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“Costly state verification with Limited Commitment”

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Costly state verification with Limited Commitment

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Abstract: This paper examines a principal-agent model that the principal mandates actions and conducts costly inspections without transfers. The principal prefers lower actions, while the agent prefers higher actions and has private information about his type. The agent is protected by ex-post participation and rejects any action below his private type. The principal faces a trade-off between mandating lower actions and the risk the the agent rejects actions and chooses his outside option. We analyze various levels of the principal's commitment ability. If the principal can commit to both inspections and actions when no inspection is performed, and if the principal's fear of ruin is greater than the agent's, then a deterministic inspection policy is optimal. Additionally, if the principal cannot commit to either inspections or actions, the highest equilibrium payoff does not involve non-deterministic inspection strategies. Finally, if the inspection cost is low and the principal commits to inspecting whenever requested by the agent, the principal can achieve the payoff of the optimal deterministic inspection policy.

Keywords: Costly state verification, mechanism design, cheap talk, inspection, limited commitment, regulation.

JEL numbers: D82, D86, M48.

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1. Introduction

Addressing the climate crisis requires widespread carbon dioxide reduction to prevent or lessen severe environmental impacts. Market incentives alone often fall short in prompting companies to cut pollution and embrace necessary technologies, making government intervention essential.

Government action can take various forms, including subsidies, regulatory standards, inspections, R&D support, and emission-related taxes or tradeable permits. Regulatory standards are the most common and encompass various approaches. Performance standards, for instance, set limits on emissions per unit of product, such as restricting CO_2 emissions to a certain amount per kilowatt-hour of electricity generated.[1]

In this paper, we examine two key regulatory tools: mandating standards and inspections. We explore the optimal regulatory behaviour for enforcing standards and conducting inspections, considering different levels of commitment for the regulator.

Firms incur different costs for adjusting their technology to meet mandated standards. The government can set these standards to foster firms to adopt new technologies. Additionally, the government has the authority to audit firms to assess the costs associated with implementing these technologies.

Our model presumes that the benefits of adopting a technology are known to the government, while the costs are known only to the polluting firms. The government can discover these costs through audits. Importantly, we assume that the government cannot compel a firm to adopt a technology. The firm can move the business to another country or state. Therefore, the government's objective is to develop a cost-effective strategy that incentivizes efficient enough firms to adopt abatement technologies.

We also analyze the impact of varying levels of commitment ability of the regulator. Specifically, we assume that the regulator cannot commit to standards once inspections have taken place. We investigate the implications of this limited commitment power and identify the key instruments the regulator should use to effectively overcome the lack of commitment.[2]

[1] For more information see: Solomon (2007), section 13.2.1.1 regulations and standards.

[2] Lack of commitment can arise from various sources. 1) Legal enforcement: For legal enforcement to be effective, the court must remain impartial and not collude with any party, and the contract must be clear and ideally complete. Furthermore, the court needs to verify

Section 2 considers the problem of a Principal (regulator) who can commit to inspection and actions (standards) in case of no inspection. The Agent (firm) prefers higher actions (more pollution and emission, lower performance standards) and the Principal prefers lower actions (less pollution and emission, higher performance standards) and cannot use transfers. The Agent is protected by Ex-post participation constraint; therefore, the Principal faces a trade-off between low actions and the risk that the Agent rejects the action, and chooses his outside option. We discuss properties of the optimal mechanism if the Principal can commit to actions post audit.

In Section 3, we present the Principal's optimization problem and demonstrate that if the Principal's fear of ruin is greater than the Agent, then deterministic inspection is the optimal choice. We also provide an upper bound on the payoff of stochastic inspection relative to deterministic inspection. This upper bound is calculated as the aggregate difference in fear of ruin between the Principal and the Agent, scaled by the Principal's marginal utility. In section 3.1 we focus on deterministic inspection. We show that the optimal mechanism with deterministic inspection is a cutoffs policy that splits types into three regions. Low types (efficient types) are never inspected and face a cap on their actions. The intermediate types are inspected and are mandated to a first best action. Finally, high types (inefficient types) are excluded. The Principal offers a low action to high types, and they refuse to undertake this action. This structure highlights the importance of inspecting intermediate types which limits the low types' rents while obtaining a low action for efficient types. Finally commitment on actions in case of inspection (post audit) does not increase the Principal's payoff.

In Section 4, we explore various levels of commitment ability of the Principal. Section 4.1 assumes that while the Principal cannot commit to inspections, she still can commit to actions in case of no inspection. We demonstrate that the equilibrium identical to the optimal deterministic policy exists. Furthermore, we show that if the Agent's utility exhibits log-supermodularity, the highest equilibrium payoff will involve only deterministic inspections and will match the payoff of the optimal deterministic policy.

deviations from the contract, which can be particularly difficult with non-deterministic promises. 2) Reputation: reputation can help ensure that the regulator adheres to pre-committed promises, if the regulator interacts with multiple agents, either sequentially or simultaneously. This interaction could enable the regulator to commit to a consistent decision-making frequency, but this may require additional assumptions, such as the absence of macroeconomic shocks and the presence of post-facto incentives to align observations with incentive schemes.

Section 4.2 contrasts with Section 4.1 by assuming that the Principal can only commit to an inspection policy. We demonstrate that if the Principal cannot commit to actions, focusing on semi-separating equilibria with only two groups of pooling types and deterministic inspections is enough to find the highest (ex-ante) equilibrium payoff for the Principal.

Furthermore, we show the highest equilbrium payoff involve inspection of inefficient types. In Section 4.3, we assume the Principal lacks the power to commit to any of her instruments. In both scenarios from Sections 4.2 and 4.3, the structure of the highest equilibrium payoff is similar to that of the optimal deterministic inspection policy, though the thresholds may differ.

Finally, in Section 4.4, we explore a partial commitment setting. If the cost of inspection is not high, and the Principal commits to inspection whenever the Agent requests, then the Principal can achieve the optimal deterministic inspection policy through this partial commitment.

Relationship to the literature. The paper contributes to two areas of literature: first, mechanism design with costly state verification (CSV), and second, CSV without commitment. The literature on mechanism design with CSV begins with the well-known paper by Becker (1968), which argues that high punishments and low probabilities of monitoring are the best policy for the Principal. However, this analysis assumes that very high punishments are enforceable.

The literature continues with an application to financial markets by Townsend (1979). Townsend examines the optimal insurance contract between a risk-neutral principal (investor) and a risk-averse agent (entrepreneur). At the time of contracting, both parties have the same information. After the contract is written, the agent privately observes the project's income. The agent reports an income, and according to the contract, must pay a cost to verify this income. A contract specifies two things for each income report: whether the agent should verify the income, and the transfer from the agent to the principal. Townsend's optimal contract maximizes the ex-ante payoff of the agent, subject to the ex-ante individual rationality (IR) constraint of the principal. This optimal contract resembles a debt contract, where the agent verifies incomes below a certain threshold.

Gale and Hellwig (1985) studies a similar problem involving a risk-neutral borrower (agent) and a risk-neutral lender (principal). The optimal mechanism maximizes the agent's expected utility under the principal's zero profit (IR constraint) and the agent's incentive compatibility constraints. Similarly, the agent incurs a cost to verify the project's income. Both Townsend (1979) and Gale and

Hellwig (1985) focus on deterministic inspection.

Border and Border and Sobel (1987) consider a more general mechanism with stochastic audits and bounded pre-audit and post-audit transfers. They assume the Principal may never make a net payment (reward) to the Agent. They show that the probability of an audit should decrease with the agent's wealth. They illustrate with an example that if the principal aims to maximize expected revenue net of audit costs, the optimal contract involves large rewards and infrequent audits.

Mookherjee and Png (1989) assume the borrower is risk-averse and demonstrate that the optimal contract should be stochastic. This paper differs from the existing literature in four ways. First, the Principal does not have full commitment power and cannot commit to the mechanism after inspection occurs. Second, it does not consider transfers as a tool for the designer. Third, most works in the CSV literature on financial markets assume a competitive borrowing market, maximizing the borrower's utility subject to the lender's outside option and the borrower's truth-telling condition. Fourth, unlike in this paper, the state of nature is not known by any party at the contract date.

Another application of mechanism design with CSV is for optimal allocations of objects and collective choice problems. Ben-Porath et al. (2014) considers a principal allocates an object to one of I Agents. The principal cannot use transfers but can check the private information of each agent at a cost. The private information is the value of the object for each agent. Mylovanov and Zapechelnyuk (2017), study a similar problem with a different verification technology, and limited punishments. They assume the principal can verify information after allocating the object, and contingent on this observation, can destroy a fraction of the agent's payoff. Li (2020) studies the connection between costly verification and limited punishment. Patel and Urgun (2022) assume money burning as a new instrument for the Principal and study the optimal allocation problem with CSV, and Erlanson and Kleiner (2020) investigates the optimal allocation and collective choice problems.

Another branch of mechanism design with CSV is in Monopoly regulation. Baron and Besanko (1984) extend Baron and Myerson (1982) to allow random and costly audit. In their setting, monopolist pricing is a two-part tariff consisting of a fixed charge and unit price. Palonen and Pekkarinen (2022) consider a CVS regulation principal-agent problem with a different approach. They assume the Agent can reduce the probability of being verified, by engaging in costly avoidance action. The paper assumes a linear and exogenous punishment function if the the agent caught being untruthful, and no reward if the agent is truthful. The principal maximizes the expected weighed sum of the agent's payoff and transfers net of monitoring costs subject to the incentive compatibility and the participation constraints. These papers departs from ours for two reasons. First, we have ex-post participation constraint, so punishments are endogenous and restricted to the utility of the agent. Second, these papers do not study different commitment ability of the principal.

The paper most closely related to ours is Halac and Yared (2020), which examines a CSV principal-agent delegation problem where the agent has a bias towards higher actions. When the agent's bias becomes extreme, their model under full commitment resembles ours. However, unlike their study, we do not limit our analysis to deterministic mechanisms, and the agent is not protected by ex-post participation constraint. Further differences are discussed following the introduction of a related result in sections 3.1 and 4.2.

Section 4 connects with the literature on cheap talk models, following the work of Crawford and Sobel (1982).[3] Khalil (1997) contributes to the literature on CSV without commitment by examining a model where the Principal cannot commit to an inspection policy. His model involves costly signaling rather than cheap talk. Melumad and Mookherjee (1989) also considers a setting without commitment on inspection. The main differences are that the Principal can commit to transfers in the event of an inspection, and the Agent cannot reject the mechanism ex-post.

Banks (1989) explores a model where an agent requests a budget. The principal lacks commitment power, utilities are linear in transfer, and inspection is imperfect. Unlike our setting (see Section 4.3), full pooling is the only equilibrium. In any equilibrium, no information is conveyed from the agent to the principal.

[3]Our model is not exactly the same as cheap talk models, since the Agent (sender) can accept or reject the proposed action by the Principal (receiver).

2. Model

Players, and information structure. There are two players, an Agent (he) and a Principal (she). The Agent has type $\theta \in [\underline{\theta}, \overline{\theta}]$ drawn from a commonly known cumulative distribution function $F(\cdot)$, and probability distribution function $f(\cdot) > 0$. The type is the Agent's private information.

Mechanism and action. The Principal chooses and commits to a mechanism.[4] The mechanism $\mathbb{M} = (M, i(m), a(m))$ has three components: the message space M, the probability of inspection and the mandated action. The probability of inspection as a function of message $m \in M$ is $i(m) \in [0, 1]$, and inspection allows learning the true type of the Agent. Action $a(m) \in \mathbb{R}_+$ is in case of no inspection which determine the Agents' actions as a function of the message m. In case of inspection the Principal, without commitment, mandates $a^I(m, \theta) \in \mathbb{R}_+$. Action $a^I(m, \theta) \in \mathbb{R}_+$ is as a function of the message m and the true type θ through inspection.

Inspection costs $\phi > 0$ to the Principal. We assume the Agent is secured by **ex-post participation**: the Agent can accept or reject the final action.

Payoffs. If the Agent rejects the mandated action, the payoffs of both players are zero. He rejects when the mandated action generates a negative payoff (the outside option is zero) for him.[5] The payoff of the Agent with type θ and action a is $\max\{u(\theta, a), 0\}$. The Principal's payoff is

$$v(\theta, a)\mathbb{1}_{u(\theta,a)\geq 0} - \phi \mathbb{1}_{\text{inspection}}.$$

Timing. The Principal commits to the mechanism. The nature draws a type θ, and the Agent learns it privately. The Agent sends a message $m \in M$. The Principal inspects with probability $i(m)$. If does not inspect, with commitment, mandates an action $a(m)$. If inspects, without commitment, mandates an action $a^I(m, \theta)$. The Agent accepts or rejects the action. Figure 1 shows the timing of the model.

We consider pure strategy Perfect Bayesian equilibrium as the solution concept and maintain the following assumptions throughout the paper.

Assumption 1 *(Agent's utility) (i) Utility $u(\theta, a)$ is C^2 for all $(\theta, a) \in [\underline{\theta}, \overline{\theta}]^2$. (ii) A type $\theta \in [\underline{\theta}, \overline{\theta}]$ of the Agent gets a **zero** payoff when $a = \theta$, and prefers **higher***

[4]In section 4, we study different commitment ability of the Principal.
[5]Another interpretation of this model is that the Agent has a private outside option θ.

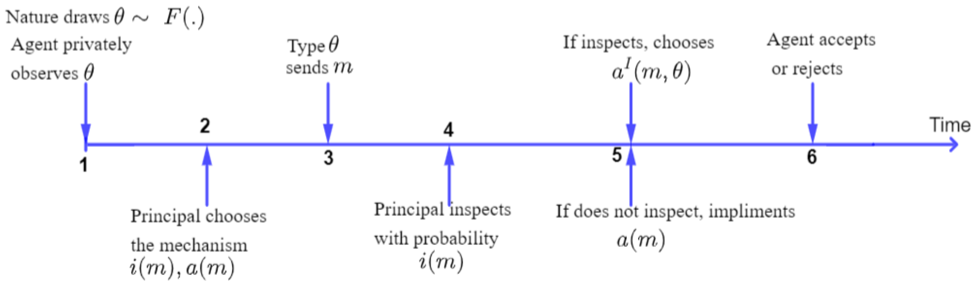

Figure 1: Timing

actions. *Formally*

$$u(\theta, \theta) = 0, \text{ and } u_a(\theta, a) > 0 \text{ for all } (\theta, a) \in [\underline{\theta}, \overline{\theta}]^2.$$

Assumption 2 *(Principal's utility) (i) Utility $v(\theta, a)$ is C^2 for all $(\theta, a) \in [\underline{\theta}, \overline{\theta}]^2$. (ii) All types of the Agent are **valuable** for the Principal. Moreover, **lower** types are more valuable for the Principal than **higher** types. Formally $v(\theta, \theta)$ is positive and weakly decreasing in θ, for all $\theta \in [\underline{\theta}, \overline{\theta}]$. (iii) The Principal prefers lower actions for all types of the Agent. Formally*

$$v_a(\theta, a) < 0 \text{ for all } (\theta, a) \in [\underline{\theta}, \overline{\theta}]^2.$$

Assumption 1 simply says the Agent's utility starts from zero (a normalization) at action equal to his type ($a = \theta$), and it is increasing in action. More precisely, the Agent prefers higher actions by only considering the support of CDF $F(.)$, i.e. $a \in [\underline{\theta}, \overline{\theta}]$. The assumption is **silent** for actions above $\overline{\theta}$.

