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“Menu Auctions Under Asymmetric Information”

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MENU AUCTIONS UNDER ASYMMETRIC INFORMATION[1]

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Abstract. We study menu auction games in which several principals influence the choice of a privately-informed agent by simultaneously offering action-contingent payments; the agent is free to accept any subset of the offers. Building on tools from non-smooth optimal control with type-dependent participation constraints, we provide necessary conditions for any equilibrium allocation as the (constrained) maximizer of an endogenous aggregate virtual-surplus program. The aggregate maximand includes an information-rent component which captures how the principals' rent-extraction motives combine. Although there is a large set of equilibria, including equilibrium allocations with discontinuities, we isolate one particular equilibrium allocation, the *maximal* allocation, which is the solution to an unconstrained maximization program. Under weak conditions, necessary conditions for a maximal allocation are also sufficient, and the corresponding equilibrium tariff offers are easily constructed. We illustrate our findings and derive some economic implications in several applications, with principals having either congruent interests (e.g., public goods collective action games), opposed interests (e.g., pork barrel politics, lobbying), and protection for sale in an international trade context.

Keywords. Menu auctions, delegated common agency, screening contracts, non-smooth optimization problems, public goods games, collective action, pork barrel politics, positive theory of regulation, protection for sale.

1. INTRODUCTION

Motivation. Economists have long been interested in strategic settings in which several interested parties (with either congruent or conflicting interests) attempt to influence a common agent through contribution schedules. In the almost four decades that have passed since the seminal strategic analysis by Bernheim and Whinston (1986), the *truthful* equilibrium of their complete-information model of menu auctions and influence games has become a workhorse in a wide range of settings. Applications include international trade (Grossman and Helpman, 1994 and 1995; Dixit, Grossman and Helpman, 1997), political economy (Grossman and Helpman, 1996; Aidt, 1998; Besley and Coate, 2001; Persson, 1998; Bellettini and Ottaviano, 2005; Felli and Merlo, 2006), public finance (Persson and Tabellini, 1994 and 2002), combinatorial auction design (Milgrom, 2007),

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industrial organization (Bernheim and Whinston, 1998; Inderst and Wey, 2008), and environmental economics (Aidt, 1998).

The menu auction game of Bernheim and Whinston (1986) owes its success, in part, to the simplicity and robustness of its equilibrium characterization, even in what may at first glance appear to be very complicated strategic settings. To review, the basic game consists of n principals and a single common agent. The agent chooses some action, $q \in \mathcal{Q}$, that has payoff consequences for each of the principals. Prior to taking an action, however, the principals may each offer the agent enforceable payment schedules – menus of promised payment-action pairs (possibly subject to constraints on feasible payments). After receiving a menu offer from each principal, the agent chooses which contracts to accept, selects an action to maximize his own utility, and the corresponding payments are enforced. Bernheim and Whinston (1986) show that there are a large number of equilibria to this influence game, but there is always an equilibrium in which the agent chooses an action which maximizes the collective surplus of the principals and the agent. Such a surplus-maximizing equilibrium can be supported with *"truthful"* menus in which each principal offers a transfer schedule whose margin is equal to the principal's marginal benefit of action. This focal property (together with the robustness of this allocation to a reasonable class of renegotiations) leads Bernheim and Whinston (1986) to argue that this collective surplus-maximizing allocation is a reasonable equilibrium to use for predicting outcomes in general menu auction games with complete information.

The novel contribution of this paper is to reconsider menu auction games under the assumption that the agent has private information. Our paper provides the first general analysis of this class of influence games. Here also, we will be interested in which allocations are candidates for an equilibrium and what kind of contribution schedules sustain those equilibrium allocations. The first concern leads us to establish a set of general necessary conditions. The second construction provides sufficient conditions.

Computing Best-Responses in Non-Smooth Environments. Characterizing the set of equilibria in our incomplete-information game of common agency raises a range of difficulties. The first one, although it appears purely technical, has far reaching economic consequences. Most previous research in common agency has imposed a refinement (sometimes implicitly) that principals offer continuously differentiable contribution schedules. Examples abound. In the scenario where common agency is *intrinsic* (i.e., the agent must choose between participating and accepting all principal offers, and not participating and rejecting all offers), Laffont and Tirole (1991), Martimort (1992, 1996), Stole (1991), Ivaldi and Martimort (1994), Mezetti (1997), Biais et al. (2000), Calzolari (2001), Olsen and Osmudsen (2003, 2011), Laffont and Pouyet (2004), Martimort and Stole (2009a) made this differentiability assumption in various contexts. In models where common agency is *delegated* (i.e., the agent is allowed to accept any subset of principal offers) and the principals contract on different actions, contributions by Martimort and Stole (2009b), Martimort and Semenov (2008) and Calzolari and De Nicolo (2013, 2015), also take this restrictive approach. To understand how restrictive these differentiability assumptions are on the set of equilibria requires us to allow principals to choose from the larger set of upper semicontinuous transfer functions. Note that in this case, if principal i expects principal j to offer a discontinuous payment schedule, principal i's objective function (which includes the surplus of the agent as a function of her action) is discontinuous and non-differentiable. One can no longer apply standard textbook control-theoretic tools which assume continuity and piecewise differentiability to study this problem. Fortunately, we are able to import results from our earlier work on non-smooth optimal control for contract theory (Martimort and Stole, 2022) to characterize each principal's best response function, providing the first step of our analysis (which culminates in Proposition 1). Roughly speaking, this approach generalizes the seminal work of Jullien (2000) on type-dependent participation constraints by concavifying each principal's objective for a given set of (possibly discontinuous) contributions of other principals. At a best response, the bilateral contract that each principal offers exhibits the familiar tradeoff between bilateral efficiency and information rent extraction. However, rent extraction is further limited by the possibility that the agent rejects the offer and obtains a type-dependent reservation payoff by contracting with only the other principals. All together, the familiar textbook distortions from the screening literature[4] are modified by taking into account the shadow value of the agent's type-dependent participation constraint.

