibly different from those of the voters (the members of the house have been elected few years before the presidential election).[40]

To speculate on the effects of this tie-breaking rule, consider the IC probability model and assume first that the preferences of the representatives are independent from those of the voters today and second that there is a clear majority in each state and a clear majority in the second tier. In such case, note that if a tie occurs in the electoral college, then the two candidates are elected both with a probability $\frac{1}{2}$. So under the above assumptions, there are no differences between the version of the Electoral College considered in this paper and the true one.

[39]The district of Columbia is not part of the game and in fact in case of a tie, the rule is to have further votes (as much as needed) as long as a fixed deadline has not expired.

[40]So truly this tie-breaking simple game is itself a compound simple game where the second tier is the ordinary majority game with 50 players $(\{1,\ldots,50\}, \mathcal{W}(q,w))$ where $w = (1,\ldots,1)$ and $q = 25$ and the first tiers are the majority games with the representatives of the states as the basic players.

5 Concluding Remarks

In this paper, dedicated to the measurement of electoral justice (or lack of) in the 2010 Electoral College, we have obtained several results. First, we have seen that the results obtained by Owen in the case of the 1960 and 1970 Electoral College, on top of which the coincidence between the conclusions drawn respectively from Banzhaf and Shapley-Shubik's probability models, remain valid in 2010. Both probability models conclude to a violation of electoral justice at the expense of small states. Second, we have also shown that this conclusion has completely flipped upside-down when we use instead May's probability model: this model concludes to a violation of electoral justice at the expense of large states. Besides unifying through a common measurement methodology disparate approaches, one main lesson is that the conclusion on electoral justice is sensitive to the probability models which are used and to the type and magnitude of correlation between voters that they carry.

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7 Appendix: Some Theoretical Notes on the Coincidence between the Shapley-Shubik (IAC) and Banzhaf (IC) Probability Models

The main purpose of this appendix is threefold. First, in appendix 1, we prove that the asymptotic coincidence of the three main probability models (and thus in particular Owen's Coincidence) holds true for a discrete version of the Shapley-Shubik probability model. Then, in appendix 2, we point out why the argument of appendix 1 does not extend to the original Shapley-Shubik probability model[41] and we sketch some arguments on the speed of convergence to 0 of the Shapley-Shubik power indices. Finally, in appendix 3, we shed some light on the relevance of the Penrose's approximation in our situation.

7.1 Appendix 1: A Discrete Version of the Shapley-Shubik Probability Model

Consider the model π_p where all the preferences P_i proceed from independent Bernoulli draws with parameter $p \in [0,1]$. In this case, for all $T \subseteq N \setminus \{i\} : \pi_{-i}(T) = p^t(1-p)^{n-1-t}$. The Banzhaf probability model is $\pi_{\frac{1}{2}}$. The probability that a voter i in a population of n individuals (with n odd)[42] is pivotal for the ordinary majority mechanism is equal to:

$$Piv(i, \pi_p, n) = \binom{\frac{n-1}{2}}{n-1} p^{\frac{n-1}{2}} (1-p)^{\frac{n-1}{2}}.$$

If n is large, by using Stirling's formula, we obtain, after some straightforward simplifications:

$$Piv(i, \pi_p, n) \simeq \sqrt{\frac{2}{\pi n}} (4p(1-p))^{\frac{n-1}{2}}. \quad (3)$$

[41] The careful reader will also notice that the arguments in appendix 1 extend to an arbitrary (i.e. not necessarily uniform) probability distribution and that lower and upper bounds similar to those derived in appendix 2 can be obtained for an arbitrary absolutely continuous probability distribution on $[0,1]$.

[42] The case where n is even is treated similarly.