Assumption 2 states lower types are more valuable for the Principal than higher types. This assumption assures that the existence of all types of the Agent is valuable for the Principal. This assumption is without loss of generality. If we assume $v(\overline{\theta}, \overline{\theta}) < 0$, then the Principal can exclude this type by mandating actions to be less that $\overline{\theta}$. The principal prefers lower actions down to θ. If the principal chooses an action less than θ, the agent will reject the action (due to the ex-post participation constraint).

By Assumption 1 we can conclude, the payoff of the Agent at final action a given type θ is $u(\theta, a)\mathbb{1}_{a \geq \theta}$, and the Principal's payoff is $v(\theta, a)\mathbb{1}_{a \geq \theta} - \phi \mathbb{1}_{\text{inspection}}$. [6]

[6] In Assumption 1, $u(\theta, \theta)$ can be changed with $u(\theta, a^*(\theta)) = \underline{u}(\theta)$, where $a^*(\theta)$ is the action that gives the Agent the same utility as his outside option, $\underline{u}(\theta)$.

Discussion on commitment ability of the Principal on $a^I(\cdot,\cdot)$: If the Principal is able to commit on $a^I(\hat{\theta},\theta)$, then $a^I(\hat{\theta},\theta)$ for $\hat{\theta}\neq\theta$ is off the equilibrium path and the Principal implements the maximum punishment: $a^I(\hat{\theta},\theta)\leq\theta$. The main challenge is when the Principal inspects and the Agent is truthful, i.e. $a^I(\theta,\theta)$. In Appendix, Reward Based Mechanisms, we show if $\lim_{a\to\infty}-v(\theta,a)=\lim_{a\to\infty}-\frac{u(\theta,a)}{v(\theta,a)}=\infty$, the optimal mechanisms does not exists.[7] Moreover, we show reward based mechanisms approximate first-best value. Reward based mechanisms set the efficient action without inspection $a(\theta)=\theta$, and inspects all types with a small probability. In case of inspection these mechanisms generate a very high payoff for the Agent, i.e, $a^I(\theta,\theta)$ a high amount. In environmental regulation, $a^I(\theta,\theta)$ means (as an example) that the regulator allows the Agent to pollute with a very high rate. However, We do not see these type of incentive schemes in practice. Non existence of the optimal solution and reward based mechanisms has been raised in settings with transfer.[8] Reward based mechanisms require a high commitment power for the Principal. One way to tackle this issue, as Border and Sobel (1987) (a setting with transfer) is to assume an exogenous upper bound on rewards, i.e. $a^I\leq\overline{a}$. An alternative way is to assume the Principal cannot commit on actions following an inspection.[9]

3. Results

In this section, we begin by formulating the Principal's problem as an optimization problem. We then derive properties of the optimal mechanism and identify the conditions under which deterministic inspection is optimal. Following this, we explore the relationship between the fear of ruin and our problem. Finally, we discuss the value of stochastic inspection when deterministic inspection is not optimal.

In any equilibrium, the Principal mandates action $a^I(m,\theta)=\theta$. This is due to the lack of commitment of the Principal after inspection occurs. By the revelation principle we can restrict the messages space M to the types space $[\underline{\theta},\overline{\theta}]$, and mechanisms to direct mechanisms. The Principal chooses a direct mechanism

[7] An example is $v(\theta,a)=-|\theta-a|^{\frac{1}{2}}$ and $u(\theta,a)=a-\theta$. Mookherjee and Png (1989) assumes (changing the notation) $\lim_{a\to\infty}-v(\theta,a)=\lim_{a\to\infty}-\frac{u(\theta,a)}{v(\theta,a)}=0$ and shows the optimal mechanism exists in a finite type space.

[8] See for example: Border and Sobel (1987) and Ahmadzadeh and Waizmann (2024).

[9] This idea has been suggested by Halac and Yared (2020) as well. See discussion on stochastic inspection in this paper.

$\mathbb{M} = \big(i(\hat{\theta}), a(\hat{\theta})\big)$, which $\hat{\theta}$ is the reported type of the Agent. The Agent's expected payoff given its type θ and the report $\hat{\theta}$ is

$$\begin{aligned}\pi(\hat{\theta}, \theta) &\equiv (1 - i(\hat{\theta}))\Big(u(\theta, a(\hat{\theta}))\Big)\mathbb{1}_{a(\hat{\theta})\geq\theta} + i(\hat{\theta})\Big(u(\theta, a^I(\hat{\theta}, \theta))\Big)\mathbb{1}_{a^I(\hat{\theta},\theta)\geq\theta} \\ &= (1 - i(\hat{\theta}))\Big(u(\theta, a(\hat{\theta}))\Big)\mathbb{1}_{a(\hat{\theta})\geq\theta}.\end{aligned}$$

The Principal's expected payoff if the Agent with type θ reports $\hat{\theta}$ is

$$(1 - i(\hat{\theta}))\Big(v(\theta, a(\hat{\theta}))\Big)\mathbb{1}_{a(\hat{\theta})\geq\theta} + i(\hat{\theta})\Big(v(\theta, \theta) - \phi\Big).$$

The Principal's problem:

Define the problem $\mathbb{P}$ as follows

$$\mathbb{P} \quad : \qquad \max_{i(\cdot), a(\cdot)} \ \mathbb{E}\Big[(1 - i(\theta))\Big(v(\theta, a(\theta))\Big)\mathbb{1}_{a(\theta)\geq\theta} + i(\theta)\Big(- \phi + v(\theta, \theta)\Big)\Big],$$

subject to combined ex-post participation constraints and the truth telling conditions (IC) for the Agent:

$$\pi(\theta, \theta) \geq \sup_{\hat{\theta}} \pi(\hat{\theta}, \theta),$$

for all $\theta \in [\underline{\theta}, \overline{\theta}]$. For simplicity, let $\pi(\theta) = \pi(\theta, \theta)$. Note that the mandated action without inspection $a(\hat{\theta})$, can be less than the true type of the Agent, which implies that the Agent rejects the offered action, or in other words, the Principal can exclude some types. It is without loss of optimality, to assume $a(\hat{\theta}) = \underline{\theta}$, if the Principal wants to exclude a type through $a(\cdot)$. We assume the solution to problem $\mathbb{P}$ exists and is piece-wise continuous. This existence of the solution is not a challenge when type space is finite.

Assumption 3 *The solution to problem $\mathbb{P}$ exists and it is piece-wise continuous.*

Types with zero payoffs:

Let us begin the analysis with types that have zero payoffs. Let $\mathbb{M} = \big(i(\cdot), a(\cdot)\big)$, satisfies IC. Define

$$\tilde{\theta} = \{\inf \theta | \pi(\theta) = 0\}.$$

The following lemma studies the structure of $\mathbb{M}$ for $\theta > \tilde{\theta}$.

Lemma 1 *(i)* $\pi(\theta) = 0$ *for all* $\theta \geq \tilde{\theta}$.

(ii) If $a(\theta) < \theta$*, then it is without loss of generality to assume* $a(\theta) = \underline{\theta}$.

(iii) If $a(\theta) = \underline{\theta}$*, then optimality of* $\mathbb{M}$ *implies that for all* $\theta' > \theta$, $a(\theta') = \underline{\theta}$.

(iv) If $a(\theta) \leq \theta$*, then for all* $\theta' > \theta$*, either* $a(\theta') = \underline{\theta}$ *or* $i(\theta') = 1$.

The proof of Lemma 1 is in Appendix. Lemma 1 (iii), and (iv) imply that a necessary condition for $\mathbb{M}$ to be optimal is that there exists $\tilde{\tilde{\theta}} \in [\tilde{\theta}, \overline{\theta}]$, such $i(\theta) = 1$ for $\theta \in [\tilde{\theta}, \tilde{\tilde{\theta}}]$, and $a(\theta) = \underline{\theta}$ for $\theta \in [\tilde{\tilde{\theta}}, \overline{\theta}]$. This means that full inspection and exclusion areas are completely separated. Using the structure of Lemma 1, one can conclude that types $\theta \leq \tilde{\theta}$ do not have incentive to mimic types higher than $\tilde{\theta}$. In addition, a necessary and sufficient condition for incentives of types higher than $\tilde{\theta}$, is that $a(\theta) \leq \tilde{\theta}$ for $\theta \leq \tilde{\theta}$. This implies $a(\tilde{\theta}) = \tilde{\theta}$.

Corollary 1 *In any optimal* $\mathbb{M}$, $a_*(\tilde{\theta}) = \tilde{\theta}$.

An upper bound on the payoff of the Principal:

Now we establish an upper bound on the Principal's payoff. Using this upper bound, we then identify the conditions under which deterministic inspection is optimal. Finally, we present the conditions where the optimal outcome matches the upper bound.

Lemma 2 *Denote* $\Big(\mathbb{M} = \big(i_*(\cdot), a_*(\cdot)\big)\Big)$ *the solution to problem* $\mathbb{P}$ *replacing global IC constraints by local IC constraints. Suppose there exists* θ^* *such that* $i_*(\theta) = 0$ *for all* $\theta \leq \theta^*$*. Then the payoff of the Principal (problem* $\mathbb{P}$ *with global IC) is (weakly) less than*

$$\int_{\underline{\theta}}^{\theta^*} \Big(v(\theta, a_*(\underline{\theta})) - \frac{u(\theta^*, a_*(\underline{\theta}))}{u_a(\theta^*, a_*(\underline{\theta}))} v_a(\theta, a_*(\underline{\theta}))\Big)\, dF(\theta) + \int_{\theta^*}^{\tilde{\tilde{\theta}}} \Big(v(\theta, \theta) - \phi\Big)\, dF(\theta).$$

The proof of Lemma 2 is in Appendix. Lemma 2 finds an upper bound on the Principal's payoff. Replacing global IC by local IC weaken the constraints. Therefore the payoff of the Principal should be weakly higher.

To grasp the intuition behind the proof of Lemma 2, we first need to reformulate problem $\mathbb{P}$ by substituting the global incentive compatibility (IC) constraints with local IC constraints.

$$\max_{i(.),a(.),\tilde{\theta},\tilde{\tilde{\theta}}} \int_{\underline{\theta}}^{\tilde{\tilde{\theta}}} \Big[(1-i(\theta))\Big(v(\theta,a(\theta))-v(\theta,\theta)+\phi\Big)+\Big(v(\theta,\theta)-\phi\Big)\ \Big]dF(\theta),$$

subject to

$$(1-i(\theta))u(\theta,a(\theta))\geq\ (1-i(\hat{\theta}))u(\theta,a(\hat{\theta})),$$

for all θ, $\hat{\theta}$ and $a(\theta)\geq\theta$, $\tilde{\theta}\leq\tilde{\tilde{\theta}}$ and $a(\tilde{\theta})=\tilde{\theta}$. The argument is analogous to methods used in the Calculus of Variations. A global variation in the inspection probability involves decreasing $(1-i(\theta))$ to $(1-\beta)(1-i(\theta))$ for types in $[\theta^*,\tilde{\theta}]$. In words β percentage decrease in probability of not inspection. After this variation types above θ^* do not mimic types above θ^* since $(1-\beta)$ cancels from both sides of IC inequalities. However, types above θ^* would like to mimic types below θ^*. In order to keep the incentives unchanged a variation on $a_*(\underline{\theta})$ is $a_*^\beta(\underline{\theta})$ such that $a_*^\beta(\underline{\theta})$ solves

$$u(\theta^*,a_*^\beta(\underline{\theta}))=(1-\beta)u(\theta^*,a_*(\underline{\theta})).$$

For small enough $\beta>0$, the marginal change in the payoff by decreasing β percentage of not inspection is

$$\int_{\theta^*}^{\tilde{\theta}}(1-i_*(\theta))\Big(v(\theta,a_*(\theta))-v(\theta,\theta)+\phi\Big)dF(\theta).$$

For enough $\beta>0$, the marginal change in the payoff by changing $a_*(\underline{\theta})$ to $a_*^\beta(\underline{\theta})$ is

$$\int_{\underline{\theta}}^{\theta^*}\frac{u(\theta^*,a_*(\underline{\theta}))}{u_a(\theta^*,a_*(\underline{\theta}))}v_a(\theta,a_*(\underline{\theta}))\ dF(\theta).$$

Similar arguments holds for $\beta>0$. At the optimal, the total marginal change should be zero, therefore

$$\int_{\theta^*}^{\tilde{\theta}}(1-i_*(\theta))\Big(v(\theta,a_*(\theta))-v(\theta,\theta)+\phi\Big)dF(\theta)=\int_{\underline{\theta}}^{\theta^*}-\frac{u(\theta^*,a_*(\underline{\theta}))}{u_a(\theta^*,a_*(\underline{\theta}))}v_a(\theta,a_*(\underline{\theta}))\ dF(\theta).$$

Using the above equality, one can derive the payoff stated in Lemma 2. The left side represents the marginal increase in payoff by reducing one percent of not inspection for types in $[\theta^*,\overline{\theta}]$. The right side represents the marginal increase in the payoff by adjusting $a_*(\underline{\theta})$ so that the utility of type θ^* decreases by one percent. At the optimal, the gain by increasing inspection should trade off the gain by decreasing $a_*(\underline{\theta})$.

Lemma 2 indicates that the net gain from stochastic inspection, as opposed to deterministic inspection, is

$$\int_{\underline{\theta}}^{\theta^*} \left(v(\theta, a_*(\underline{\theta})) - \frac{u(\theta^*, a_*(\underline{\theta}))}{u_a(\theta^*, a_*(\underline{\theta}))} v_a(\theta, a_*(\underline{\theta})) - v(\theta, \theta) \right) dF(\theta)$$

If the above payoff is not strictly positive, then deterministic inspection is optimal. Particularly, if for all $\theta^* \leq a_*(\underline{\theta})$ ($\underline{\theta} \leq \theta^* \leq a_*(\underline{\theta}) \leq \overline{\theta}$) the integral is negative, then the optimal inspection is deterministic.

Fear of ruin:

Theorem 1 *If fear of ruin of the principal is higher than the agent then deterministic inspection is optimal.*

An immediate consequence of Lemma 2 is if

$$\frac{v(\theta, a_*(\underline{\theta})) - v(\theta, \theta)}{v_a(\theta, a_*(\underline{\theta}))} \geq \frac{u(\theta^*, a_*(\underline{\theta})) - u(\theta^*, \theta^*)}{u_a(\theta^*, a_*(\underline{\theta}))},$$

then there is no benefit from stochastic inspection. The debate on the sub-optimality of deterministic inspection was initiated by Townsend (1979). He provided an example demonstrating that stochastic inspection yields a higher payoff than deterministic inspection. However, many questions remain unanswered. For instance, under what conditions is deterministic inspection optimal, and to what extent is stochastic inspection superior to deterministic inspection?