A second difficulty introduced by asymmetric information is that an individual principal may only choose to actively influence a strict subset of types in equilibrium. By influence, we mean that the principal's offer induces an agent to choose an action that the type would not have chosen otherwise. The sets of types for which each principal is active must be determined in order to construct equilibrium tariffs, but the equilibrium tariffs, in turn, determine the regions of activity. In short, the equilibrium activity sets must be jointly determined as part of a fixed point of the principals' best-response correspondences. To obtain a straightforward characterization of the relevant type-dependent participation constraint and to better understand the structure of the influence areas of principals, we focus our attention on economic environments and equilibria that that satisfy a mild *active-interval* property. If the preferences of principals are linear in the agent's action, we will see the agent's type-dependent participation constraint for principal i will always bind on a single interval of types (possibly degenerate). For more general nonlinear principal preferences, we will focus on equilibria with this property; such equilibria always exist.[5]

[4]Laffont and Martimort (2002, Chapter 3).

[5]It is an open question as to whether or not the requirement of an active-interval is an equilibrium refinement with nonlinear principal preferences that are independent of type. We have failed to find any

To put this into a more economic context, principals are classified into two subsets: those principals who like more of the agent's action and those principals who like less. We demonstrate that the influence of a given principal can be summarized by means of a *virtual surplus* function that includes the preferences of the principal, an information rent term, and the impact of the agent's outside option when rejecting the principal's offer. Intuitively, whenever the marginal virtual surplus of a principal's virtual surplus is positive (resp. negative), the principal pays the agent to increase (resp. decrease) his action relative to the outcome in which principal i does not make any offer. In active-interval equilibria, a principal who values more agent action offers nonnegative marginal transfers and a principal who values less agent action offers nonpositive marginal transfers.

Equilibrium Necessary Conditions. Our menu auction game is an *aggregate game*[6] because the agent's choice depends only upon the aggregate payment he receives from all principals, and each principal's preferences over strategy profiles can be reduced to preferences over the aggregate payment function and her own payments. Although the menu auction game has infinite-dimensional strategies and asymmetric information, it also has the convenient property that it is quasi-linear in strategies (i.e., payoffs are linear in payments). Following Martimort and Stole (2012), we can thus apply the *Aggregate Concurrence Principle*. In a nutshell, this principle states that if an equilibrium action solves the best-response problems for each principal, it must also solve the sum of those objectives. This leads us to show that any possible equilibrium allocations must satisfy a simpler optimization problem that conflates the influence of all principals. Roughly, such an allocation should maximize the sum of the virtual valuations of all principals together with the agent's surplus. We identify this objective with that of a surrogate principal whose choice replicates the non-cooperative decision process. Because virtual valuations for each principal are computed with respect to the agent's outside option when not dealing with that principal, this objective still depends on the allocation itself and the surrogate principal's optimization problem still features a fixed-point.

As in Bernheim and Whinston's (1986) complete information game, there are multiple equilibrium allocations in our incomplete information setting. We provide necessary conditions satisfied by every such equilibrium allocation. Our main theoretical contribution (Theorem 1) and its corollary demonstrates that all equilibria exhibit the same kind of informational-rent distortion with differences in equilibria characterized by action domain restrictions in the surrogate maximization problem.

Maximal Equilibrium and Sufficient Conditions. We begin by characterizing and focusing on a particular allocation – what we call the *maximal* allocation. The adjec-

example of equilibria which fails to satisfy this active-interval property.

[6]See Jensen (2018) for an overview of the literature.

tive *maximal* is used here because this allocation is the solution to the surrogate program with no constraints on the action domain (i.e., the domain is maximal). Beyond this aesthetic feature, this allocation has also many desirable properties. Under weak conditions, it alway exists (Theorem 2). It is easy to compute, continuous in type, and exhibits maximal separation across types compared to all equilibria. In contrast, discontinuous equilibrium allocations exhibit bunching around discontinuities. Furthermore, *any* equilibrium allocation that is fully separating over an open interval of types must equal the maximal allocation over that interval (Corollary 1). Lastly, the maximal allocation is an equilibrium allocation which is implemented with continuously-differentiable schedules (Theorem 3). These smooth equilibrium transfer functions are reminiscent of the *truthful equilibria* found in complete information settings but now, the schedules account for informational distortions. Generalizing a well-known property of *truthful* schedules to a world of asymmetric information, the marginal *maximal contribution* of a given principal perfectly reflects her marginal virtual surplus.