It is easy to see that the function $\phi(p) \equiv (4p(1-p))^{\frac{n-1}{2}}$ is single peaked and symmetric on $[0,1]$ with a peak in $p = \frac{1}{2}$ with $\phi(\frac{1}{2}) = 1$. This implies that for all $p^* \in \left]0, \frac{1}{2}\right[$, there exist $\alpha(p^*) > 0$ such that:

$$Piv\,(i, \pi_p, n) \leq \sqrt{\frac{2}{\pi n}} e^{-\alpha(p^*)\frac{n-1}{2}}, \text{ for all } p \in [0, p^*] \cup [1-p^*, 1]\,. \tag{4}$$

Consider now the following discrete version π of the Shapley-Shubik model defined as follows. Conditionally to a draw of the parameter p in the finite set $\left\{p_1, \ldots, p_m, \frac{1}{2}, 1-p_m, \ldots, 1-p_1\right\}$ according to the uniform distribution, the preferences P_i proceed from independent Bernoulli draws with parameter p.[43]

Let us assume that N is partitioned into K states: $N = \cup_{1 \leq k \leq K} N^k$ and n^k voters of state k are endowed with w^k electoral votes. We consider the replications of the vector $\mathbf{n} = \left(n^1, n^2, \ldots, n^K\right)$ defined, for any odd[44] integer r, as $\mathbf{n}_r = \left(rn^1, rn^2, \ldots, rn^K\right)$. Consider any r, any k and any individual i in state $k = k(i)$. We obtain:[45]

$$Piv\,(i, \pi, r) = \frac{2\sum_{j=1}^{m} Piv\,(i, \pi_{p_j}, r)}{2m+1} + \frac{Piv\left(i, \pi_{\frac{1}{2}}, r\right)}{2m+1}.$$

Since the multiplicative formula applies pointwise, we deduce:

$$Piv\,(i, \pi, r) = \frac{2\sum_{j=1}^{m} \underline{Piv}\,(i, \pi_{p_j}, r) \times \overline{Piv}\,(k(i), \pi_{p_j})}{2m+1} + \frac{\underline{Piv}\left(i, \pi_{\frac{1}{2}}, r\right) \times \overline{Piv}\left(k(i), \pi_{\frac{1}{2}}\right)}{2m+1}. \tag{5}$$

In section 3, we have argued that :

$$\underline{Piv}\left(i, \pi_{\frac{1}{2}}, r\right) \simeq \sqrt{\frac{2}{\pi r n^{k(i)}}} \text{ and } \overline{Piv}\left(k(i), \pi_{\frac{1}{2}}\right) \text{ is proportional to } w^{k(i)},$$

where $w^{k(i)}$ represents the weight (number of votes) of State $k(i)$ in the second tier.

To bound $Piv\,(i, \pi, r)$, note first that (3) and (5) imply the following trivial lower bound:

$$\sqrt{rn^{k(i)}}\,Piv\,(i, \pi, r) \geq \frac{1}{2m+1}\sqrt{\frac{2}{\pi}}\;\overline{Piv}\left(k(i), \pi_{\frac{1}{2}}\right).$$

On the other hand, from (4) we deduce that:

$$\underline{Piv}\,(i, \pi_{p_j}, r) \leq \sqrt{\frac{2}{\pi r n^{k(i)}}} e^{-\alpha^*(p_m)\frac{rn^{k(i)}-1}{2}} \text{ for all } j = 1, \ldots, m.$$

[43]Note that the probability model is symmetric around $\frac{1}{2}$.
[44]The case where r is even is treated similarly.
[45]With an abuse in notation, as we write $Piv\,(i, \pi, r)$ instead of $Piv\,(i, \pi, n)$ with $n = r\sum_{k=1}^{K} n^k$.

Therefore for any $\epsilon > 0$, there exists $r(\epsilon)$ large enough such that:

$$\sqrt{rn^{k(i)}}\underline{Piv}\left(i, \pi_{p_j}, r\right) \leq \epsilon \text{ for all } r \geq r(\varepsilon) \text{ and for all } j = 1, \ldots, m.$$

By combining this inequality with (5), we deduce that:

$$\sqrt{rn^{k(i)}}Piv\left(i, \pi, r\right) \leq \epsilon + \frac{1}{2m+1}\sqrt{\frac{2}{\pi}}\ \overline{Piv}\left(k(i), \pi_{\frac{1}{2}}\right) \text{ for all } r \geq r(\varepsilon)$$

i.e. asymptotically:

$$Piv\left(i, \pi, r\right) \simeq \frac{1}{2m+1}B\left(i, r\right).$$

We have demonstrated that if the population of voters is large enough and if the original Shapley-Shubik model is replaced by a discrete version, then the "discrete Shapley-Shubik" power of voter i is $\frac{1}{2m+1}$ times the Banzhaf power index B_i of voter i. An immediate conclusion is that the ratios $\frac{Piv(i,\pi,r)}{Piv(j,\pi,r)}$ coincide with the ratios $\frac{B(i,r)}{B(j,r)}$.