The concept of the fear of ruin was introduced by Aumann and Kurz (1977) in the context of taxation policies. We explain this concept in our setting. Suppose an agent's type is θ and the agent is considering a bet where he risks his entire fortune $a(\theta)$ for a potential small increase to $a(\theta) + a'$. The probability q of ruin must be very small for the Agent to be indifferent between taking the bet and keeping his current fortune, i.e., $u(\theta, a(\theta) + a') = (1 - q)u(\theta, a(\theta)) + qu(\theta, \theta)$. Thus, the more unwilling the agent is to risk ruin, the smaller q will be. Therefore, q serves as an inverse measure of the agent's aversion to risking ruin. It can be shown that the probability of ruin (q) per potential extra gain (a') represents the fear of ruin, i.e., $\lim_{q \to 0} q/a' = u(\theta, a(\theta))/u_a(\theta, a(\theta))$. It is well-known that both $v(\cdot, a)$ and $u(\cdot, a)$ are concave in a and if Arrow-Pratt coefficient of the Principal is higher than the Agent, then fear of ruin of the Principal is also higher.[10]

It seems intuitive that if the Principal has a greater fear of ruin than the Agent, it is less favorable for her to use stochastic inspection. However, employing stochastic inspection might reduce the Agent's information rent, potentially offsetting the risk the Principal assumes.

Theorem 1 asserts that regardless of the inspection cost, if the Principal's fear of ruin exceeds that of the Agent, the optimal inspection strategy is deterministic. However, this does not imply that the optimal mechanism is unaffected by the inspection cost. In Section 3.1, we identify the optimal deterministic inspection policy.

Example 1 *(linear utilities) Suppose $v(\theta, a) = \alpha(a - \theta) + b$ and $u(\theta, a) = a - \theta$, where $\alpha < 0$, and $b > 0$. Then fear of ruin of the Principal is equal to the Agent. Therefore the optimal using Theorem 1 is deterministic.*

When utilities are linear in actions, they can be interpreted as transfers. The literature on CSV with transfers does not predict the optimality of deterministic inspection for several reasons. In Border and Sobel (1987) and Chander and Wilde (1998), the Principal can commit to an action (transfer) in the event of an inspection. Unlike our setting, the Agent's payoff is higher when inspected (truthful report) than when not inspected. Commitment to action in the case of inspection enhances the efficiency of stochastic inspection. The Principal can leverage rewards for truthful reporting.[11]

Example 2 *(Principal risk-averse) Suppose $v(\theta, a) = \alpha(a - \theta)^{\frac{1}{2}} + b$ and $u(\theta, a) = a - \theta$, where $\alpha < 0$, and $b > 0$. Then fear of ruin of the Principal is higher than the Agent. Therefore the optimal mechanism is deterministic.*

Harris and Raviv (1996) and Harris and Raviv (1998) study a principal-agent model in the context of capital budgeting and delegation. Agent's utility is linear but Principal has a concave utility. These papers deal with finite types (maximum three types) and predict optimality of stochastic inspection. Their

[10] See Proposition 4 of Foncel and Treich (2005).

[11] Palonen and Pekkarinen (2022) assumes no reward for the Agent following the inspection. Among other differences to our paper their punishment is linear (exouenously fixed). They predict inspection probabilities take two values. This result is due to the specific punishment.

argument depends on the discreteness of the type space and continuity of inspection probability.

Theorem 1 is silent when the Principal has less fear of ruin. In Mookherjee and Png (1989) and Melumad and Mookherjee (1989) the principal is risk neutral and the agent is risk-averse. The principal maximizes the agent's payoff subject to IR of the principal and IC of the Agent. Both papers predicts the optimality of stochastic inspection. Their argument depends on the discreteness of the type space and continuity of inspection probability. In addition, this result, leave the question that how different is the payoff of optimal stochastic inspection to the deterministic inspection.

Lemma 2 provides an intuitive upper bound on the payoff of stochastic inspection.

$$\int_{\underline{\theta}}^{\theta^*} v_a(\theta, a_*(\underline{\theta}))\Big(\frac{v(\theta, a_*(\underline{\theta})) - v(\theta, \theta)}{v_a(\theta, a_*(\underline{\theta}))} - \frac{u(\theta^*, a_*(\underline{\theta}))}{u_a(\theta^*, a_*(\underline{\theta}))}\Big)\, dF(\theta).$$

The upper bound is the aggregate difference of fear of ruin of the Principal for types below θ^* and the Agent for type θ^* evaluated at $a_*(\underline{\theta})$ scaled by the marginal utility of the Principal.

Claims 1 and 2 in Appendix, given θ^* and $\tilde{\theta}$, provide the solution of problem $\mathbb{P}$ if local IC is sufficient condition for global IC and if the solution does not involve bunching in interval $[\theta^*, \tilde{\theta}]$. Next Lemma provides a condition that local IC is sufficient for global IC.

Lemma 3 *Suppose $u(\cdot,\cdot)$ is log-Supermodular: For all $\theta \le a$, and (θ, a) in $[\underline{\theta}, \overline{\theta}]^2$*

$$\frac{\partial^2 \ln\big(u(\theta, a)\big)}{\partial\theta\,\partial a} \ge 0.$$

Then (i) IC implies that $a(\theta)$ and $i(\theta)$ are weakly increasing for all $\theta \le \tilde{\theta}$. (ii) Local IC and $a(\cdot)$ increasing, implies global IC.

The proof of Lemma 3 is in Appendix. Log-Supermodularity closely resembles Supermodularity in the standard mechanism design literature. The reason for incorporating the logarithm is that, unlike transfers, the inspection probability multiplies the Agent's utility. Thus, a logarithmic transformation adjusts the Agent's (IC) to fit the standard mechanism design framework. However, this transformation is not applicable to the Principal's objective.

If the optimal mechanism, as derived using Claims 1 and 2 in the Appendix, does not involve bunching, then the Principal's optimal payoff will match the upper bound established in Lemma 2.

3.1. Deterministic Inspection

In this section we derive the optimal policy restricting inspection policy to deterministic inspection. In order to state the optimal policy, we need to define two thresholds. Define the problem $\mathbb{P}_D$ as follows

$$\mathbb{P}_D : \max_{\theta^* \in [\underline{\theta}, \overline{\theta}]} \left\{ \int_{\underline{\theta}}^{\theta^*} \Big(v(\theta, \theta^*) \Big) \, dF(\theta) + \int_{\theta^*}^{\overline{\theta}} \Big(v(\theta, \theta) - \phi \Big) \mathbb{1}_{v(\theta,\theta) \geq \phi} \, dF(\theta) \right\}.$$

Let Θ^* be the set of the solutions of $\mathbb{P}_D$.[12] For $\theta^* \in \Theta^*$, define $\theta^{**} \geq \theta^*$ such that

$$\theta^{**} = \begin{cases} \theta^* & \text{if} \quad v(\theta^*, \theta^*) \leq \phi \\ \overline{\theta} & \text{if} \quad v(\overline{\theta}, \overline{\theta}) > \phi \end{cases},$$

otherwise define θ^{**} as the solution of $v(\theta^{**}, \theta^{**}) = \phi$. Using θ^*, and θ^{**}, the below proposition expresses the optimal policy.

Theorem 2 *The optimal policy for all $\theta \in [\underline{\theta}, \overline{\theta}]$ is*

$$i_*(\theta) = \begin{cases} 0 & \theta < \theta^* \\ 1 & \theta^{**} \geq \theta \geq \theta^* \\ 0 & \theta > \theta^{**}, \end{cases}$$

$$a_*(\theta) = \begin{cases} \theta^* & \theta < \theta^* \\ \theta & \theta^{**} \geq \theta \geq \theta^* \\ \underline{\theta} & \theta > \theta^{**}. \end{cases}$$

Moreover, commitment on $a^I(\cdot, \cdot)$ does not have any benefit for the Principal.

The proof of Theorem 2 is in Appendix. First, Theorem 2 states that there are two thresholds. We call the first threshold (θ^*), *inspection threshold*, and the second threshold (θ^{**}), *exclusion threshold*. The optimal policy divides types in to three areas. Efficient types ($\theta < \theta^*$), intermediate types ($\theta \in [\theta^*, \theta^{**}]$), and inefficient

[12] Problem $\mathbb{P}_D$ is continuous in $\theta^* \in [\underline{\theta}, \overline{\theta}]$, therefore it admits a maximizer.

types ($\theta > \theta^{**}$). If the inspection cost is high ($v(\theta^*, \theta^*) \leq \phi$), then there is no inspection region. If the inspection cost is low ($v(\overline{\theta}, \overline{\theta}) > \phi$), there is no exclusion region.

Second, Theorem 2 expresses that the principal mandates the efficient action after inspection, i.e. $a^I(\theta, \theta) = \theta$. The Principal does not give a reward for telling the truth in case of inspection. The reason is by excluding inefficient types and inspecting intermediate types, the Principal can decrease the informational rent (the set of types that can mimic) of efficient types and leave a zero for intermediate types. Thus intermediate types cannot mimic higher types, and the Principal optimally can set $a^I(\theta, \theta) = \theta$.

Third, the Theorem says the Principal does not waste resources (cost of inspection) for inefficient types ($\theta > \theta^{**}$), so inspection is zero. Instead she mandates a low action without inspection ($a(\cdot)$) and excludes the inefficient types. By inspecting intermediate types, the optimal policy hits two goals. First, if does not allow efficient types ($\theta < \theta^*$) to mimic intermediate types. Second by having full information on the intermediate types, she can mandate efficient action ($a^I(\theta, \theta) = \theta$) for these types.

Fourth, Theorem 2 argues that the optimal mandated action without inspection ($a(\cdot)$) sets a fixed action equal to θ^* for efficient types. This is in contrast to CSV literature with deterministic inspection. The mandated action for efficient types is fixed (the same as ours), but strictly less than the efficient action for the lowest type that is inspected ($a < \theta^*$).[13] In our paper, if a for efficient types is less than θ^*, then types in interval $[a, \theta^*]$ will reject the mandated action ex-post. This means that the Principal is excluding some efficient types. The aforementioned structure cannot be optimal, since the Principal can shift the inspection area toward more efficient types, and instead exclude more inefficient types.

The trade-off at the inspection threshold, is between inspecting the marginal type at the benefit of decreasing a for all efficient types. The trade-off at the exclusion threshold, is between inspecting the marginal type at the benefit of mandating the efficient action, instead of exclusion of the marginal type.

Figure 2 illustrates the optimal policy. The Principal sets a **cap** on actions equal to θ^*. She inspects types between θ^* and θ^{**}, and mandates the efficient action for these types. The Principal excludes types above θ^{**} by mandating a very low action $a(\hat{\theta}) = \underline{\theta}$.

[13] See Halac and Yared (2020), section IV, definition of TEC.

Threshold structure of the optimal policy does not come as a surprise due to deterministic inspection. Similar structure exists in Townsend (1979), Gale and Hellwig (1985), Malenko (2019) (the static case), and Halac and Yared (2020). Beside the differences in the modeling assumptions of these papers with the current paper, the prediction of these models are different compare to ours. First, There is no exclusion region in these papers. The reason is either because the inspection cost is borne by the Agent (Townsend (1979) and Gale and Hellwig (1985)), or the first best value of high types is increasing (Malenko (2019) and Halac and Yared (2020)), as opposed to ours which $v(\theta, \theta)$ is decreasing in θ.

Second, In these papers,[14] at the contracting date, the Principal and the Agent have similar information. Therefore naturally these papers assume ex-ante IR as opposed to ex-post participation constraint in our paper. Hence inspection is used for two reasons.[15] Protecting inefficient types from excessive losses, and reducing the rent of efficient types. Therefore the inspection area is affected by the outside option (ex-ante) of the Agent, whereas, in our model inspection does not act as an insurance to excesses losses for the agent, since the agent can always rejects the mandated action ex-post.

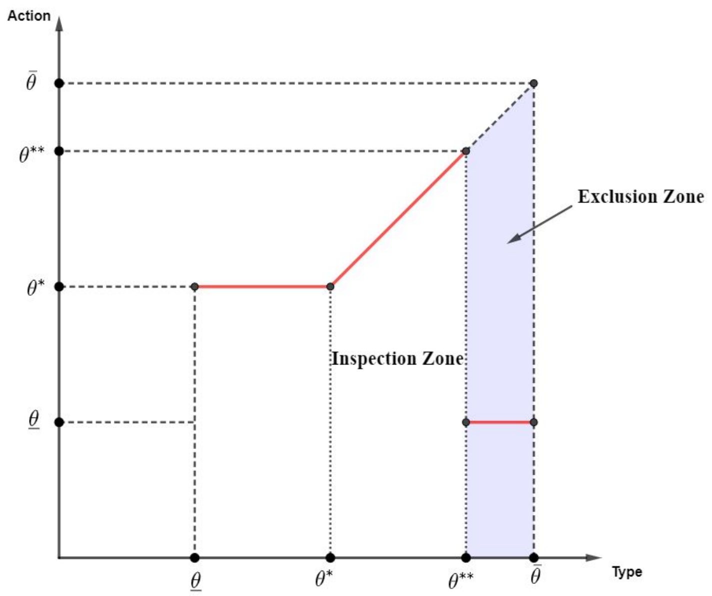

Figure 2: optimal policy

[14]Except Halac and Yared (2020).

[15]Note that in Townsend (1979) and Gale and Hellwig (1985), the Agent is borne the cost of inspection, in Malenko (2019) the Principal and in Halac and Yared (2020) both.

4. Different commitment ability

In this section, we examine the varying commitment capabilities of the Principal. There are two motivations for this analysis.

Firstly, it helps to identify which instruments are essential for the Principal to commit to. Understanding this is crucial for determining the effectiveness of the Principal's commitment ability across different tools.

Secondly, in practice, the Principal's commitment can vary depending on the application. For instance, in some cases, a government department specifies only audit policies without setting standards (actions in our context). In other cases, a regulator has an independent audit agency that does not specify when and under what conditions inspections are conducted. We view these different environments as representing different commitment abilities for the Principal.

By studying these variations, we can gain deeper insights into the strategic importance of each instrument and how the Principal can best utilize them in practice.

4.1. Commitment only to the action

From an ex post perspective, the principal may lack the incentive to conduct inspections. When the agent sends a message to the principal, it might not be optimal for the principal to inspect the agent, making it challenging to ensure adherence to the contract. Additionally, if the principal's inspection efforts are unobservable, they might reduce their effort to save on inspection costs. Especially when the optimal inspection policy is stochastic, it becomes difficult to monitor whether the principal is adhering to the contract, and it is hard to enforce and verify deviations in court.

In this section, we examine a scenario where the Principal can commit to an action if no inspection occurs but does not commit to inspection policies. For example, consider an environmental regulator that sets standards (action in our setting) when there is no inspection. However, the regulator does not specify when inspections will occur, under what conditions they will take place, or what the mandated standards will be after an inspection.

Suppose the Principal can commit only to $a(\cdot) : \mathbb{M} \to \mathbb{R}_+$. Timing is as follows: The Principal commits to $a(\cdot)$. Agent privately observes type $\theta \sim F(\cdot)$, and sends a message m. Principal observes m, and decides to inspect or not: with inspection observes the true type and mandates a (without commitment). without inspection mandates $a(m)$ (with commitment). Agent accepts or rejects the mandated standard.

Agent's strategy is: $m(\theta) \in \mathbb{R}_+$. Principal's strategy is $\big(i(m) \in [0,1], a^I(m,\theta) \in \mathbb{R}_+\big)$. Principal's belief is: $\beta(\theta|m)$.