Discontinuous Equilibria. Our general approach also allows us to characterize discontinuous equilibria. Because in any equilibrium of the game, the solution to the surrogate program is a subset of the actions induced by the maximal allocation, the characterization of the solution bears similarities with the allocations found in the mechanism design literature on delegation without transfers (Holmström, 1984; Melumad and Shibano, 1991; Martimort and Semenov, 2006; Alonso and Matouschek, 2008; Amador and Bagwell, 2013). We borrow from this literature techniques that allow us to characterize necessary conditions for any discontinuous equilibrium (Theorem 4), and we provide sufficient conditions that ensure that allocations satisfying these necessary conditions arise as equilibria (Theorem 5). Roughly, the ranges of these equilibria are obtained by introducing gaps in the range of the maximal equilibrium and having principals not paying if the agent were to choose actions in those gaps. Importantly, among all such discontuous equilibria, the maximal equilibrium maximizes the agent's payoff.

Maximal Equilibrium at Work. We consider several applications below that reflect many of the settings to which the menu auction literature has been applied. Although we focus much of our attention on the maximal equilibrium allocation for the reasons enumerated above, we will also illustrate the importance of discontinuous equilibria in one setting.

Public Good Games and Collective Action. We first consider a public good game where principals are contracting with an agent with privately-known marginal cost of providing a public good. Principals have linear surplus functions, possibly heterogeneous in marginal values. From a technical viewpoint, linearity brings a significant simplification of our analysis: virtual surplus functions become independent of the agent's outside option and the maximal allocation becomes a simple solution to an unconstrained surrogate optimization problem.

In a world with a single principal, this setting is analogous to the government regulation of a monopolist with unknown marginal cost; a workhorse model in regulatory economics since Baron and Myerson (1982). With multiple principals, however, we will see that there are additional effects that generates an allocation considerably different from either the first-best allocation (which is also the *truthful equilibrium* of Bernheim and Whinston's (1986) complete information game) or the Baron and Myerson (1982) optimal allocation. Instead, the maximal equilibrium allocaation is a solution to a virtual version of the Lindahl-Samuelson conditions for public good provision.

The maximal allocation is simple to characterize and provides a number of interesting comparative statics. Among others, the equilibrium outcome features *non-neutrality.* In contrast with the scenario of complete information, the equilibrium allocation is now sensitive to *ex ante* redistributions of the marginal surplus across principals. This result is analogous to a result found in the public finance literature (Bergstrom, Blume and Varian (1986)), that a set of neutral taxes and subsidies on Cournot competitors will have a non-neutral aggregate price effect whenever the public intervention impacts the set of active firms. We extend this insight to a setting of incomplete information. We find that although a mean-preserving spread of the principal's marginal preferences does not affect the efficient amount of public provision, it leads to an increase the provision of the public good.

Collective Action. The Olsonian collective action problem can also be viewed as a menu auction game. Consider several individuals, with heterogenous (and possibly opposed) linear preferences for agent action, each of whom wants to influence the agent to take an action. Under complete information, the truthful equilibrium allocation is efficient, maximizing the sum of the player's payoffs. Under asymmetric information, however, a version of the *"tragedy of the commons"* arises. Each principal maximizes the virtual bilateral surplus between the principal and the agent, given the other principals' offers. The principal introduces inefficiencies to extract the agent's information rents on the margin, ignoring the impact that harvesting the information rent has on the other principals. In the maximal equilibrium, there is over harvesting of the agent's rents leading to greater distortions relative to the setting in which all principals could cooperate. We demonstrate that although a mean-preserving spread of the principal's marginal preferences does not affect the efficient policy choice, it leads to a mean-preserving spread in the distribution of equilibrium policies. We also consider discontinuous equilibria in this setting with two opposed principals, and characterize an equilibrium with a gap in the middle of the agent's action space. Over this gap, each principal is inactive, with zero payment for these actions; a positive mass of agent types choose actions on the boundary of the gap. Thinking of the principal's as opposed lobbyists and the agent as a politician, the effect of competition in this equilibrium is extreme polarization with middle-of-the policies abandoned by legislators in favor of extreme policies to the left and the right which are well remunerated.

Pork Barrel Politics and Lobbying for Influence. We next consider a simplified model of decision-making within a legislature. A legislator (the agent) allocates a fixed budget between two opposing interest groups. Each group has the same constant marginal value of receiving funds from the legislator who as an unknown bias in favor of one of the groups, but also values any campaign contributions the groups may provide. With complete information, the truthful equilibrium is efficient and maximizes social welfare (the sum of the groups and legislator's preferences). Under incomplete information, however, the maximal allocation is more polarized. This comes from the interaction of each principal's virtual surplus. To extract information rent from legislator's types on their own side of the political spectrum, a principal is less eager to influence types on the other side. As a result, asymmetric information weakens competition between interest groups and reduces the legislator's payoffs.

Protection for Sale. Finally, we adapt Grossman and Helpman (1994)'s celebrated model of protection for sale to a context where the domestic government is privately informed on how it evaluates the trade-off between social welfare and the private contributions it receives from lobbying groups. In this setting, domestic producers of an intermediate good are heterogenous in terms of their productivity. They compete with foreign producers and may lobby the government for an import tariff. We derive the optimal tariff implemented in this context when domestic groups collectively determine influence the government and compare it to the setting in which the lobbying groups individually exert their influence. While there would not be any difference between these two settings in the truthful equilibrium under complete information, under incomplete information producers who cooperate exert greater influence and induce larger tariffs than when producers cannot cooperate. The structure of protection is shown to depend on the more or less dispersed nature of lobbying groups.