A slight modification of the above argument shows that the asymptotic coincidence extends to a discrete version of the May's Model π' defined as follows. Conditionally to a sequence of K draws $p^1, p^2, \ldots, p^K$ of the parameter p in the finite set:

$$\left\{p_1, p_2, \ldots, p_m, \frac{1}{2}, 1 - p_m, \ldots, 1 - p_2, 1 - p_1\right\}$$

according to the uniform distribution, the preferences P_i in state $k(i)$ proceed from independent Bernoulli draws with parameter $p^{k(i)}$. Consider any individual i in state $k = k(i)$. We obtain:

$$Piv\left(i, \pi', r\right) = \frac{\underline{Piv}\left(i, \pi_{\frac{1}{2}}, r\right) + 2\sum_{j=1}^{m}\underline{Piv}\left(i, \pi_{p_j}, r\right)}{2m+1} \times \overline{Piv}\left(k(i), \pi_{\frac{1}{2}}\right). \tag{6}$$

From (6), we can proceed along the same lines as before. Note that this general coincidence between the three models is asymptotic. When we compare (5) and (6), under the conjecture that $\overline{Piv}\left(k(i), \pi_p\right) < \overline{Piv}\left(k(i), \pi_{\frac{1}{2}}\right)$ for all $k = 1, \ldots, K$ and all $p \neq \frac{1}{2}$, we deduce that:

$$Piv\left(i, \pi, r\right) < Piv\left(i, \pi', r\right) < Piv\left(i, \pi_{\frac{1}{2}}, r\right) \text{ for all } i.$$

For the sake of illustration, consider the case of three equipopulated states, $m = 1$ and $p_1 = \frac{1}{2} - \delta$ where δ is a small positive number. For any voter in state 1, we obtain:

$$Piv\left(i, \pi_{\frac{1}{2}}, r\right) \simeq \sqrt{\frac{2}{\pi rn^1}} \times \frac{1}{2},$$

$$Piv\left(i,\pi',r\right)\simeq\sqrt{\frac{2}{\pi rn^{1}}}\times\frac{1}{2}\times\left(\frac{1+2\left(1-4\delta^{2}\right)}{3}\right)=\sqrt{\frac{2}{\pi rn^{1}}}\times\left(\frac{3-8\delta^{2}}{6}\right)$$

and:

$$Piv\left(i,\pi,r\right)\simeq\sqrt{\frac{2}{\pi rn^{1}}}\times\left(\frac{\frac{1}{2}+2\left(1-4\delta^{2}\right)\times p(\delta)}{3}\right),$$

where:

$$p(\delta)=2\left[\sum_{l=\frac{rn^{1}+1}{2}}^{rn^{1}}\binom{rn^{1}}{l}\left(\frac{1}{2}-\delta\right)^{l}\left(\frac{1}{2}+\delta\right)^{rn^{1}-l}\right]\times\left[\sum_{l=0}^{\frac{rn^{1}-1}{2}}\binom{rn^{1}}{l}\left(\frac{1}{2}-\delta\right)^{l}\left(\frac{1}{2}+\delta\right)^{rn^{1}-l}\right].$$

When δ is different from 0 and r is large, $p(\delta)$ can be much smaller than $\frac{1}{2}$. For instance, even when $\delta=10^{-1}$ and $rn^{1}=10^{3}+1$, we obtain:

$$p(\delta)=1.616\times10^{-10}$$

and therefore:

$$Piv\left(i,\pi,r\right)\simeq0.16667\sqrt{\frac{2}{\pi rn^{1}}}$$

while:

$$Piv\left(i,\pi_{\frac{1}{2}},r\right)=Piv\left(i,\pi',r\right)=0.48667\sqrt{\frac{2}{\pi rn^{1}}}.$$