A similar observation is that the Principal is opportunistic in case of inspection: $a^I(m,\theta) = \theta$.

Theorem 3 *(i) The Equilibrium similar to the optimal deterministic policy (Theorem 2) exists. (ii) Suppose $u(\cdot,\cdot)$ is log-Supermodular, then the maximum ex-ante payoff is equal to the optimal deterministic policy (Theorem 2).*

The proof of Theorem 3 is in Appendix. Theorem 3 asserts that the principal can guarantee a payoff equivalent to deterministic inspection by committing solely to $a(\cdot)$. The equilibrium strategies are as follows: types $[\underline{\theta}, \theta^*] \cup [\theta^{**}, \overline{\theta}]$ sends message m_N and types in $[\theta^*, \tilde{\theta}]$ send m_I. The Principal sets the cap θ^* on actions without inspection i.e., $a(m) = \theta^*$ and inspects only when receives message m_I i.e., $i(m_I) = 1$. The principal's off the equilibrium path belief is $\beta(\theta|m)$ is $\underline{\theta}$ with probability one.

We explain intuitively why the above strategies form an equilibrium. Types below θ^* do not deviate to m_I, otherwise they receive a zero payoff. Moreover, types above θ^* receive a zero payoff anyhow.

The principal adheres to her inspection strategy. If does not inspect intermediate types (types that send m_I), then she will exclude types through the pre-committed action $a(m_I) = \theta^*$. By committing to this action, she effectively penalizes herself. She has committed to a low action when she does not inspect and receives m_I. The Principal does not inspect m_N. If inspecting m_N would yield a higher payoff, then inspecting both messages could generate an even higher payoff than optimal deterministic inspection (Theorem 2). However, this implies that in the optimal deterministic inspection, full inspection (inspecting all types) should be optimal. This cannot be the case unless the inspection cost, ϕ, is zero.

Note that Theorem 3 does not claim that there is no equilibrium yielding a higher payoff for the Principal than the optimal deterministic inspection. This might seem counterintuitive, as one might expect that without the ability to commit to an inspection policy, the Principal would be indifferent between inspecting and not inspecting, thus simplifying the policy to a deterministic one. However, this is not entirely accurate. While it is true that the Principal should be indifferent to inspection when the probability of inspection is between zero and one, stochastic inspection can influence the Agent's incentives and reduce the information rent. Theorem 3 (*ii*) states that if the Agent has log-supermodular utility, the Principal cannot use stochastic inspection to lower the Agent's information rent. In fact, the proof of Theorem 3 (*ii*) relies on the following monotonicity property of an equilibrium: if $m \in m(\theta)$ and $m' \in m(\theta')$, where $\tilde{\theta} > \theta' > \theta$, then $i(m') \geq i(m)$ and $a(m') \geq a(m)$. The monotonicity condition can be shown using IC condition of the agent for log-supermodular utilities.

Khalil (1997) examines a model of CSV where the Principal cannot commit to an inspection policy. Inspections occur ex-post (after production), and the Principal cannot adjust allocations post-inspection, with penalties being exogenous. Therefore, his model is one of costly signaling rather than cheap talk. An ex-post audit implies that if auditing is optimal, then inspections are stochastic and compliance is random (with the agent playing a mixed strategy due to the presence of two types). In an audit contract, the agent does not receive rent in either state of nature because the inefficient type is indifferent. This paper finds that there is more auditing under no-commitment than under commitment. Conversely, our paper shows that the level of auditing remains the same whether there is no-commitment (section 4.1) or commitment to inspection (section 3.1).

Melumad and Mookherjee (1989) is also consider a setting with no commitment on inspection. The main differences are that the principal can commit on transfers in case of inspection and the agent cannot reject the mechanism ex-post. In our model the principal can punish her by mandating a very low action if receives m_I and does not inspect. Then the agent will reject. This helps the principal to decrease the value of not inspecting without generate a big loss for the agent. Therefore in our model the principal can achieve the full (deterministic) inspection payoff.

4.2. Commitment only to inspection

In this section, we examine a scenario where a Principal commits to an inspection policy but does not commit to actions. For instance, consider an inspector agency that outlines the timing and conditions under which inspections will occur and provides this information to a regulatory body. However, the regulatory body does not set specific standards for these inspections.

Suppose the Principal can only commit to $i(\cdot) : \mathbb{M} \to [0,1]$. Timing is as follows: Principal commits to $i(\cdot)$. Agent privately observes type $\theta \sim F(\cdot)$, and sends a message m. Principal inspects or not based on $i(m)$. With inspection observes the true type of the Agent and chooses an action a. without inspection chooses an (possible different) action a. Agent accepts or rejects the mandated standard. Agent's strategy is $m(\theta) \in \mathbb{R}_+$. Principal's strategy is $\big(a(m) \in \mathbb{R}_+, a(m,\theta) \in \mathbb{R}_+\big)$. Principal's belief: $\beta(\theta|m)$. Strategies and beliefs are measurable. A similar observation is that the Principal is opportunistic in case of inspection: $a^I(m,\theta) = \theta$.

Lemma 4 *Fix $i(\cdot)$. In any equilibrium, (i) $a(m)$ is constant for all on-path messages m such that $i(m) < 1$. (ii) For on-path messages m_1 and m_2, if $i(m_1) < 1$ and $i(m_2) < 1$, then $i(m_1) = i(m_2)$.*

The proof of Lemma 4 is in Appendix. Lemma 4 asserts that the mandated action, in the absence of inspection, should be uniform across all messages. The reasoning is as follows: if there are two distinct mandated actions, types that are very close to the lower action will attempt to imitate those with the higher action. This phenomenon occurs regardless of the probability of inspection (as long as it is less than one). The underlying reason is that there will always be a type very close to an action; otherwise, the Principal can reduce that action.

As a result of Lemma 4, we can consolidate all messages with an inspection value less than 1 into a single message, m_N, and all messages with an inspection value of exactly 1 into another single message, m_I. Consequently, we can focus solely on equilibria with at most two messages on the equilibrium path. The following Corollary formally states this observation.

Corollary 2 *For any equilibrium, there exists an outcome (ex-ante payoff of the Principal and ex-ante payoff of all types of the Agent) equivalent equilibrium with maximum two different messages m_I and m_N. $i(m_I) = 1$, $i(m_N) < 1$.*

An interpretation of these two messages in Corollary 2 is that when the Agent sends m_I, he is requesting an inspection, whereas when he sends m_N, he is not requesting an inspection.

Theorem 4 *(i) The highest ex-ante payoff of the principal is achieved by strategies such that: Agent's strategy is: types in $[\underline{\theta}, s^*] \cup [s^{**}, \overline{\theta}]$ send m_N and types in $[s^*, s^{**}]$*

send m_I. Principal's strategy is: $i(m_I) = 1$, $i(m_N) = 0$ and $a(m_N) = s^*$. (ii) $\theta^{**} \leq s^{**}$. (iii) If $\phi < v(\overline{\theta}, \overline{\theta})$, (no exclusion region), is equivalent to optimal deterministic inspection policy.

The proof of Theorem 4 is in Appendix. Theorem 4 indicates that the equilibrium yielding the highest payoff shares a similar threshold structure with the optimal deterministic inspection policy (Theorem 2). However, the thresholds differ. The equilibrium structure does not involve stochastic inspection, suggesting that commitment to stochastic inspection is beneficial only if the Principal can also commit to an action ($a(\theta)$).

It is intuitive that the exclusion area is not in the middle of the type space. In other words, the Principal does any inspects type θ' that is higher than θ type that excludes. Otherwise, replacing the messages send by these types ($m(\theta) = m_I$ and $m(\theta') = m_N$) both increases the Principal's payoff and does not change the best reply of the Principal when receives message m_N. The payoff increases since $v(\theta, \theta)$ is decreasing in θ. The best reply, $a(m_N)$, does not change since for all $a \leq a(m_N)$

$$\int_{\underline{\theta}}^{a} \left[v\left(\theta, a\right) \mathbb{1}_{a \geq \theta} \right] \beta(\theta | m_N) \, d\theta \leq \int_{\underline{\theta}}^{a(m_N)} \left[v\left(\theta, a(m_N)\right) \mathbb{1}_{a(m_N) \geq \theta} \right] \beta(\theta | m_N) \, d\theta.$$

The inequality remains valid because both sides are unchanged before and after the modification. This inequality holds for all $a \geq a(m_N)$ as well since the left side (weakly) decreases.

The Theorem states that to incentivize the Principal to mandate sufficiently low actions when the Agent sends m_N, the Principal should commit to inspect inefficient types, i.e., $s^{**} \geq \theta^{**}$. The benefit from exclusion area should be small enough such that the Principal does not deviate in $a(m_N)$ and chooses a higher action. Moreover, if the inspection cost is sufficiently low, meaning the exclusion area does not exist as per Theorem 2, then the highest payoff and equilibrium structure align with Theorem 2.

In the limited commitment section, Halac and Yared (2020) examines a delegation model where the Principal can commit to inspection but may alter the allowable action. This model shares some similarities with the one discussed in this section. However, Halac and Yared (2020) focuses on deterministic inspections, and $v(\theta, \theta)$ increases with θ. Consequently, there is no exclusion area.

4.3. No commitment

Now we analysis the case that the Principal does not have any commitment power. Timing is as follows: Agent privately observes type $\theta \sim F(\cdot)$, and sends a message m. Principal observes m, and decides to inspect or not. With inspection observes the true type and chooses action a. Without inspection chooses action (possibly different) a. Agent accepts or rejects the mandated standard.

Agent's strategy is: $m(\theta) \in \mathbb{R}_+$. Principal's strategy is: $\big(i(m) \in [0,1], a(m) \in \mathbb{R}_+, a(m,\theta) \in \mathbb{R}_+\big)$. Principal's belief is: $\beta(\theta|m)$.

By Lemma 4, since the Principal cannot commit on action without inspection, one can construct all equilibria outcomes by at most two messages. We focus on interval strategies Agent's strategy $m(\theta)$ alternates between m_I, and m_N in finite intervals.

Assumption 4 *For all $(\theta, \theta', s) \in [\underline{\theta}, \overline{\theta}]^3$, such that $\theta' \leq \theta \leq s$,*

$$v(\theta, s) - v(\theta, \theta) \geq v(\theta', s) - v(\theta', \theta').$$

It is easy to see that Assumption 4 is a weaker condition than supermodularity.

Theorem 5 *Suppose the Principal's utility satisfies Assumption 4. Then the equilibrium with highest ex-ante payoff has a structure similar to optimal deterministic inspection policy (with different thresholds).*

The proof of Theorem 5 is in Appendix. Banks (1989) examines a model such that an agent requests a budget. In the open procedure section, the principal may choose to inspect or not and then decides on a budget for the agent accordingly. Utilities are linear, and inspection is imperfect; there is a probability that the inspection does not yield any information. Imperfect inspection incentivizes types just below but very close to s^* to send m_I, therefore (full) pooling are the only equilibria. In any equilibrium, no information is transmitted from the agent to the principal.

In contrast, our paper focuses on finding the highest payoff for the principal and with perfect inspection there are equilbria like Theorem 5 that reveals more information to the Principal. Non existence of no new information will be transmitted from the agent to the principal. In contrast, our paper focuses on finding the highest payoff for the principal and with perfect inspection there are equilbria like Theorem 5 that reveals more information to the Principal. Then non existence of semi-separating equilbria in Banks (1989) is due to imperfect inspection. However, this is a knife edge case. Even with imperfect inspection, the Principal can incentives types below s^* to not send m_I. If a type below s^* sends m_I and if the inspection is successful, then the agent should pay a small part of inspection cost.

4.4. Partial commitment

Suppose the Principal is obligated to conduct an inspection only if the Agent requests it. In numerous applications, there are institutions in place to ensure that principals cannot refuse an audit once it has been requested. Thus, the agent always has the option to ask for an audit. We demonstrate that with this minimal level of commitment, and if the inspection cost is not high, then the highest equilibrium payoff matches with the payoff described in Theorem 2.

Theorem 6 *(Partial commitment) Suppose the inspection cost is not high, i.e. $\phi < v(\overline{\theta}, \overline{\theta})$, and the Principal can commit to inspect message m_I. There exists an equilibrium such that the ex-ante payoff of the Principal is the same as Theorem 2.*

The proof of Theorem 6 is in Appendix. The intuition behind the above theorem is as follows. Since the Principal cannot commit to actions, by employing Lemma 4 and Theorem 4, we can focus on equilibria with deterministic inspection and two messages, $\{m_I, m_N\}$. Suppose types in $[\theta^*, \overline{\theta}]$ send message m_I, while other types send m_N. The Principal has no incentive to inspect m_N. If inspecting m_N would yield a higher payoff, then inspecting both messages could generate an even higher payoff than the optimal deterministic inspection. The Principal optimally chooses action $a(m_N) = \theta^*$ after observing m_N. The reasoning is as follows: first, $a(m_N) \leq \theta^*$, since the Principal assigns zero probability to types above θ^* after observing m_N. Second, if $a(m_N) < \theta^*$, then a policy that caps actions at $a(m_N)$ and inspects types in the interval $[\theta^*, \overline{\theta}]$ yields a strictly higher payoff than the optimal policy in problem $\mathbb{P}_D$, which is not possible.

5. Conclusion

In conclusion, this paper provides a comprehensive analysis of the regulatory tools of mandating standards and conducting inspections within a principal-agent framework. We have examined how different levels of commitment by the regulator affect the effectiveness of these tools. Our findings indicate that when the regulator's fear of ruin exceeds firm's fear of ruin, a deterministic inspection policy proves to be optimal. This policy involves a structured approach where types are segmented into efficient, intermediate, and inefficient categories. This segmentation helps in balancing the trade-off between encouraging technology adoption and managing the risk of firm rejecting standards.

Moreover, our analysis of varying commitment levels reveals that while full commitment to both actions and inspections yields higher payoffs, partial commitment scenarios also show promising outcomes. When the cost of inspections is not high and the regulator commits to inspecting upon request, the optimal deterministic policy can still be achieved. These insights highlight the importance of strategic commitment and the potential for tailored regulatory approaches to effectively incentivize technology adoption. Overall, our study underscores the need for regulators to carefully design and commit to their policies to maximize their effectiveness in achieving desired environmental and performance standards.

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A. Appendix

A.1. Reward Based Mechanisms

Fix $\alpha > 0$. For all $n \in \mathbb{N}$, define mechanism $\mathbb{M}_n(\alpha)$ as follows: $a^I(\theta,\theta) = n$, $i(\theta) = \frac{-\alpha}{v(\theta,n)}$, $a(\theta) = \theta$, and $a^I(\hat{\theta},\theta) = \underline{\theta}$ if $\hat{\theta} \neq \theta$ for all $(\theta,\hat{\theta}) \in [\underline{\theta},\overline{\theta}]^2$.