Organization. The basic menu auction game with asymmetric information is presented in Section 2. Section 3 analyzes the best-response of a given principal to other principals' offers. The key building block of our analysis is the characterization of best responses using non-smooth optimal control techniques. Necessary conditions that are satisfied by any candidate equilibrium allocation are presented in Section 4. Section 5 focuses on the properties of the maximal allocation and provides sufficient conditions for such allocation to be an equilibrium. Justifying our concern for non-smooth optimization techniques, Section 6 also characterizes a class of discontinuous equilibria. We also show that the maximal equilibrium has attractive welfare properties. Section 7 compares the equilibrium of delegated agency games to alternative outcomes, including the case where principals cooperate, the scenario of intrinsic agency and the scenario of *ex ante* contracting. Section 8 provides the analysis of several applications of our framework to highlight the economic insights that are now available. Proofs are relegated to three different appendices: non-smooth optimal control techniques are adapted for our contracting problem in Appendix A; the proofs of main theorems of the paper are in Appendix B; the proofs and computations for the applied results are presented in Appendix C.

2. A MODEL OF MENU AUCTIONS WITH INCOMPLETE INFORMATION

PREFERENCES AND INFORMATION. Our menu auction game is a setting in which n principals (pronouns "she/they") simultaneously offer individual non-negative payment schedules to influence a common agent (pronoun "he") for the choice of an action.

Each principal has preferences that are concave in the agent's choice of q and linear in monetary transfers. Given a transfer t_i to the agent and a choice of q by the agent, we denote principal i's payoff simply as

$$S_i(q) - t_i$$

where, in full generality, S_i is upper semi-continuous on the set of feasible actions $q \in \mathcal{Q}$, which is a closed and bounded interval. Throughout, we assume that principals have strictly monotonic preferences (i.e., S_i is either strictly increasing or decreasing); we denote the former principals who prefer more action as $i \in \mathcal{A}$ and the latter principals who prefer less action $i \in \mathcal{B}$, where $\mathcal{A} \cup \mathcal{B} = \mathcal{N}$. Below for applications we will sometimes specialize these assumptions to require that S_i is also continuously differentiable and concave, and for other applications we will further assume that each principal's preferences are linear, $S_i(q) = s_i q$.

The agent has heterogeneous preferences over actions and monetary transfers, indexed by a type parameter $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$. Agent preferences are quasi-linear in transfers and represented by

$$S(\theta, q) + t,$$

where $S(\theta, q)$ is upper semi-continuous on $\mathcal{Q}$ and t is the aggregate payment the agent receives from the principals. For ease of presentation, we further assume that S is linear in θ,

$$S(\theta, q) = S_0(q) - \theta q,$$

where S_0 is an upper-semicontinuous function of action q. Assuming linearity in type allows a crisp characterization of incentive compatibility by means of convexity arguments which are familiar, at least since Rochet (1986).[7] EXAMPLES 1 TO 4 below show that the type linearity restriction is readily satisfied in a rich set of economic settings, many of which have become standard models in applied theory.

The type parameter θ is drawn from a commonly-known distribution function, F, with an associated positive, atomless and differentiable density function f. To guarantee that the solutions to a relaxed optimization problem satisfy the standard monotonicity condition of screening models, we assume the familiar *Monotone Hazard Rate Condition* (thereafter *MHRC*), requiring that the distribution function, F, and its complement, $1 - F$, are log-concave.[8] For future reference, we define the agent's stand-alone action $\overline{q}_0(\theta)$ and payoff $U_0(\theta)$ in the absence of any principal influence as

$$\overline{q}_0(\theta) \equiv \arg\max_{q\in\mathcal{Q}} \ S(\theta, q) \text{ and } U_0(\theta) \equiv \max_{q\in\mathcal{Q}} \ S(\theta, q). \tag{2.1}$$

CONTRACTS. Each principal i may present to the agent any upper semi-continuous function, $t_i : \mathcal{Q} \to \mathbb{R}_+$, as her contract offer. Requiring the schedules to be non-negative is without loss of generality if the agent has the option to reject any subset of the offered schedules, which is the scenario of *delegated common agency* and the setting of this paper.[9] Let $\mathcal{T}$ denote the set of non-negative, upper semi-continuous functions on $\mathcal{Q}$. Requiring that t_i only be upper semi-continuous allows for possible discontinuities in transfers, possibly supporting equilibria with discontinuities in action.

TIMING AND EQUILIBRIUM CONCEPT. The timing of our delegated common agency game has three stages. First, nature chooses the agent's type. Second, each principal i chooses a transfer function, $t_i \in \mathcal{T}$. We will denote $T(q) \equiv \sum_{i=1}^{n} t_i(q)$ as the associated aggregate transfers of the principals from this stage and define $T_{-i}(q) \equiv \sum_{j\neq i} t_j(q)$ when this aggregate is taken over all principals except i. Third, the agent chooses an optimal action given the aggregate transfers offered in the second stage. We will denote such a best response as $\overline{q}(\theta\,|\,T)$ but sometimes omit the dependency on the aggregate T when obvious. Finally, payments are made by the principals in accord with their contractual

[7] It is straightforward to extend our analysis to the case where $S(\theta, q) = S_0(q) - \theta c(q)$, for some function c increasing and convex. A simple change of variables (letting $q' = c(q)$ and $\mathcal{S}_i(q') = S_i(c^{-1}(q'))$) would allow us to again apply our methodology *mutatis mutandis*. Beyond, our arguments could also be easily extended to more general functional forms with the notion of S-convexity and S-differentiability developed in Carlier (2001) but at the cost of added complexity in the characterization of incentive compatibility.