When instead $\delta=0.25$ and $rn^{1}=10^{3}+1$, we obtain:

$$p(\delta)=1.279\times10^{-64}$$

and:

$$\begin{aligned}Piv\left(i,\pi_{\frac{1}{2}},r\right)&\simeq0.48667\sqrt{\frac{2}{\pi rn^{1})}}\\Piv\left(i,\pi',r\right)&\simeq0.41667\sqrt{\frac{2}{\pi rn^{1}}}\\Piv\left(i,\pi',r\right)&\simeq0.16667\sqrt{\frac{2}{\pi rn^{1}}}.\end{aligned}$$

7.2 Appendix 2: Back to the Original Shapley-Shubik Probability Model

In the standard Shapley-Shubik probability model, the parameter p is drawn in $[0,1]$ according to the continuous uniform distribution. We cannot use anymore the argument developed in appendix 1 since the parameter p can be arbitrarily closed to $\frac{1}{2}$. Consider voter i in state $k = k(i)$. By definition:

$$\begin{aligned} Sh_i(r) &= \int_0^1 Piv(i,\pi_p,r)dp = \int_0^{\frac{1}{2}-\frac{c}{\sqrt{rn^k}}} Piv(i,\pi_p,r)dpdp + \\ & \int_{\frac{1}{2}+\frac{c}{\sqrt{rn^k}}}^1 Piv(i,\pi_p,r)dpdp + \int_{\frac{1}{2}-\frac{c}{\sqrt{rn^k}}}^{\frac{1}{2}+\frac{c}{\sqrt{rn^k}}} Piv(i,\pi_p,r)dpdp, \end{aligned} \tag{7}$$

where:

$$Piv(i,\pi_p,r) = \underline{Piv}(i,\pi_p,r) \times \overline{Piv}\,(k(i),\pi_p) \text{ and } c \text{ is an arbitrary positive constant.}$$

Let us consider first $\underline{Piv}(i,\pi_p,r)$ when $p = p_{rkc} = \frac{1}{2} - \frac{c}{\sqrt{rn^k}}$. From (3), we obtain:

$$\underline{Piv}(i,\pi_p,r) \simeq \sqrt{\frac{2}{\pi rn^k}}(1-\frac{4c^2}{rn^k-1})^{\frac{rn^k-1}{2}}.$$

Since:

$$(1-\frac{4c^2}{rn^k})^{\frac{rn^k-1}{2}} \text{ tends to } \sqrt{e^{-4c^2}} = e^{-2c^2} \text{ when } r \to \infty,$$

we deduce:

$$Piv(p,rn^k) \simeq \sqrt{\frac{2}{\pi rn^k}}e^{-2c^2} \text{ when } r \to \infty,$$

and

$$\underline{Piv}(i,\pi_p,r) \geq \sqrt{\frac{2}{\pi rn^k}}e^{-2c^2} \text{ when } r \to \infty \text{ for all } p \in \left[\frac{1}{2}-\frac{c}{\sqrt{rn^k}}, \frac{1}{2}+\frac{c}{\sqrt{rn^k}}\right]. \tag{8}$$

Consider now the second term $\overline{Piv}\,(k(i),\pi_p)$. To evaluate $\overline{Piv}\,(k(i),\pi_p)$ when $p = p_{rkc}$, we consider first the probability $P(l,r,c)$ that a majority of voters in state l vote D when the probability model is π_p with $p = p_{rkc}$. By definition:

$$P(l,r,c) = \text{Prob}\left(S_{l,r,c} \geq \frac{rn^l}{2}+\frac{1}{2}\right) =$$

$$\text{Prob}\left(\frac{S_{l,r,c}-p_{rkc}rn^l}{\sqrt{p_{rkc}(1-p_{rkc})rn^l}} \geq \frac{c}{\sqrt{p_{rkc}(1-p_{rkc})}}\sqrt{\frac{n^l}{n^k}} + \frac{1}{2\sqrt{p_{rkc}(1-p_{rkc})rn^k}}\right)$$

with:

$$S_{l,r,c} = \sum_{i=1}^{rn^l} X_i,$$

where the X_i are independent Bernoulli random variables with parameter $p = p_{rkc}$. From the central limit theorem for triangular arrays, we deduce that:

$$P(k,r,c) \text{ converges to } \int_{2c}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \equiv 1 - F(2c\sqrt{\frac{n^l}{n^k}}), \tag{9}$$

where F denotes the CDF of the standard Gaussian random variable.