Lemma 5 *Let* $\overline{u} = \sup_{(\hat{\theta},\theta)\in[\underline{\theta},\overline{\theta}]^2} u(\theta,\hat{\theta})$, *and* $\alpha > 0$. *Suppose* $\lim_{a\to\infty} v(\theta,a) = -\infty$, *and* $\lim_{a\to\infty} -\frac{u(\theta,a)}{v(\theta,a)} > \frac{\overline{u}}{\alpha}$, *for all* $\theta \in [\underline{\theta},\overline{\theta}]$. *Then the payoff of mechanisms* $\mathbb{M}_n(\alpha)$ *converges to* $\mathbb{E}\big[v(\theta,\theta)\big] - \alpha$, *as n goes large.*

Proof. For n large enough the mechanism $\mathbb{M}_n$ is incentive compatible, i.e,

$$\begin{aligned}
&(1 - i(\theta))\Big(u(\theta,a(\theta))\Big)\mathbb{1}_{a(\theta)\geq\theta} + i(\theta)\Big(u(\theta,a^I(\theta,\theta))\Big)\mathbb{1}_{a^I(\theta,\theta)\geq\theta}\\
&= \frac{-\alpha}{v(\theta,n)}u(\theta,n) \geq \sup_{\hat{\theta}}(1 - i(\hat{\theta}))\Big(u(\theta,a(\hat{\theta}))\Big)\mathbb{1}_{a(\hat{\theta})\geq\theta}\\
&= \sup_{\hat{\theta}}(1 - i(\hat{\theta}))u(\theta,\hat{\theta}).
\end{aligned}$$

The reason is $\lim_{n\to\infty} \frac{-\alpha}{v(\theta,n)}u(\theta,n) > \overline{u}$, and $\sup_{\hat{\theta}}(1-i(\hat{\theta}))u(\theta,\hat{\theta})$ is weakly less than $\overline{u}$. Note that all types accept $\mathbb{M}_n$ ex-post. Now we compute the payoff of $\mathbb{M}_n$ for the Principal.

$$\begin{aligned}
&\mathbb{E}\Big[(1 - i(\theta))\Big(v(\theta,a(\theta))\Big)\mathbb{1}_{a(\theta)\geq\theta} + i(\theta)\Big(-\phi + v(\theta,a^I(\theta,\theta))\mathbb{1}_{a^I(\theta,\theta)\geq\theta}\Big)\Big]\\
&= \mathbb{E}\Big[(1 - \frac{-\alpha}{v(\theta,n)})\Big(v(\theta,\theta)\Big) + \frac{-\alpha}{v(\theta,n)}\Big(-\phi + v(\theta,n)\Big)\Big]
\end{aligned}$$

Fixing α, the limit of the above payoff as n goes large is $\mathbb{E}\big[v(\theta,\theta) - \alpha\big]$. ■

As a result if $\lim_{a\to\infty} -\frac{u(\theta,a)}{v(\theta,a)} = \infty$, then $\lim_{a\to\infty} -\frac{u(\theta,a)}{v(\theta,a)} > \frac{\overline{u}}{\alpha}$ for all $\alpha > 0$. Therefore mechanism $\mathbb{M}_n(\alpha)$ approximates first-best as n goes large and α goes small.

Lemma 6 *Suppose* $\lim_{a\to\infty} v(\theta,a) = -\infty$, *and* $\lim_{a\to\infty} -\frac{u(\theta,a)}{v(\theta,a)} = \infty$, *for all* $\theta \in [\underline{\theta},\overline{\theta}]$. *Then the solution to problem* $\mathbb{P}$ *does not exist.*

Proof. Assume it exits, therefore the payoff of the Principal should be $\mathbb{E}\big(v(\theta,\theta)\big)$. Based on the objective of the principal in problem $\mathbb{P}$, if the payoff is $\mathbb{E}\big(v(\theta,\theta)\big)$, then $i(\theta) \overset{\text{a.e}}{=} 0$, $a(\theta) \overset{\text{a.e}}{=} 0$. This mechanism clearly cannot satisfy the truth-telling condition. ■

A.2. Proof of Lemma 1

Proof.

(i) By contradiction let $\theta_2 > \theta_1$ and $\pi(\theta_2) > \pi(\theta_1) = 0$. Then $i(\theta_2) < 1$ and $a(\theta_2) > \theta_2$. But θ_1 can mimic θ_2 and receive a strictly positive payoff. A contradiction.

(ii) When $a(\theta) < \theta$, it means that the Principal excludes type θ, if she does not inspect. In this case, type θ's expected payoff does not change if the principal decreases $a(\theta)$ to $\underline{\theta}$.

(iii) Let $a(\theta) = \underline{\theta}$. Then for all types $\theta^{'} > \theta$, we have $a(\theta^{'}) \leq \theta^{'}$. If $a(\theta^{'}) < \theta^{'}$, then we can assume $a(\theta^{'}) = \underline{\theta}$. If $a(\theta^{'}) = \theta^{'}$, then $i(\theta^{'}) = 1$, otherwise type θ, can mimic $\theta^{'}$. If $i(\theta^{'}) = 1$, the Principal can strictly gain by switching the policies for type θ, and $\theta^{'}$. Excluding type θ, while keeping $\theta^{'}$ is not efficient; i.e $v(\theta, \theta) - \phi > v(\theta^{'}, \theta^{'}) - \phi$.

(iv) If $a(\theta) \leq \theta$, then either $a(\theta^{'}) \leq \theta^{'}$ or $i(\theta^{'}) = 1$ for all $\theta^{'} > \theta$. If $a(\theta^{'}) < \theta^{'}$, we can assume $a(\theta^{'}) = \underline{\theta}$. If $a(\theta^{'}) = \theta^{'}$, then $i(\theta^{'}) = 1$, otherwise θ can mimic $\theta^{'}$.

■

A.3. Proof of Lemma 2

Proof. For ease of notation we drop $*$. For $\theta \leq \tilde{\tilde{\theta}}$, $a(\theta) \geq \theta$. Therefore the optimization becomes

$$\max_{i(.), a(.), \tilde{\tilde{\theta}}} \int_{\underline{\theta}}^{\tilde{\tilde{\theta}}} \Big[(1 - i(\theta))\Big(v(\theta, a(\theta)) - v(\theta, \theta) + \phi\Big) + \Big(v(\theta, \theta) - \phi\Big)\Big] f(\theta) d\theta,$$

subject to the IC condition. By Lemma 1, there exists $\tilde{\theta}$ such that for all $\theta < \tilde{\theta}$, $a(\theta) > \theta$, $a(\tilde{\theta}) = a(\tilde{\theta})$. For types $\theta > \tilde{\theta}$, $a(\cdot)$ is irrelevant. Therefore the optimization becomes

$$\max_{i(.), a(.), \tilde{\theta}, \tilde{\tilde{\theta}}} \int_{\underline{\theta}}^{\tilde{\theta}} \Big[(1 - i(\theta))\Big(v(\theta, a(\theta)) - v(\theta, \theta) + \phi\Big)\Big] f(\theta) d\theta + \int_{\underline{\theta}}^{\tilde{\tilde{\theta}}} \Big[\Big(v(\theta, \theta) - \phi\Big)\Big] f(\theta) d\theta$$

subject to

$$(1-i(\theta))u(\theta,a(\theta)) \geq \ (1-i(\hat{\theta}))u(\theta,a(\hat{\theta})),$$

$a(\theta) \geq \theta$ and $\tilde{\theta} \leq \tilde{\tilde{\theta}}$. Fixing $\tilde{\theta}$ and $\tilde{\tilde{\theta}}$, we relax the global IC constraint by local IC as follows. Given a type θ and $\hat{\theta} \in (\theta, a(\theta))$. Therefore $a(\hat{\theta}) > \theta$. The local IC condition is

$$(1-i(\theta))u(\theta,a(\theta)) \geq \ (1-i(\hat{\theta}))u(\theta,a(\hat{\theta})).$$

By employing the Envelope Theorem, the local truth telling (except those points that $a(\cdot)$ has a discontinuity) condition gives

$$\frac{\partial \ln\big[\pi(\theta)\big]}{\partial\theta} = \frac{\partial ln\big[u(\theta,a(\theta))\big]}{\partial\theta},$$

or equivalently for $\pi(\theta) > 0$ ($\theta < \tilde{\theta}$),

$$\dot{\pi}(\theta) = \pi(\theta)\frac{u_\theta(\theta,a(\theta))}{u(\theta,a(\theta))}.$$

The relaxed problem of $\mathbb{P}$ by replacing global IC with local IC is

$$\max_{\pi(\cdot),a(\cdot)} \int_{\underline{\theta}}^{\tilde{\theta}} \Big[\frac{\pi(\theta)}{u(\theta,a(\theta))}\Big(v(\theta,a(\theta)) - v(\theta,\theta) + \phi\Big)\Big] f(\theta)d\theta + \int_{\underline{\theta}}^{\tilde{\tilde{\theta}}} \Big(v(\theta,\theta) - \phi\Big) f(\theta)d\theta,$$

subject to (for $\theta < \tilde{\theta}$)

$$\dot{\pi}(\theta) = \pi(\theta)\frac{u_\theta(\theta,a(\theta))}{u(\theta,a(\theta))},$$

$$\pi(\theta) \leq u(\theta,a(\theta)).$$

Note that the state equation does not need to be valid at discontinuity points of $a(\cdot)$. We analysis the relaxed problem by using the Pontryagin's maximum principle (π is the state and a is the control variable).[16] The Hamiltonian for $\theta < \tilde{\theta}$ is

$$H(a,\pi,\mu,w,\theta) = \frac{\pi}{u(\theta,a)}\Big(v(\theta,a) - v(\theta,\theta) + \phi\Big) f(\theta) + \mu\, \pi\frac{u_\theta(\theta,a)}{u(\theta,a)} + w(u(\theta,a) - \pi),$$

where the Lagrangian multiplier for $\pi(\theta) \leq u(\theta,a(\theta))$ is $w(\theta)$. From the Pontryagin principle for the co-state variable $\mu(\theta)$ we have (except at discontinuity points of

[16] Note that we can write the Pontryagin's maximum principle for piece-wise continuous control, and piece-wise differentiable state. For more information see Seierstad and Sydsaeter (1986), Chapter 2, Theorem 2, or Kamien and Schwartz (2012), section 14.

$a(\cdot))$

$$\dot{\mu}(\theta) = -\frac{\partial H}{\partial \pi} = -\frac{1}{u(\theta, a(\theta))}\Big(v(\theta, a(\theta)) - v(\theta, \theta) + \phi\Big) f(\theta) - \mu(\theta)\ \frac{u_\theta(\theta, a(\theta))}{u(\theta, a(\theta))} + w(\theta).$$

Since $u(\theta, a(\theta)) > 0$, then

$$\dot{\mu}(\theta) u(\theta, a(\theta)) + \mu(\theta)\ u_\theta(\theta, a(\theta)) - w(\theta) u(\theta, a(\theta)) = -\Big(v(\theta, a(\theta)) - v(\theta, \theta) + \phi\Big) f(\theta). \tag{1}$$

For types $\theta \in [\theta^*, \tilde{\theta}]$, $w(\theta) = 0$. Hence

$$\frac{d\ \mu(\theta)\pi(\theta)}{d\ \theta} = -\frac{\pi(\theta)}{u(\theta, a(\theta))}\Big(v(\theta, a(\theta)) - v(\theta, \theta) + \phi\Big) f(\theta).$$

Employing Newton–Leibniz theorem, and the fact that $\mu(\cdot)$, and $\pi(\cdot)$ are continuous functions:

$$\int_{\theta^*}^{\tilde{\theta}} \frac{d\ \mu(\theta)\pi(\theta)}{d\ \theta}\ d\theta = \int_{\theta^*}^{\tilde{\theta}} -\frac{\pi(\theta)}{u(\theta, a(\theta))}\Big(v(\theta, a(\theta)) - v(\theta, \theta) + \phi\Big) f(\theta)\ d\theta.$$

Since $\pi(\tilde{\theta}) = 0$,

$$\mu(\theta^*)\pi(\theta^*) = \int_{\theta^*}^{\tilde{\theta}} \frac{\pi(\theta)}{u(\theta, a(\theta))}\Big(v(\theta, a(\theta)) - v(\theta, \theta) + \phi\Big) f(\theta)\ d\theta. \tag{2}$$

In the next step we compute $\mu(\theta^*)$. From the Pontryagin's maximum principle, we know for all $\theta < \tilde{\theta}$, a maximizes the following

$$\frac{\pi(\theta)}{u(\theta, a)}\Big(v(\theta, a) - v(\theta, \theta) + \phi\Big) f(\theta) + \mu(\theta)\ \pi(\theta)\frac{u_\theta(\theta, a)}{u(\theta, a)} + w(\theta)(u(\theta, a) - \pi(\theta)).$$

Sine the above is a C^1 function of a, as a necessary condition we can write

$$\begin{aligned} H_a = \pi(\theta)\Bigg(&\frac{u(\theta, a(\theta))\big(v_a(\theta, a(\theta)) f(\theta) + \mu(\theta) u_{\theta a}(\theta, a(\theta))\big)}{u^2(\theta, a(\theta))} \\ &- \frac{u_a(\theta, a(\theta))\Big((v(\theta, a(\theta)) - v(\theta, \theta) + \phi) f(\theta) + \mu(\theta)\ u_\theta(\theta, a(\theta))\Big)}{u^2(\theta, a(\theta))}\Bigg) \\ &+ w(\theta) u_a(\theta, a(\theta)) = 0. \end{aligned} \tag{3}$$

Pluging 1, we get

$$H_a = \pi(\theta)u(\theta, a(\theta))\Big(\frac{v_a(\theta, a(\theta))f(\theta) + \mu(\theta)u_{\theta a}(\theta, a(\theta)) + u_a(\theta, a(\theta))\big(\dot{\mu}(\theta) - w(\theta)\big)}{u^2(\theta, a(\theta))}\Big) + w(\theta)u_a(\theta, a(\theta)) = 0.$$

Simple algebra gives us

$$\pi(\theta)u(\theta, a(\theta))\Big(\frac{v_a(\theta, a(\theta))f(\theta) + \mu(\theta)u_{\theta a}(\theta, a(\theta)) + u_a(\theta, a(\theta))\dot{\mu}(\theta)}{u^2(\theta, a(\theta))}\Big) = u_a(\theta, a(\theta))w(\theta)\big(\frac{\pi(\theta)}{u(\theta, a(\theta))} - 1\big) = 0.$$

The last equality is due to the fact that $w(\theta)(u(\theta, a(\theta)) - \pi(\theta)) = 0$. Finally since $\pi(\theta)$, and $u(\theta, a(\theta))$ are strictly positive we have

$$\mu(\theta)u_{\theta a}(\theta, a(\theta)) + u_a(\theta, a(\theta))\dot{\mu}(\theta) = -v_a(\theta, a(\theta))f(\theta).$$

The above equation for $\theta \leq \theta^*$ implies

$$\int_{\underline{\theta}}^{\theta} \frac{d\ \mu(t)u_a(t, a(t))}{d\ t}\ dt = \int_{\underline{\theta}}^{\theta} -v_a(t, a_*(t))f(t)\ dt.$$

The transversality condition at $\underline{\theta}$ is $\mu(\underline{\theta}) = 0$. Hence

$$\mu(\theta)u_a(\theta, a(\underline{\theta})) = \int_{\underline{\theta}}^{\theta} -v_a(t, a(\underline{\theta}))f(t)\ dt. \tag{4}$$