[8] Bagnoli and Bergstrom (2005).

[9] *"Contracts on contracts"* are ruled out for verifiability reasons. For instance, in a lobbying context a given interest group may not have all the relevant information on other groups' offers to condition her own contributions. Szentes (2015) analyzes such settings.

obligations.

Our solution concept is pure-strategy Perfect Bayesian equilibria. We say that the strategy profile $\{\overline{q}, \overline{t}_1, \dots, \overline{t}_n\}$ is an *equilibrium* of the influence game if and only if

$$\overline{q}(\theta \,|\, T) \in \arg\max_{q \in \mathcal{Q}} \; S(\theta, q) + T(q) \quad \forall \theta \in \Theta, \tag{2.2}$$

$$\overline{t}_i \in \arg\max_{t_i \in \mathcal{T}} \int_{\underline{\theta}}^{\overline{\theta}} \big(S_i(\overline{q}(\theta \,|\, \overline{T}_{-i} + t_i)) - t_i(\overline{q}(\theta \,|\, \overline{T}_{-i} + t_i))\big) f(\theta) d\theta \quad \forall i \in \mathcal{N}. \tag{2.3}$$

Condition (2.2) is the agent's optimality condition. That $\overline{t}_i$ (and thus $\overline{T}$) is upper semi-continuous ensures existence of such best response $\overline{q}(\theta \,|\, T)$ for the agent's optimization problem over the compact set of actions $\mathcal{Q}$. Condition (2.3) is principal i's optimality condition given the agent's best response.

For any aggregate transfer function, $\overline{T}$, we will refer to the *allocation* $(\overline{U}, \overline{q})$, as defined by $\overline{q}(\theta) \equiv \overline{q}(\theta \,|\, \overline{T})$ and $\overline{U}(\theta) = S(\theta, \overline{q}(\theta)) + \overline{T}(\overline{q}(\theta))$ for all $\theta \in \theta$. The equilibrium range of agent's choices is defined by $\overline{q}(\Theta) \equiv \{q \in \mathcal{Q} \,|\, \exists \theta \in \Theta \,\text{s.t.}\; q = \overline{q}(\theta)\}$ which we will sometimes refer to more succinctly as $\overline{\mathcal{Q}}$.

2.1. *Illustrations*

Our general framework covers a number of economic settings of interest.

Example 1: Public Good Games. Consider the following public good game inspired by the seminal work of Bergstrom, Blume and Varian (1986), and its later developments in a complete information common agency context by Bernheim and Whinston (1986), Laussel and Le Breton (2001) and Le Breton and Salanié (2003). There are n principal-citizens and a privately-informed supplier of a public good. Each principal values the public good, but the principals may differ in terms of the intensities of their preferences. Principals have linear surplus functions for the public good of the form $S_i(q) = s_i q$.[10] We also order the n principals such that $s_1 \geq \ldots \geq s_n$ and denote a configuration of principals' preferences by the vector $\mathbf{s} \equiv (s_1, \dots, s_n)$.

In this context, we take the agent's type to be an unknown, positive marginal cost of production, $\theta \in \Theta$, and the domain of possible public goods to be $\mathcal{Q} = [0, q_{\max}]$, with $q_{\max}$ being sufficiently large to avoid boundary solutions at $q = q_{\max}$. The agent's cost of production contains a known component, $C(q)$, which is continuously differentiable, increasing, convex, and $C(0) = 0$, and a private component depending upon type, θq.

[10]This linearity assumption is standard in the mechanism design literature on public good provision (Mailath and Postlewaite, 1990; Ledyard and Palfrey, 1999, among others).

Thus, we have

$$S(\theta, q) = -\theta q - C(q),$$

and the stand-alone action and payoffs are $\overline{q}_0(\theta) = \overline{U}_0(\theta) = 0$. ■

Example 2: Collective Action. Example 1 can be recast *mutatis mutandis* as a model of collective action in the spirit of Olson (1965). A decision-maker, the common agent, chooses a policy q from the domain $\mathcal{Q}$. Absent any influence, this agent would maximize $S(\theta, q) = S_0(q) - \theta q$ and choose a stand-alone action $\overline{q}_0(\theta)$ (his bliss point) that, for simplicity, is assumed to be interior (i.e., $\overline{q}_0(\theta) \in \mathring{\mathcal{Q}}$ for all $\theta \in \Theta$) and thus uniquely defined by the first-order condition,

$$S_0'(\overline{q}_0(\theta)) = \theta, \tag{2.4}$$

with the corresponding stand-alone payoff $\overline{U}_0(\theta) = S_0(\overline{q}_0(\theta)) - \theta \overline{q}_0(\theta)$.