When r is large, $Piv_k\,(p_{rk})$ is equivalent to the pivotality $\varphi(k,c)$ of player k in the weighted majority game $(\{1,\ldots,K\}\,,w)$ for the probability model $\pi \equiv \pi_{p^1\ldots p^K}$ where the votes of the K players are independent and player l vote D with probability $p^l \equiv 1 - F(2c\sqrt{\frac{n^l}{n^k}})$ for all $l = 1,\ldots,K$. From (7), (8) and (9), under the conjecture[46] that the function $\varphi(k,c)$ is decreasing with respect to c, we deduce that for r large enough:

$$Sh(i,r) \geq \sqrt{\frac{2}{\pi n^{k(i)} r}} e^{-2c^2} \varphi(k,c) \int_{\frac{1}{2}-\frac{c}{\sqrt{n^{k(i)}r}}}^{\frac{1}{2}+\frac{c}{\sqrt{n^{k(i)}r}}} dp = \frac{2}{n^{k(i)}r}\sqrt{\frac{2}{\pi}} c e^{-2c^2} \varphi(k,c).$$

To provide estimates of this lower bound for all $k = 1,\ldots,K$, we need an analysis of the function $ce^{-2c^2}\varphi(k,c)$ for all k. It is equal to 0 when $c = 0$ and tends rapidly to 0 when $c \to \infty$.[47] Therefore, the supremum is well defined and we can write the lower bound as:

$$Sh(i,r) \geq \frac{2}{n^{k(i)}r}\sqrt{\frac{2}{\pi}} \underset{c\in[0,+\infty[}{Sup}\; ce^{-2c^2}\varphi(k,c).$$

On the other hand, since $\overline{Piv}\,(k,\pi_p) \leq 1$ for all k and all p, we obtain:[48]

$$Sh(i,r) \leq \underset{p\in[0,1]}{Sup}\,\overline{Piv}\,(k,\pi_p) \times \int_0^1 \underline{Piv}(i,\pi_p,r)dp = \underset{p\in[0,1]}{Sup}\,\overline{Piv}\,(k,\pi_p) \times \frac{1}{n^{k(i)}r}.$$

The two bounds are distinct but display the same speed of convergence to 0, namely $\frac{1}{n^{k(i)}r}$. We conjecture that $rSh(i,r)$ converges to a limit when r tends to ∞.

To conclude this appendix, let us point out that in the symmetric case (equipopulated states (say $n^k = 1$, for all $k = 1,\ldots,K$) and equal weights), the evaluation is trivial. This follows from the fact to which we already alluded, that for any simple game the vector of Shapley-Shubik powers is the vector of Shapley values of the players for the TU simple game constructed from

[46]This conjecture is driven by the fact that the function $1 - F(2c\sqrt{\frac{n^l}{n^k}})$) is obviously decreasing with respect to c and that the intuition that the pivotality of any player in any simple game and probability model $\pi_{p^1\ldots p^K}$ with $p_l < \frac{1}{2}$ for all $l = 1,\ldots,K$ is increasing with respect to p_l.

[47]In the equipopulated case, $\varphi(k,c) = \varphi(c) = \binom{K-1}{\frac{K-1}{2}}\left(1-F(2c)\right)^{\frac{K-1}{2}}\left(F(2c)\right)^{\frac{K-1}{2}}$.