Employing equations 2 and 4, and the fact that $\pi(\theta^*) = u(\theta^*, a(\underline{\theta}))$, the Principal's payoff becomes

$$\int_{\underline{\theta}}^{\theta^*} \Big(v(\theta, a(\underline{\theta})) - v(\theta, \theta) + \phi\Big)f(\theta)\ d\theta + \frac{u(\theta^*, a(\underline{\theta}))}{u_a(\theta^*, a(\underline{\theta}))}\int_{\underline{\theta}}^{\theta^*} -v_a(\theta, a(\underline{\theta}))f(\theta)\ d\theta + \int_{\underline{\theta}}^{\tilde{\tilde{\theta}}} \Big(v(\theta, \theta) - \phi\Big)f(\theta)d\theta.$$

Equivalently

$$\int_{\underline{\theta}}^{\theta^*} \Big(v(\theta, a(\underline{\theta})) - \frac{u(\theta^*, a(\underline{\theta}))}{u_a(\theta^*, a(\underline{\theta}))} v_a(\theta, a(\underline{\theta})) \Big) f(\theta)\, d\theta + \int_{\theta^*}^{\tilde{\tilde{\theta}}} \Big(v(\theta, \theta) - \phi \Big) f(\theta) d\theta.$$

■

Claim 1 *Suppose local IC is sufficient for global IC. Assume $a_*(\theta)$ is strictly increasing in $[\theta^*, \tilde{\theta}]$, then there exists a C^1 function $\mu(\theta)$ such that $a_*(\theta)$ satisfy the below equations:*

$$-\dot{\mu}(\theta) = \frac{v(\theta, a_*(\theta)) - v(\theta, \theta) + \phi}{u(\theta, a_*(\theta))} f(\theta) + \mu(\theta) \frac{u_\theta(\theta, a_*(\theta))}{u(\theta, a_*(\theta))},$$

$$\frac{\partial \left(\frac{v(\theta, a_*(\theta)) - v(\theta, \theta) + \phi}{u(\theta, a_*(\theta))} \right)}{\partial a} f(\theta) + \mu(\theta) \frac{\partial^2 \ln \big(u(\theta, a_*(\theta)) \big)}{\partial \theta\, \partial a} = 0,$$

for all $\theta \in [\theta^*, \tilde{\theta}]$. Moreover, if $a_*(\cdot)$ is piecewise C^1 then $a_*(\cdot)$ satisfies (at all differentiable and continuous points of a^*) the below first order differential equation

$$\dot{a}(\theta) = \Big[\psi(\theta, a_*(\theta)) - \frac{\partial \Psi(\theta, a_*(\theta))}{\partial \theta} \Big] \Big(\frac{\partial \Psi(\theta, a_*(\theta))}{\partial a} \Big)^{-1},$$

with end point condition $a_*(\tilde{\theta}) = \tilde{\theta}$.

Proof. For all $\theta \in [\theta^*, \tilde{\theta}]$, from equation 3:

$$\mu(\theta) = \frac{- \frac{\partial \left(\frac{v(\theta, a_*(\theta)) - v(\theta, \theta) + \phi}{u(\theta, a_*(\theta))} \right)}{\partial a} f(\theta)}{\frac{\partial^2 \ln \big(u(\theta, a_*(\theta)) \big)}{\partial \theta\, \partial a}} = \Psi(\theta, a_*(\theta)),$$

and from equation 1:

$$-\dot{\mu}(\theta) = \frac{v(\theta, a_*(\theta)) - v(\theta, \theta) + \phi}{u(\theta, a_*(\theta))} f(\theta) + \mu(\theta) \frac{u_\theta(\theta, a_*(\theta))}{u(\theta, a_*(\theta))}.$$

Therefore

$$-\dot{\mu}(\theta) = \frac{\big(u_\theta(\theta, a_*(\theta)) v_a(\theta, a_*(\theta)) - u_{a\theta}(\theta, a_*(\theta)) (v(\theta, a_*(\theta)) - v(\theta, \theta) + \phi) \big) f(\theta)}{-u_a(\theta, a_*(\theta)) u_\theta(\theta, a_*(\theta)) + u(\theta, a_*(\theta)) u_{\theta a}(\theta, a_*(\theta))}$$
$$\equiv \psi(\theta, a_*(\theta)).$$

Finally a simple calculation gives us the differential equation. ∎

Claim 2 *(The optimal inspection) Let $a_*(.)$ be the optimal mandated action in case of no inspection. Then the expected payoff for the Agent, and the optimal inspection policy for $\theta \le \tilde{\theta}$, are*

$$\pi(\theta) = \pi(\underline{\theta})\ exp\Big(\int_{\underline{\theta}}^{\theta} \frac{u_\theta(t, a_*(t))}{u(t, a_*(t))}\ dt\Big),$$

and

$$i_*(\theta) = 1 - \frac{\pi(\underline{\theta})\ exp\big(\int_{\underline{\theta}}^{\theta} \frac{u_\theta(t,a_*(t))}{u(t,a_*(t))}\ dt\big)}{u(\theta, a_*(\theta))}.$$

Proof.

Using Newton–Leibniz theorem, and the fact that $\pi(.)$ is a continuous function:

$$ln(\pi(\theta)) - ln(\pi(\underline{\theta})) = \int_{\underline{\theta}}^{\theta} \frac{d\ ln(\pi(t))}{d\ t}\ dt = \int_{\underline{\theta}}^{\theta} u_\theta(t, a(t))dt.$$

Note that the above equality is valid even if $i_*(.)$ or $a_*(.)$ has discontinuities. Finally

$$\pi(\theta) = \pi(\underline{\theta})\ exp\Big(\int_{\underline{\theta}}^{\theta} \frac{u_\theta(t, a_*(t))}{u(t, a_*(t))}\ dt\Big).$$

We know $\pi(\theta) = (1 - i_*(\theta))u(\theta, a_*(\theta))$, so $i_*(\theta) = 1 - \frac{\pi(\theta)}{u(\theta, a_*(\theta))}$. ∎

A.4. Proof of Lemma 3

Proof. (i) For all $\theta < \tilde{\theta}$, $i(\theta) < 1$ and $a(\theta) > \theta$. First we show $a(\cdot)$ is weakly increasing. For all θ, and $\hat{\theta} \in (\theta, a(\theta))$, IC conditions imply

$$\ln(1 - i(\theta)) + \ln\big(u(\theta, a(\theta))\big) \ge\ \ln(1 - i(\hat{\theta})) + \ln\big(u(\theta, a(\hat{\theta}))\big),$$

and

$$\ln(1 - i(\hat{\theta})) + \ln\big(u(\hat{\theta}, a(\hat{\theta}))\big) \ge\ \ln(1 - i(\theta)) + \ln\big(u(\hat{\theta}, a(\theta))\big).$$

Assume $\theta > \hat{\theta}$. Adding up the above inequalities gives us

$$\frac{\ln\big(u(\theta, a(\theta))\big) - \ln\big(u(\hat{\theta}, a(\theta))\big)}{\theta - \hat{\theta}} \ge \frac{\ln\big(u(\theta, a(\hat{\theta}))\big) - \ln\big(u(\hat{\theta}, a(\hat{\theta}))\big)}{\theta - \hat{\theta}}.$$

Using log-Supermodular property of $u(\cdot,\cdot)$, we conclude $a(\theta) \geq a(\hat{\theta})$. Now we show $i(\cdot)$ is weakly increasing. By contradiction assume there exist types $\theta < \theta^{'}$ such that $i(\theta) > i(\theta^{'})$. Since $a(\cdot)$ is a weakly increasing function, then type θ prefers to mimic type $\theta^{'}$, i.e.

$$(1 - i(\theta))u(\theta, a(\theta)) \leq (1 - i(\theta^{'}))u(\theta, a(\theta^{'})).$$

A contradiction.

(ii) Define $y(\theta) = ln(\pi(\theta))$, and $u(\theta, a(\hat{\theta})) = ln(u(\theta, a(\hat{\theta})))$. A logarithm transformation of the local truth-telling condition for all t gives us

$$\dot{y}(t) = u_\theta(t, a(t)),$$

for all t such that $a(\cdot)$ is continuous at t. At discontinuity points of $i(.)$, and $a(.)$ we should replace the derivatives with left or right derivatives, so

$$\dot{y}(t^-) = u_\theta(t, a(t^-)), \text{ and } \dot{y}(t^+) = u_\theta(t, a(t^+)).$$

Let $t > t^{'}$. Using Newton–Leibniz theorem, and the fact that $y(.)$ is a continuous function:

$$y(t) - y(t^{'}) = \int_{t^{'}}^{t} u_\theta(s, a(s))ds, \tag{5}$$

We need to show for all t, and $t^{'}$.

$$\pi(t) \geq \pi(t^{'})\frac{u(t, a(t^{'}))}{u(t^{'}, a(t^{'}))},$$

or

$$y(t) - y(t^{'}) \geq u(t, a(t^{'})) - u(t^{'}, a(t^{'})) = \int_{t^{'}}^{t} u_\theta(s, a(t^{'}))ds$$

Employing 5, we should show

$$\int_{t^{'}}^{t} \left[u_\theta(s, a(s)) - u_\theta(s, a(t^{'}))\right]ds \geq 0.$$

By log-Supermodular property of $u(\cdot,\cdot)$, the integral is positive. Now let $t^{'} > t$. We need to show

$$y(t) - y(t^{'}) \geq u(t, a(t^{'})) - u(t^{'}, a(t^{'})) = -\int_{t}^{t^{'}} u_\theta(s, a(t^{'}))ds.$$

Rewrite equation 5, $y(t) - y(t^{'}) = -\int_t^{t^{'}} u_\theta(s, a(s))ds$, and plugging in the above inequality, we can conclude

$$\int_t^{t^{'}} \left[u_\theta(s, a(t^{'})) - u_\theta(s, a(s))\right] ds \geq 0.$$

Again using log-Supermodular property of $u(\cdot, \cdot)$ the integral is positive. ■

Now given $a_*(.)$, we can find the payoff of the Agent $\pi(.)$, and the optimal inspection policy $i_*(.)$.

A.5. Proof of Theorem 2

Proof. Due to the maximum punishment rule, and Assumption 1, $a^I(\hat{\theta}, \theta) \leq \theta$ if $\hat{\theta} \neq \theta$. Observe that if $a(\theta) < \theta$, then the mechanism can choose $a(\theta) = \underline{\theta}$. Now given a mechanism $(i(\hat{\theta}), a^I(\hat{\theta}, \theta), a(\hat{\theta}))$, define $\theta^* = \sup\{a(\theta) | i(\theta) = 0\}$. Assumption 1, and the global IC imply that

$$\begin{aligned} i(\theta)\Big(u(\theta, a^I(\theta, \theta))\Big)\mathbb{1}_{a^I(\theta,\theta)\geq\theta} + (1 - i(\theta))\Big(u(\theta, a(\theta))\Big)\mathbb{1}_{a(\theta)\geq\theta} \\ \geq i(\hat{\theta})\Big(u(\theta, a^I(\hat{\theta}, \theta))\Big)\mathbb{1}_{a^I(\hat{\theta},\theta)\geq\theta} + (1 - i(\hat{\theta}))\Big(u(\theta, a(\hat{\theta}))\Big)\mathbb{1}_{a(\hat{\theta})\geq\theta} \\ = (1 - i(\hat{\theta}))\Big(u(\theta, a(\hat{\theta}))\Big)\mathbb{1}_{a(\hat{\theta})\geq\theta}. \end{aligned}$$

for all $\hat{\theta} \neq \theta$, and $(\theta, \hat{\theta}) \in [\underline{\theta}, \overline{\theta}]^2$. The last inequality comes from the fact that either $a^I(\hat{\theta}, \theta) = \theta$, then $u(\theta, \theta) = 0$ (by Assumption 1), or $a^I(\hat{\theta}, \theta) < \theta$, then $u(\theta, a^I(\hat{\theta}, \theta)\mathbb{1}_{a^I(\hat{\theta},\theta)\geq\theta} = 0$. Thus we have

$$i(\theta)\Big(u(\theta, a^I(\theta, \theta))\Big)\mathbb{1}_{a^I(\theta,\theta)\geq\theta} + (1 - i(\theta))\Big(u(\theta, a(\theta))\Big)\mathbb{1}_{a(\theta)\geq\theta} \geq \Big(u(\theta, \theta^*)\Big)\mathbb{1}_{\theta^*\geq\theta}.$$

First it means that if $i(\theta) = 0$, and $a(\theta) = \underline{\theta}$, then $\theta^* \leq \theta$. Second if $i(\theta) = 0$, and $a(\theta) \neq \underline{\theta}$ means that $a(\theta) \geq \theta$, and then $a(\theta) \geq \theta^*$. However, by the definition of θ^*, we conclude $a(\theta) = \theta^*$. Therefore if $i(\theta) = 0$, then

$$a(\theta) = \begin{cases} \theta^* & \theta > \theta^* \\ \{\theta^*, \underline{\theta}\} & \theta = \theta^* \\ \underline{\theta} & \theta < \theta^*. \end{cases} \tag{6}$$

Third if $i(\theta) = 1$, and $\theta \leq \theta^*$, then $a^I(\theta, \theta) \geq \theta^*$. On the other hand, we know if $i(\theta) = 1$, then $a^I(\theta, \theta) \geq \theta$. To see this by contradiction assume $a^I(\theta, \theta) < \theta$, then if the Principal chooses $i(\theta) = 0$, and $a(\theta) = \underline{\theta}$, will have higher payoff with no effect on the global IC. The higher payoff comes from the fact that inspection has a positive cost $\phi > 0$. Thus from IC, two necessary conditions are $a^I(\theta, \theta) \geq \max\{\theta^*, \theta\}$, and condition 6.