In this setting, the principals are lobbyists with preferences that are linear in action, $S_i(q) = s_i q$, but unlike the public goods case preferences are not congruent: some subset of principals prefers higher actions while the complement prefers lower ones. Each principal non-cooperatively promises the agent a payment as a function of the action the agent chooses. ■

Example 3: Pork Barrel Politics. Consider a legislator who has to allocate a budget of unit size between two socio-economic groups, represented by principals 1 and 2 respectively. Principals compete for shares of the overall budget by means of offering campaign contributions to the legislator as a function of his chosen budget allocation. To model a possible bias towards either group, we assume that the legislator (the common agent) has preferences of the form

$$S_0(\theta, q) = \left(\frac{1}{2} - \theta\right) q - \frac{q^2}{2},$$

where $q \in \mathcal{Q} = [0, 1]$ denotes the share of the budget captured by group 1. The bias parameter θ is uniformly distributed on $\Theta = \left[-\frac{1}{2}, \frac{1}{2}\right]$. A positive (resp. negative) value of θ thus means a bias towards principal 2 (resp. 1). Groups have the same constant marginal benefit for money, $b > 0$. Because they split a constant pie, principals have conflicting interests with their preferences being expressed as

$$S_1(q) = bq \text{ and } S_2(q) = b(1 - q).$$

To illustrate, the complete information collective surplus maximizing budget share for principal 1, which is also the agent's stand-alone action here, would be given by

$$q^{fb}(\theta) = q_0(\theta) = \frac{1}{2} - \theta.$$

In particular, the legislator with the strongest bias towards group 1 (i.e., $\theta = -\frac{1}{2}$) allocates all the budget to that group.

Up to a change of variable, this simple model can be interpreted *mutatis mutandis* as a model of split award auctions in lines with Anton and Yao (1989, 1992). There, a buyer wants to procure one unit of the good and might split that unit between two providers. This simple setting can also be useful to think about models of competitive nonlinear pricing when the buyer has some taste for diversity as in Hoernig and Valletti (2011) and optimally splits his purchase between two sellers. Additionally, this reinterpretation could also be viewed as a model of market-share pricing where sellers offer discounts to their customers based on their consumption mix, as in Calzolari and Denicolo (2013). ■

EXAMPLE 4: PROTECTION FOR SALE. Grossman and Helpman (1994) have applied Bernheim and Whinston (1986)'s methodology to successfully explain the design of trade policy, arguing that the choice of trade instruments reflects the influence of various interest groups. To cast this approach into the framework of our model, consider a small country that trades a good with the rest of the world and takes as given the world price p. The domestic policy maker, here the common agent in the setting, may set an import tariff γ on the traded good. The policy maker cares about domestic welfare (the sum of domestic producer and consumer surpluses), tariff revenue, and transfers from the protected industries. Let domestic demand at price p be $\mathcal{D}(p)$. We focus on the setting in which the country is a net importer with domestic price determined by $p+\gamma$. Preferences of domestic consumers are quasi-linear in money and thus consumer surplus with tariff γ and world price p is given by

$$\int_0^{\mathcal{D}(p+\gamma)} \mathcal{D}^{-1}(x)dx - (p+\gamma)\mathcal{D}(p+\gamma).$$

On the supply side, domestic producers are endowed with two different kinds of technology indexed by the subscript $i = 1, 2$. Producing y units of the traded good requires, for producers of type i, $C_i(y)$ units of the numeraire, with $C_i'(y) \geq 0$ and $C_i''(y) > 0$. We assume that producers in group 1 are more efficient, i.e., $C_2'(y) > C_1'(y) > 0$ for all $y > 0$. Domestic producers are competitive and earn profits $\pi_i(\gamma) = (p+\gamma)y_i(p+\gamma) - C_i(y_i(p+\gamma))$ where producer i's supply curve is $y_i(p+\gamma) = C_i^{-1'}(p+\gamma)$. We denote domestic supply as $\mathcal{S}(p+\gamma) \equiv \sum_{i=1}^{2} y_i(p+\gamma)$ and imports as $\mathcal{M}(p+\gamma) \equiv \mathcal{D}(p+\gamma) - \mathcal{S}(p+\gamma)$, which we assume is positive at $\gamma = 0$. Producers are organized as two interest groups with congruent interests; they both enjoy greater profits from higher import tariffs, but with different intensities.

The policy maker cares about the contributions he receives from each protected industry as well as domestic social welfare (including tariff revenue). We shall use the level of domestic social welfare as the variable under control of the policy maker and, with some awkwardness, denote this action by q as in other examples:

$$q = \int_0^{\mathcal{D}(p+\gamma)} \mathcal{D}^{-1}(x)dx - (p+\gamma)\mathcal{D}(p+\gamma) + \gamma\mathcal{M}(p+\gamma) + \sum_i \pi_i(\gamma),$$

which defines implicitly a tariff function of welfare level, $\Gamma(q)$. Note that q ranges from the lowest surplus which is generated by a tariff that chokes off all imports, to the maximum surplus which is generated by no tariff, $\gamma = 0$, which we denote q^C. The policy-maker cares directly about the received monetary contributions from the protected industries, but also places a welfare weight of $\alpha \in [0,1]$ on domestic welfare (including tariff revenue), where α is private information. Redefining $\theta = -\alpha$, we can again write the agent's direct payoff function as

$$S(\theta, q) = -\theta q \tag{2.5}$$

where θ is distributed on $\Theta = [-1, 0]$. With regard to the principals' objectives, we assume that they maximize profits less transfers to the policy maker, but further assume that transfers are inefficient with each transferred Euro costing $(1+\lambda)$ Euros; $\lambda > 0$ is symmetric across principals. Thus, the payoff of principal i as a function of q can be compactly written as

$$S_i(q) = \frac{\pi_i(\Gamma(q))}{1+\lambda}.$$

Because the tariff function $\Gamma(q)$ is non-increasing, we have $S_i'(q) < 0$; i.e., producers prefer higher tariffs. In the absence of any lobbying, the policy-maker would maximize welfare by opting for free trade, $\Gamma(q^C) = 0$ for all $\theta \in [0, 1)$. The stand-alone action and payoffs are thus $\overline{q}_0(\theta) = q^C$ and $\overline{U}_0(\theta) = -\theta q^C$. ■