[48]To do so, we use the formula $\int_0^1 p^k(1-p)^r dp = \frac{k!r!}{(k+r+1)!}$. When $k = r = \frac{n-1}{2}$, we deduce: $\binom{n}{\frac{n-1}{2}}\int_0^1 p^k(1-p)^r dp = \frac{1}{n}$.

the original simple game. Since the total sum to allocate is equal to 1 and all players are the same, we deduce trivially that:

$$Sh\left(i,r\right)=\frac{1}{n}=\frac{1}{Kr}\text{ for all }i.$$

In the asymmetric case, we can deduce by the same token that for all r, there is at least one i such that:[49]

$$Sh\left(i,r\right)\geq\frac{1}{n}=\frac{1}{r\sum_{k=1}^{K}n^{k}}=\frac{1}{n^{k(i)}r}\times\frac{1}{\sum_{k=1}^{K}\frac{n^{k}}{n^{k(i)}}}$$

and at least one i such that:

$$Sh\left(i,r\right)\leq\frac{1}{n}=\frac{1}{r\sum_{k=1}^{K}n^{k}}=\frac{1}{n^{k(i)}r}\times\frac{1}{\sum_{k=1}^{K}\frac{n^{k}}{n^{k(i)}}},$$

but we cannot deduce directly any further information from that this simple fact.

8 Appendix 3: Validity of the Penrose's Approximation

In this third appendix, we study the validity of the Penrose's approximation for the second tier of the electoral mechanism (called the state game in Owen, 1975). The second tier is a weighted majority game with 51 players/voters. In the case where the Bernoulli parameter is $p=\frac{1}{2}$, we obtain the Banzhaf/IC measure of influence B^k of each state k for $k=1,\ldots,51$. The exact values of these numbers can be obtained by using the friendly program conceived by Leech and Leech. The results are summarized in the following table.

Weight	B	Weight	B
55	0.471147	11	0.083202
38	0.298862	10	0.075594
29	0.223975	9	0.067999
20	0.152464	8	0.060416
18	0.136921	7	0.052842
16	0.121475	6	0.045277
15	0.113784	5	0.037720
14	0.106113	4	0.030169
13	0.098460	3	0.022622
12	0.090823	.	.

Table 1: Weights and Banzhaf Values in the 2010 U.S. Electoral College

[49] In fact, by symmetry, this lower bound applies to all voters in state $k(i)$.

The Penrose's approximation approximates the Banzhaf value of a player by a value depending upon the vector of weights. Several variants of the Penrose's approximation exist (Penrose, 1946, 1954, Lindler and Machover, 2004) and many scholars have addressed through simulations or mathematical results the scope of validity of these approximation(s) (Chang Chua and Machover, 2006, Lindner and Machover, 2004, Lindner and Owen, 2007). Let us focus on the approximation:

$$B^k \simeq \sqrt{\frac{2}{\pi}} \times \frac{w^k}{\sqrt{\sum_{1 \leq j \leq K} \left(w^k\right)^2}}.$$

From the perspective of ratios, note that if this approximation is correct, then the ratios that we obtain are the ratios of weights discussed at the end of section 2. The approximation is decent but far from perfect. For instance the ratio of the first two largest Banzhaf values is equal to $\frac{0.471147}{0.298862} = 1.5765$ while the ratio of weights is equal to $\frac{55}{38} = 1.4474$. For the two smallest values the ratio of their Banzhaf values is equal to $\frac{0.032981}{0.024727} = 1.3338$ while the ratio of weights is equal to $\frac{4}{3} = 1.3333$.

A complementary way of comparing states is obtained through the computation of their relative powers and weights.

Normalized Weight	Normalized B	Normalized Weight	Normalized B
0.10223	0.113608	0.020446	0.020062
0.070632	0.072065	0.018587	0.018228
0.053903	0.054007	0.016729	0.016397
0.037175	0.036764	0.014870	0.014568
0.033457	0.033016	0.013011	0.012742
0.029740	0.029291	0.011152	0.010918
0.027881	0.027437	0.0092937	0.009095
0.026022	0.025587	0.0074349	0.007275
0.024164	0.023742	0.0055762	0.005455
0.022305	0.021900	.	.

Table 2: Normalized Weights and Banzhaf Values in the 2010 U.S. Electoral College

We see that the normalized weights under estimate the Banzhaf values for the first 3 largest weights (i.e. 4 states) but over estimate the Banzhaf values of all other states.[50]

[50] A similar observation was made by Lindner and Machover (2004) for the 1970 U.S. Electoral College.