Rewrite the Principal's problem

$$\max_{i(.),a^I(.,.),a^N(.)} \int_{\underline{\theta}}^{\overline{\theta}} \Big[i(\theta) v(\theta, a^I(\theta, \theta)) + (1 - i(\theta)) v(\theta, a(\theta)) \mathbb{1}_{a(\theta) \geq \theta} - \phi i(\theta) \Big] \, dF(\theta),$$

subject to the IC constraints. For the moment we consider weaker conditions, that are $a^I(\theta, \theta) \geq \max\{\theta^*, \theta\}$, and condition 6. Later we check the global IC condition. By Assumption 2, the objective function is decreasing in $a^I(\theta, \theta)$. Therefore $a^I(\theta, \theta) = \max\{\theta^*, \theta\}$. Rewriting the objective function

$$\max_{i(.),\theta^*} \left\{ \int_{\underline{\theta}}^{\overline{\theta}} \Big[i(\theta) \Big(v(\theta, \max\{\theta^*, \theta\}) - \phi \Big) + (1 - i(\theta)) \Big(v(\theta, \theta^*) \Big) \mathbb{1}_{\theta^* \geq \theta} \Big] \, df(\theta) \right\}.$$

Now if $\theta \leq \theta^*$, then

$$v(\theta, \max\{\theta^*, \theta\}) - \phi < v(\theta, \theta^*),$$

therefore the optimal policy chooses $i(\theta) = 0$ for all $\theta \leq \theta^*$. For $\theta > \theta^*$, the optimal policy chooses $i(\theta) = 1$, iff $v(\theta, \theta) - \phi \geq 0$. Therefore the objective becomes to solve problem $\mathbb{P}_D$

$$\max_{\theta^* \in [\underline{\theta}, \overline{\theta}]} \left\{ \int_{\theta^*}^{\overline{\theta}} \Big(v(\theta, \theta) - \phi \Big) \mathbb{1}_{v(\theta,\theta) \geq \phi} \, dF(\theta) + \int_{\underline{\theta}}^{\theta^*} \Big(v(\theta, \theta^*) \Big) \, dF(\theta) \right\}.$$

The above explanation, and the optimal θ^* for problem $\mathbb{P}_D$, together suggests that the following policy is optimal for the weaker IC conditions.

$$a^I(\hat{\theta}, \theta) = \theta,$$

$$i(\hat{\theta}) = \begin{cases} 0 & \hat{\theta} \leq \theta^* \\ 1 & \theta^{**} \geq \hat{\theta} > \theta^* \\ 0 & \hat{\theta} > \theta^{**}, \end{cases}$$

$$a(\theta) = \begin{cases} \theta^* & \hat{\theta} \le \theta^* \\ \theta & \theta^{**} \ge \hat{\theta} > \theta^* \\ \underline{\theta} & \hat{\theta} > \theta^{**}, \end{cases}$$

Now we have to show the above policy is globally IC. The argument is as follows. Types $\theta \le \theta^*$, cannot mimic types $\hat{\theta} \ge \theta^*$, since they are either inspected or excluded. Types $\theta \le \theta^*$, are indifferent to mimic types $\hat{\theta} < \theta^*$, since $a(\hat{\theta}) = \theta^*$. Types $\theta > \theta^*$, cannot mimic types $\hat{\theta} \ge \theta^*$, since they are either inspected or excluded. Types $\theta > \theta^*$, cannot mimic types $\hat{\theta} < \theta^*$, since $a(\hat{\theta}) = \theta^* < \theta$. ■

A.6. Proof of Theorem 3

Proof. (i) Let m_I and m_N be two different messages. Consider the following strategy for the agent. $m(\theta) = m_N$ for $\theta \in [\underline{\theta}, \theta^*] \cup [\theta^{**}, \overline{\theta}]$ and $m(\theta) = m_I$ for $\theta \in (\theta^*, \theta^{**})$. The Principal commits to $a(\cdot)$ as follow: $a(m_N) = \theta^*$ and $a(m) = \underline{\theta}$ for all $m \neq m_I$. The inspection strategy of the Principal is as follows: $i(m_I) = 1$ and $i(m) = 0$ for all $m \neq m_I$. Let the principal's belief for off the equilibrium path messages be (with probability one) equal to type $\underline{\theta}$.

The Agent does not have any deviation strategy. Types in $[\underline{\theta}, \theta^*]$ strictly prefer to send message m_N. Types in $(\theta^*, \overline{\theta}]$ are indifferent to send any message since their payoff will be zero anyhow.

The Principal does not have any deviation strategy. If the Principal does not inspect message m_I, then types (θ^*, θ^{**}) will reject the action $a(m_I) = \underline{\theta}$. The payoff of inspection is strictly higher since $v(\theta, \theta) - \phi > 0$ for all $\theta \in (\theta^*, \theta^{**})$. The Principal does not inspect message m_N. By contradiction, suppose the ex-ante payoff of inspecting types in $[\underline{\theta}, \theta^*] \cup [\theta^{**}, \overline{\theta}]$ is higher than mandating action $a(\theta) = \theta^*$. We know, the ex-ante payoff of inspection types in (θ^*, θ^{**}) is strictly higher than mandating action $a(\theta) = \theta^*$ for these types. This implies that inspecting all types yields a higher payoff than mandating action $a(\theta) = \theta^*$ for types $[\underline{\theta}, \theta^*] \cup [\theta^{**}, \overline{\theta}]$ and inspecting types in (θ^*, θ^{**}). Hence inspecting all types has a higher payoff than the maximizer to problem $\mathbb{P}_D$. A contradiction.

(ii) Suppose $u(\cdot, \cdot)$ is log-Supermodular. We find all equilibria (ex-ante) payoffs for the Principal and show payoffs are (weakly) less than the ex-ante payoff of the equilibrium in (i). Fix an equilibrium E. Define $\tilde{\theta} = \sup\{\theta | \pi(\theta) > 0\}$. Let $\mathcal{M}$ be the set of on equilibrium path messages. By a similar argument to Lemma 3, one can show if $m \in \mathcal{M}(\theta)$ and $m' \in \mathcal{M}(\theta')$, where $\tilde{\theta} > \theta' > \theta$, then $i(m') \ge i(m)$ and

$a(m') \geq a(m)$. Denote $\check{i}(\theta) = i(m(\theta))$ and $\check{a}(\theta) = a(m(\theta))$. Therefore both $\check{i}(\cdot)$ and $\check{a}(\cdot)$ are weakly increasing. For ease of notation we use $a(\cdot)$ instead of $\check{a}(\cdot)$ and $i(\cdot)$ instead of $\check{i}(\cdot)$. Hence without loss of generality we can assume $i(\cdot)$ and $a(\cdot)$ are weakly increasing.

Define $\Theta_I = \{\theta | i(m(\theta)) > 0\}$, $\Theta_N = \{\theta | i(m(\theta)) = 0\}$. It is easy to see $a^* \equiv a(m(\theta)) = a(m(\theta'))$ for all θ, θ' in Θ_N. The ex-ante payoff of The Principal under E is

$$\mathbb{E}\big[v(\theta, a^*)\mathbb{1}_{a^* \geq \theta} | \theta \in \Theta_N\big]\mathfrak{M}(\Theta_N) + \mathbb{E}\big[v(\theta, \theta) - \phi | \theta \in \Theta_I\big]\mathfrak{M}(\Theta_I),$$

where $\mathfrak{M}(\Theta_N)$ and $\mathfrak{M}(\Theta_I)$ are the probability measure of Θ_N and Θ_I respectively. $a(\cdot)$ and $i(\cdot)$ are increasing therefore for all $\theta_I \in \Theta_I$ and $\theta_N \in \Theta_N$, $\theta_N \leq \theta_I$ if $\pi(\theta_N) > 0$. Moreover, $\theta_N \leq a^* \leq \theta_I$ if $\pi(\theta_N) > 0$. Therefore all equilibria payoffs for the Principal can be represented by the value of problem $\mathbb{P}_a$.

$$\mathbb{P}_a(a^*, \Theta_I, \Theta_N): \quad \mathbb{E}\big[v(\theta, a^*)\mathbb{1}_{a^* \geq \theta} | \theta \in \Theta_N\big]\mathfrak{M}(\Theta_N) + \mathbb{E}\big[v(\theta, \theta) - \phi | \theta \in \Theta_I\big]\mathfrak{M}(\Theta_I),$$

subject to the incentive of the Agent: $\Theta_I \subset [a^*, \overline{\theta}]$ and the incentive of the Principal:

$$\mathbb{E}\big[v(\theta, \theta) - \phi | \theta \in \Theta_N\big] \leq \mathbb{E}\big[v(\theta, a^*)\mathbb{1}_{a^* \geq \theta} | \theta \in \Theta_N\big].$$

Note that we do not need to check the incentive of the Principal to inspect m_I since the Principal can commit on a very inefficient action $a(m_I)$ and excludes all types.

In order to find the maximum of $\mathbb{P}_a$, fix a^*. The optimal Θ_I, by Assumption 2 is an interval starting from a^*. Thus the above optimization yields a similar payoff to problem $\mathbb{P}_D$.

The constraint $\mathbb{E}\big[v(\theta, \theta) - \phi | \theta \in \Theta_N\big] \leq \mathbb{E}\big[v(\theta, a^*)\mathbb{1}_{a^* \geq \theta} | \theta \in \Theta_N\big]$, implies that the minimum of $\mathbb{P}_a$ is $\max\{\mathbb{E}[v(\theta, \theta) - \phi], 0\}$. Now suppose $\Theta_I = [a^*, a^{**}]$. The $\mathbb{P}_a$ is continuous in a^* and a^{**}. Therefore $\mathbb{P}_a$ contains all values between the min and max. ■

A.7. Proof of Lemma 4

Proof. (i) Let m_1 and m_2 be two messages on the equilibrium path that the Principal inspects with probability less than one, i.e. $i(m_1) < 1$ and $i(m_2) < 1$.

Let $\Theta_i = \{\theta \in [\underline{\theta}, \overline{\theta}] : m(\theta) = m_i\}$ for $i \in \{1, 2\}$. Let

$$\theta_{m_i} = \sup \Big\{\theta | m = m_i, \text{ and } \int_{t<\theta} \beta(\theta | m = m_i)\, d\theta = 0\Big\}.$$

By definition, the probability measure of types less than θ_{m_i} that send message m_i is zero. By a simple computation, for a small enough $\delta > 0$

$$\begin{aligned}
&\int_{\underline{\theta}}^{\theta_{m_i}+\delta} \Big[v(\theta, \theta_{m_i} + \delta)\Big] \beta(\theta|m = m_i)\, d\theta \\
&= \int_{\underline{\theta}}^{\theta_{m_i}} \Big[v(\theta, \theta_{m_i} + \delta)\Big] \beta(\theta|m = m_i)\, d\theta + \int_{\theta_{m_i}}^{\theta_{m_i}+\delta} \Big[v(\theta, \theta_{m_i} + \delta)\Big] \beta(\theta|m = m_i)\, d\theta \\
&= \int_{\theta_{m_i}}^{\theta_{m_i}+\delta} \Big[v(\theta, \theta_{m_i} + \delta)\Big] \beta(\theta|m = m_i)\, d\theta > 0.
\end{aligned}$$

From second line to third line is by the fact that probability measure of types less than θ_{m_i} is zero and $v(\cdot, \cdot)$ is bounded. The above value is strictly positive since by Assumption 2, $v(\theta, \theta) > 0$ and the probability measure of types between θ_{m_i} and $\theta_{m_i} + \delta$ is strictly positive. As a consequence, for $i \in \{1, 2\}$ we have

$$\max_{\tilde{a}} \int_{\underline{\theta}}^{\tilde{a}} \Big[v(\theta, \tilde{a})\Big] \beta(\theta|m = m_i)\, d\theta > 0.$$

Best reply of the Principal $a(m)$ is

$$a(m) = \operatorname*{argmax}_{\tilde{a}} \int_{\underline{\theta}}^{\overline{\theta}} \Big[V(\theta, \tilde{a})\, \mathbb{1}_{\tilde{a} \geq \theta}\Big] \beta(\theta|m)\, d\theta = \int_{\underline{\theta}}^{\tilde{a}} \Big[V(\theta, \tilde{a})\Big] \beta(\theta|m)\, d\theta.$$

The maximum is strictly positive, therefore $a(m_i) > \inf\{\Theta_i\}$. Moreover for all $\epsilon > 0$ small enough we can find a type $\theta_i \in \Theta_i$ such that $a(m_i) - \theta_i < \epsilon$. Otherwise $a(m_i)$ is not optimal and the Principal can decrease it.

Now by contradiction assume $a(m_1) > a(m_2)$, then types in Θ_2 that are very close but less that $a(m_2)$ want to deviate and send message m_1. Formally

$$(1 - i(m_1))u(\theta, a(m_1))\mathbb{1}_{a(m_1)\geq\theta} > (1 - i(m_2))u(\theta, a(m_2))\mathbb{1}_{a(m_2)\geq\theta},$$

for $\theta \in \Theta_2$ and small ϵ such that $\theta \in (a(m_2) - \epsilon, a(m_2))$. The reason is for small enough ϵ, the right side goes to zero, and the left side is always higher than a positive amount. A contradiction. Hence $a(m_1) = a(m_2)$.

(ii) By contradiction assume $i(m_1) > i(m_2)$. By (i) $a(m_1) = a(m_2)$. Let $\Theta_i = \{\theta \in [\underline{\theta}, \overline{\theta}] : m(\theta) = m_i\}$ for $i \in \{1, 2\}$. Then types in Θ_1 that are less than $a(m_1)$ want to deviate to m_2. A contradiction. ■

A.8. Proof of Theorem 4

Proof. By Lemma 4 we can focus on equilibria with two messages. Fix an equilibrium E. Define $\Theta_I = \{\theta | m(\theta) = m_I\}$ and $\Theta_N = \{\theta | m(\theta) = m_N\}$ under E. Denote $\mathfrak{M}(\Theta_N)$ and $\mathfrak{M}(\Theta_I)$ are the probability measure of Θ_N and Θ_I respectively.

Claim 3 *Suppose the Principal cannot commit on $a(\cdot)$. Then in any equilibrium $\Theta_I \subset [a(m_N), \overline{\theta}]$. Moreover, the payoff of any equilibrium is equivalent to the value of problem $\mathbb{P}_i$*

$$\begin{aligned}\mathbb{P}_i : &\mathbb{E}\big[(1 - i(m_N))v(\theta, a(m_N))\mathbb{1}_{a(m_N)\geq\theta} | \theta \in \Theta_N\big]\mathfrak{M}(\Theta_N)\\ &+ \mathbb{E}\big[i(m_N)(v(\theta, \theta) - \phi) | \theta \in \Theta_N\big]\mathfrak{M}(\Theta_N) + \mathbb{E}\big[v(\theta, \theta) - \phi | \theta \in \Theta_I\big]\mathfrak{M}(\Theta_I),\end{aligned}$$

subject to $\Theta_I \subset [a(m_N), \overline{\theta}]$ and

$$a(m_N) \in \underset{\tilde{a}}{\operatorname{argmax}} \int_{\underline{\theta}}^{\overline{\theta}} \left[v\left(\theta, \tilde{a}\right) \mathbb{1}_{\tilde{a}\geq\theta}\right] \beta(\theta | m_N)\, d\theta.$$

Proof. $a(m_N)$ should be less than all types in Θ_I, otherwise types $\theta \in (\inf \Theta_I, a(m_N))$ deviate and send message m_N. $\Theta_I \subset [a(m_N), \overline{\theta}]$ is no deviation constraint for the Agent and the last equation of the Claim is no deviation constraint for the Principal. ■

(i) Let $a(m_N) = s^*$. The value of $i(m_N)$ in $\mathbb{P}_i$ does not affect any constraint. Therefore, in order to find the maximum, $i(m_N)$ should be 0 if $\mathbb{E}\big[v(\theta, s^*)\mathbb{1}_{s^*\geq\theta} | \theta \in \Theta_N\big] \geq \mathbb{E}\big[(v(\theta, \theta) - \phi) | \theta \in \Theta_N\big]$, otherwise $i(m_N)$ should be 1. If $i(m_N) = 1$, then full inspection will be the maximum payoff. Therefore let $i(m_N) = 0$. Second we show $\Theta_I = [s^*, s^{**}]$ for some $s^{**} < \overline{\theta}$. Fix an optimal Θ_I. By chaining Θ_I to $[s^*, s^{**}]$ (keeping the same probability measure), s^* does not change but the ex-ante value of the principal will (weakly) increase since $v(\theta, \theta)$ is an increasing function. After change, s^* does not change since for all $s \leq s^*$

$$\int_{\underline{\theta}}^{s} \left[v\left(\theta, s\right) \mathbb{1}_{s\geq\theta}\right] \beta(\theta | m_N)\, d\theta \leq \int_{\underline{\theta}}^{s^*} \left[v\left(\theta, s\right) \mathbb{1}_{s^*\geq\theta}\right] \beta(\theta | m_N)\, d\theta.$$