3. PRELIMINARIES

3.1. *Statement of the Best-Response Problem*

We begin with a consideration of principal i's best response under the belief that the other principals offer the aggregate tariff schedule $\overline{T}_{-i}$. From principal i's vantage point, it is as if she is designing a contract for an agent with residual preferences given by

$$S(\theta, q) + \overline{T}_{-i}(q).$$

Absent principal i, the agent can secure the following payoff when contracting with the remaining $n-1$ principals:

$$\overline{U}_{-i}(\theta) \equiv \max_{q \in \mathcal{Q}} \; S(\theta, q) + \overline{T}_{-i}(q)$$

by choosing an action

$$\overline{q}_{-i}(\theta) \in \arg\max_{q \in \mathcal{Q}} S(\theta, q) + \overline{T}_{-i}(q).$$

When principal i offers a non-negative transfer schedule t_i, the agent obtains utility

$$U(\theta) \equiv \max_{q \in \mathcal{Q}} \; S(\theta, q) + t_i(q) + \overline{T}_{-i}(q),$$

with an optimal action satisfying

$$q(\theta) \in \arg\max_{q \in \mathcal{Q}} S(\theta, q) + t_i(q) + \overline{T}_{-i}(q).$$

If the agent is offered a non-negative schedule by principal i, it necessarily follows that the agent's indirect utility of contracting with principal i weakly exceeds $\overline{U}_{-i}$. Similarly, if the agent's indirect utility exceeds $\overline{U}_{-i}$, then the agent must choose an action for which principal i has offered a positive payment. Hence, we can replace the requirement that $t_i \geq 0$ with the following individual rationality requirement for each principal i:

$$U(\theta) \geq \overline{U}_{-i}(\theta) \quad \forall \theta \in \Theta. \tag{3.1}$$

Because the agent's preferences are bilinear in q and θ, $U(\theta)$ so defined is a maximum of linear functions of θ. Following Rochet (1987), incentive compatibility can be expressed as the tandem requirements

$$-q(\theta) \in \partial U(\theta), \tag{3.2}$$

$$U(\theta) \text{ convex.} \tag{3.3}$$

Condition (3.2) is a general statement of the agent's first-order envelope condition.[11] The requirement in (3.3) that U is convex is equivalent to the requirement that q is a non-increasing selection in the agent's best-response correspondence. Of course, conditions analogous to (3.2) and (3.3) apply to the allocation $(\overline{U}_{-i}, \overline{q}_{-i})$ absent principal i.

Framed in this manner, principal i's problem of choosing an optimal t_i can be reformulated as choosing an allocation (U, q) that is individually rational and incentive compatible for the agent relative to some outside option, $\overline{U}_{-i}$, and that solves the following program:

$$(\mathcal{P}_i): \quad \max_{(U,q)} \int_{\underline{\theta}}^{\overline{\theta}} \left\{ S_i(q(\theta)) + S(\theta, q(\theta)) + \overline{T}_{-i}(q(\theta)) - U(\theta) \right\} f(\theta) d\theta$$

s.t. (3.1)-(3.2)-(3.3).

If $\overline{T}_{-i}$ were known to be continuous and piecewise differentiable, and if the integrand were known to be concave, we could apply standard optimal control techniques to characterize the optimal contract. Assuming that $\overline{T}_{-i}$ is continuous and almost everywhere differentiable, however, imposes an equilibrium refinement that is worth explicit consideration.

To provide a general solution to $(\mathcal{P}_i)$ that requires only that $\overline{T}_{-i}$ be upper semi-continuous, we utilize necessary and sufficient conditions for non-smooth control programs with type-dependent participation constraints developed in Martimort and Stole (2022). Intuitively, one can show that the solution to the program in which the objective function is replaced with its concavification is also a solution to the original program. The concavification, while continuous, is possibly non-differentiable at some points, and so tools from non-smooth optimal control may be applied. These tools, fortunately, allow us to state necessary and sufficient conditions using a distribution of Lagrange multipliers that is reminiscent of the work of Jullien (2000) for the smooth scenario.[12]

3.2. *Monotonic Common Agency Games*

Because U and $\overline{U}_{-i}$ are two implementable rent profiles, they are both convex. The type-dependent participation constraint (3.1) thus amounts to comparing two convex functions – a scenario which might lead to a variety of patterns for the set of types

[11]Here, ∂U represents the sub-differential of a convex function, allowing for the possibility that, at a countable number of values of θ, U may fail to be differentiable. If U is differentiable at θ, then $\partial U(\theta) = \{\dot{U}(\theta)\}$ and thus $\dot{U}(\theta) = -q(\theta)$. At any point θ of non-differentiability, an incentive-compatible allocation q must nonetheless lie between the right and left derivatives of U at this point.