The above inequality holds since both sides remain the same before and after the change. The above inequality is correct for all $s \geq s^*$. The reason is the left side (weakly) decreases. Therefore we can rewrite the optimization problem as follows

$$\mathbb{P}_2: \max_{s^*, s^{**}} \left\{ \int_{\underline{\theta}}^{s^*} \Big(v(\theta, s^*) \Big)\, dF(\theta) + \int_{s^*}^{s^{**}} \Big(v(\theta, \theta) - \phi \Big)\, dF(\theta) \right\},$$

subject to

$$s^* \in \underset{\tilde{s}}{\operatorname{argmax}} \int_{\theta \in [\underline{\theta}, s^*] \cup [s^{**}, \overline{\theta}]} \Big[v\left(\theta, \tilde{s}\right) \mathbb{1}_{\tilde{s} \geq \theta} \Big]\, dF(\theta).$$

(ii) Now we show if $\phi < v(\overline{\theta}, \overline{\theta})$, (no exclusion region), then the maximum payoff is equivalent to optimal deterministic inspection with full commitment. In this case, the payoff of $\mathbb{P}_2$ is maximized when $s^* = \theta^*$ and $s^{**} = \overline{\theta}$. We just need to show $a(m_N) = \theta^*$. By contradiction, first assume $a(m_N) < \theta^*$, then

$$\int_{\underline{\theta}}^{a(m_N)} \Big[v(\theta, a(m_N)) \Big]\, f(\theta) d\theta > \int_{\underline{\theta}}^{\theta^*} \Big[v(\theta, \theta^*) \Big]\, f(\theta) d\theta.$$

Therefore

$$\begin{aligned}
&\int_{\underline{\theta}}^{a(m_N)} \Big[v(\theta, a(m_N)) \Big]\, f(\theta) d\theta + \int_{a(m_N)}^{\overline{\theta}} \Big[v(\theta, \theta) - \phi \Big]\, f(\theta) d\theta \\
&\quad > \int_{\underline{\theta}}^{a(m_N)} \Big[v(\theta, a(m_N)) \Big]\, f(\theta) d\theta + \int_{\theta^*}^{\overline{\theta}} \Big[v(\theta, \theta) - \phi \Big]\, f(\theta) d\theta \\
&\qquad > \int_{\underline{\theta}}^{\theta^*} \Big[v(\theta, \theta^*) \Big]\, f(\theta) d\theta + \int_{\theta^*}^{\overline{\theta}} \Big[v(\theta, \theta) - \phi \Big]\, f(\theta) d\theta.
\end{aligned}$$

The value of the first line cannot be higher than the third line, since θ^* is the solution to $\mathbb{P}_D$. A contradiction.

Second, assume $a(m_N) > \theta^*$. We can conclude $a(m_N) > \theta^{**}$ since after observing m_I by Principal she puts zero probability in interval (θ^*, θ^{**}). However, $a(m_N) > \theta^{**}$ is impossible since $\theta^{**} = \overline{\theta}$.

(iii) Now we show $s^{**} \geq \theta^{**}$. By contradiction assume $s^* \leq s^{**} < \theta^{**}$. By changing s^{**} to θ^{**}, the value of $\mathbb{P}_2$ increases. We show this change cannot affect the constraint of $\mathbb{P}_2$. After the change, s^* does not change since for all $s \leq s^*$

$$\int_{\underline{\theta}}^{s} \left[v\left(\theta, s\right) \mathbb{1}_{s \geq \theta} \right] \beta(\theta|m_N)\, d\theta \leq \int_{\underline{\theta}}^{s^*} \left[v\left(\theta, s\right) \mathbb{1}_{s^* \geq \theta} \right] \beta(\theta|m_N)\, d\theta.$$

The above inequality holds since by multiplying both sides by $\frac{F(s^*)+1-F(\theta^{**})}{F(s^*)+1-F(s^{**})}$, the inequality transforms to the inequality before the change. The above inequality for the same reason is correct for all $s \geq \theta^{**} > s^{**}$. ■

A.9. Proof of Theorem 5

Proof. Note that since the Principal cannot commit on $a(\cdot)$, we can use both Lemma 4, and Claim 3. Let $a(m_N) = s^*$. By Assumption 4, the Agent's strategy $m(\theta)$ alternates between m_I, and m_N in finite intervals for types in $[s^*, \overline{\theta}]$.

We show $\Theta_I = [s^*, s^{**}]$ for some $s^{**} \leq \overline{\theta}$. If there exists $s^{**} \in [s^*, \overline{\theta}]$ such that $m(\theta) = m_I$ for $\theta \in [s^*, s^{**})$ and $m(\theta) = m_N$ for $\theta \in [s^{**}, \overline{\theta}]$, the claim has been proved. Otherwise, by contradiction, there exist two intervals $[t_1, t_2]$ and $[t_2, t_3]$ such that $m(t) = m_N$ for $t \in [t_1, t_2]$ and $m(t) = m_I$ for $t \in [t_2, t_3]$. The idea is to switch two small sub intervals. Let $[t_1', t_2] \subset [t_1, t_2]$ and $[t_2, t_2'] \subset [t_2, t_3]$ such that the probability measure of $[t_1', t_2]$ and $[t_2, t_2']$ are the same in $F(\cdot)$. Therefore after the switch $m(t) = m_I$ for $t \in [t_1', t_2]$ and $m(t) = m_N$ for $t \in [t_2, t_2']$. Denote β_s and β the posterior belief of the Principal after and before the switch respectively. This switch, does not change $a(m_N)$ since for all $\theta \leq s^*$, $\beta_s = \beta$ and for all $a \leq s^*$

$$\int_{\underline{\theta}}^{a} \left[v\left(\theta, s\right) \mathbb{1}_{a \geq \theta} \right] \beta_s(\theta|m_N)\, d\theta \leq \int_{\underline{\theta}}^{s^*} \left[v\left(\theta, s^*\right) \mathbb{1}_{s^* \geq \theta} \right] \beta_s(\theta|m_N)\, d\theta.$$

The above inequality holds since both sides remain the same before and after the switch. The above inequality is also correct for all $a \geq s^*$. The reason is the value of left side (weakly) decreases after the switch and the value of the right side remains the same. This implies that the value of not inspecting m_N does not change. However, the value of inspecting m_N strictly decreases after the switch.

Now we need to show after the switch, the Principal still wants to inspect message m_I. Denote $a_s(m_I)$, the action of the Principal after the switch and when observes m_I. First $a_s(m_I) \notin (t_1', t_2')$. The reason is interval (t_1', t_2') is very small, and by decreasing $a_s(m_I)$ to t_1 the Principal's value strictly increases. The value strictly increases since the Principal puts zero probability (posterior belief) on types in interval $[t_1, t_1')$ after observing m_I. For the same reason $a_s(m_I) \notin (t_1, t_1')$. Second if $a_s(m_I)$ is less than t_1, then $a(m_I)$ is also less than t_1 (before the switch). The reason is after the switch, the value of not inspecting m_I (weakly) increases, but action $a = a_s(m_I)$ is available for the Principal before the switch. Action $a = a_s(m_I)$ yields the same payoff as the payoff of not inspecting after the switch. Therefore the switch does not change the value of not inspecting messages m_I. However, the value of inspecting m_I strictly increases. Third, let $a_s(m_I)$ be higher than t_2'. The switch increases the value of inspection by

$$\Big(\int_{t_1'}^{t_2} v(\theta,\theta)\; dF(\theta) - \int_{t_2}^{t_2'} v(\theta,\theta)\; dF(\theta)\Big)\frac{1}{\mathfrak{M}(\Theta_I)}.$$

We argue the value of not inspecting m_I increases (the value after minus before) at most

$$\Big(\int_{t_1'}^{t_2} v(\theta,a_s(m_I))\; dF(\theta) - \int_{t_2}^{t_2'} v(\theta,a_s(m_I))\; dF(\theta)\Big)\frac{1}{\mathfrak{M}(\Theta_I)}.$$

The reason is

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V\left(\theta, a(m_I)\right) \mathbb{1}_{a(m_I)\geq\theta}\Big]\, \beta(\theta|m_I)\; d\theta \geq \int_{\underline{\theta}}^{\overline{\theta}} \Big[V\left(\theta, a_s(m_I)\right) \mathbb{1}_{a_s(m_I)\geq\theta}\Big]\, \beta(\theta|m_I)\; d\theta,$$

and the value of not inspecting m_I is at most

$$\begin{aligned}&\int_{\underline{\theta}}^{\overline{\theta}} \Big[V\left(\theta, a_s(m_I)\right) \mathbb{1}_{a_s(m_I)\geq\theta}\Big]\, \big(\beta_s(\theta|m_I) - \beta(\theta|m_N)\big)\; d\theta \\ &= \Big(\int_{t_1'}^{t_2} v(\theta,a_s(m_I))\; dF(\theta) - \int_{t_2}^{t_2'} v(\theta,a_s(m_I))\; dF(\theta)\Big)\frac{1}{\mathfrak{M}(\Theta_I)}.\end{aligned}$$

Finally by Assumption 4

$$\begin{aligned}&\Big(\int_{t_1'}^{t_2} v(\theta,\theta)\; dF(\theta) - \int_{t_2}^{t_2'} v(\theta,\theta)\; dF(\theta)\Big)\frac{1}{\mathfrak{M}(\Theta_I)} \\ &\geq \Big(\int_{t_1'}^{t_2} v(\theta,a_s(m_I))\; dF(\theta) - \int_{t_2}^{t_2'} v(\theta,a_s(m_I))\; dF(\theta)\Big)\frac{1}{\mathfrak{M}(\Theta_I)},\end{aligned}$$

which implies the value of inspecting message m_I after the switch is still higher than not inspecting. We conclude $\Theta_I = [s^*, s^{**}]$. ∎

Therefore we can rewrite the optimization problem as follows

$$\mathbb{P}_3: \max_{s^*, s^{**}} \left\{ \int_{\underline{\theta}}^{s^*} \Big(v(\theta, s^*)\Big)\, dF(\theta) + \int_{s^*}^{s^{**}} \Big(v(\theta,\theta) - \phi\Big)\, dF(\theta) \right\},$$

subject to

$$s^* \in \operatorname*{argmax}_{\tilde{s}} \int_{\theta \in [\underline{\theta}, s^*] \cup [s^{**}, \overline{\theta}]} \Big[v\left(\theta, \tilde{s}\right) \mathbb{1}_{\tilde{s} \geq \theta}\Big]\, dF(\theta),$$

and

$$\mathbb{E}\big[v(\theta,\theta) - \phi | \theta \in [s^*, s^{**}]\big] \mathfrak{M}(\Theta_I) \geq \max_{\tilde{s}} \int_{[s^*, s^{**}]} \Big[v\left(\theta, \tilde{s}\right) \mathbb{1}_{\tilde{s} \geq \theta}\Big]\, dF(\theta).$$

A.10. Proof of Theorem 6

Proof. An ideal equilibrium suggestion for the highest equilibrium payoff for the Principal is as follows

- Strategies:

$$m(\theta) = \begin{cases} m_{NI} & \theta \in [\underline{\theta}, \theta^*], \\ m_I & \theta \in (\theta^*, \overline{\theta}] \end{cases}$$

$$i(m) = \begin{cases} 1 & m = m_I, \\ 0 & \text{otherwise} \end{cases}$$

$$a(m) = \begin{cases} \theta^* & m = m_{NI}, \\ \underline{\theta} & \text{otherwise} \end{cases}$$

$$a^I(m, \theta) = \theta.$$

- Beliefs: On the equilibrium path, beliefs are consistent with strategies and the off-path belief puts probability 1 on $\underline{\theta}$, i.e.

$$\beta(\underline{\theta} | m \neq m_I, m_{NI}) = 1.$$

The above strategy and beliefs generates the commitment payoff according to the Proposition 2. We need to check incentives of both players. The Agent does not have any incentive to deviate. Types above θ^* do not want to deviate to m_{NI} since the mandated action becomes less than their types. Types above θ^* will not deviate to m_I since they will be inspected and their payoff become zero. The principal can commit to inspect m_I, so we do not check the incentive of the Principal after observing m_I. Now we show $a(m_N) = \theta^*$, where

$$a(m_N) = \underset{\tilde{a}}{\text{argmax}} \int_{\underline{\theta}}^{\tilde{a}} \Big[V(\theta, \tilde{a}) \Big] \beta(\theta|m_{NI})\, d\theta.$$

By contradiction, first assume $a(m_N) < \theta^*$, then

$$\int_{\underline{\theta}}^{a(m_N)} \Big[v(\theta, a(m_N)) \Big] f(\theta) d\theta > \int_{\underline{\theta}}^{\theta^*} \Big[v(\theta, \theta^*) \Big] f(\theta) d\theta.$$

Therefore

$$\int_{\underline{\theta}}^{a(m_N)} \Big[v(\theta, a(m_N)) \Big] f(\theta) d\theta + \int_{\theta^*}^{\overline{\theta}} \Big[v(\theta, \theta) - \phi \Big] f(\theta) d\theta$$
$$> \int_{\underline{\theta}}^{\theta^*} \Big[v(\theta, \theta^*) \Big] f(\theta) d\theta + \int_{\theta^*}^{\overline{\theta}} \Big[v(\theta, \theta) - \phi \Big] f(\theta) d\theta.$$

The above strict inequality is a contradiction since by the definition, the threshold θ^* should generate higher value than the threshold $a(m_N)$. Thus the left side should not be higher than the right side. A contradiction. Second, assume $a(m_N) > \theta^*$. We can conclude $a(m_N) > \theta^{**}$ since after observing m_{NI} by Principal she puts zero probability in interval (θ^*, θ^{**}). However, $a(m_N) > \theta^{**}$ is impossible since $\theta^{**} = \overline{\theta}$. Now we need to show by observing m_{NI} the principal does not inspect, formally

$$\int_{\underline{\theta}}^{a(m_N)} \Big[v(\theta, a(m_N)) \Big] \beta(\theta|m_{NI}) d\theta \geq \int_{\underline{\theta}}^{\theta^*} \Big[v(\theta, \theta) - \phi \Big] \beta(\theta|m_{NI}) d\theta,$$

we can replace $a(m_N)$ by θ^*

$$\int_{\underline{\theta}}^{\theta^*} \Big[v(\theta, \theta^*) \Big] \beta(\theta|m_{NI}) d\theta \geq \int_{\underline{\theta}}^{\theta^*} \Big[v(\theta, \theta) - \phi \Big] \beta(\theta|m_{NI}) d\theta.$$

By contradiction assume the reverse, thus we have

$$\int_{\underline{\theta}}^{\theta^*} \Big[v(\theta,\theta) - \phi\Big] f(\theta)d\theta + \int_{\theta^*}^{\overline{\theta}} \Big[v(\theta,\theta) - \phi\Big] f(\theta)d\theta$$
$$> \int_{\underline{\theta}}^{\theta^*} \Big[v(\theta,\theta^*)\Big] f(\theta)d\theta + \int_{\theta^*}^{\overline{\theta}} \Big[v(\theta,\theta) - \phi\Big] f(\theta)d\theta.$$

This is a contradiction by the definition θ^*. So the Principal does not any incentive to inspect message m_{NI}.

■