[12]Jullien (2000) provides necessary and sufficient conditions for control problems with pure type-dependent state constraints under the assumption that the objective function is continuous and piecewise differentiable. Martimort and Stole (2022) demonstrate that a variation of Jullien (2000)'s conditions can be applied to discontinuous models as well. It is worth noting that the simplicity of these conditions is a consequence of the assumption that the objective function is linear in the state variable. Because the preferences of the players are quasi-linear in money, this assumption is satisfied in the present setting.

where the constraint binds for principal i which, recall, we denote Ω_i.[13] As we will see below, when principal payoffs are linear in agent action as in Examples 1-3, the binding set of types for principal i must be either $\Omega_i = [\hat{\theta}_i, \overline{\theta}]$ for principals who prefer higher action ($i \in \mathcal{A}$), or $\Omega_i = [\underline{\theta}, \hat{\theta}_i]$ for principals who prefer lower action ($i \in \mathcal{B}$). More generally, it is intuitive that a principal $i \in \mathcal{A}$ who prefers greater agent actions would offer nondecreasing payments in q. With such monotone payments, it follows that the set of inactive types for this principal is an upper interval (possibly degenerate), $[\hat{\theta}_i, \overline{\theta}]$ as in the linear case. A symmetric intuition holds for principals who prefer lower actions. Indeed, as we will show below, when principal preferences are nonlinear (as in Example 4), there always exist equilibria with each Ω_i characterized as either an upper and lower partition of the type space. That said, when principal payoffs are nonlinear, we are unable to show that Ω_i necessarily has this property. We have also been unable to find an example of an equilibrium with more complex binding type sets. For this reason, we restrict our attention to equilibria which have the following monotonicity property, where it is understood that this is might be restrictive for the case of nonlinear principal preferences:

Definition 1 *Let $\{\overline{\Omega}_i\}_{i\in\mathcal{N}}$ be a collection of equilibrium inactive-type sets where*

$$\overline{\Omega}_i = \{\theta \in \Theta \mid \overline{U}(\theta) = \overline{U}_{-i}(\theta)\}.$$

An equilibrium to the common agency game satisfies the Monotonicity Property (hereafter MP) if for each i there is a $\hat{\theta}_i \in [\underline{\theta}, \overline{\theta}]$ such that

$$\overline{\Omega}_i = \begin{cases} [\hat{\theta}_i, \overline{\theta}] & \text{for } i \in \mathcal{A}, \\ [\underline{\theta}, \hat{\theta}_i] & \text{for } i \in \mathcal{B}. \end{cases}$$

To the extent that (MP) is restrictive in a nonlinear environment, this refinement selects equilibria in which differences of payments across alternatives reflect the ranking of the alternatives for each principal. Conveniently, when MP holds, there is a clear segmentation of the principals' areas of influence and we can refer to $\overline{\Omega}_i^c$ (the complement of $\overline{\Omega}_i$) as the set of types for whom principal i is active and influences the agent's choice.

3.3. *Virtual Surplus: Definition, Properties*

While valuation functions are the object of interest to analyze how a principal influences the agent under complete information, virtual surplus turns out to play the same role under asymmetric information. When constructing best-responses for principal i given $\overline{T}_{-i}$, the principal's virtual surplus will largely determine the principal's best response. In what follows, we introduce some standard notation from nonsmooth analysis. Consider

[13] This point is well-known from the literature on countervailing incentives (see, for example, Lewis and Sappington (1989), Maggi and Rodriguez (1995), Jullien (2000)).

any upper-semicontinuous function $H(\cdot)$ on domain $\mathcal{X}$. Define $\overline{co}(H)$ as the concave envelope of the function H over $\mathcal{X}$ and $\overline{co}(H)(x)$ as the value of this concave function at x. Lastly define $\partial\overline{co}(H)(x)$ as the sup-differential of $\overline{co}(H)$. Because $\overline{co}(H)(\cdot)$ is concave, it is almost everywhere differentiable, and thus sup-differential is equal to the derivative of $\overline{co}(H)(\cdot)$ almost everywhere. At points of non-differentiability, the sup-differential is an interval with endpoints equal to the right and left derivatives. We are now prepared to define principal i's virtual bilateral surplus. We motivate its construction below.

DEFINITION 2 *Principal i's* **virtual surplus** *relative to an allocation $(\overline{U}_{-i}, \overline{q}_{-i})$ is defined, for $\overline{q}_{-i}(\theta) \in \mathring{\mathcal{Q}}$,*[14] *as*

$$\mathcal{V}_i(\theta,q)[\overline{q}_{-i}] \equiv \begin{cases} S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}, \partial\overline{co}(S_i)(\overline{q}_{-i}(\theta))\right\} q & \text{if } i \in \mathcal{A}, \\ S_i(q) - \max\left\{\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)), -\frac{1-F(\theta)}{f(\theta)}\right\} q & \text{if } i \in \mathcal{B}.^{15} \end{cases} \tag{3.4}$$

Note that the above expression can be written more succinctly as

$$S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}, \max\left\{\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q,$$

which we will often use to economize on notation in proofs.

Several remarks are in order. First, in many applications it is reasonable to assume that S_i is concave and continuously differentiable (Example 4), in which case our expression for virtual surplus simplifies to

$$\mathcal{V}_i(\theta,q)[\overline{q}_{-i}] = \begin{cases} S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}, S_i'(\overline{q}_{-i}(\theta))\right\} q & \text{if } i \in \mathcal{A}, \\ S_i(q) - \max\left\{S_i'(\overline{q}_{-i}(\theta)), -\frac{1-F(\theta)}{f(\theta)}\right\} q & \text{if } i \in \mathcal{B}. \end{cases}$$

Differentiating (3.4) with respect to q yields

$$\mathcal{V}_{iq}(\theta,q)[\overline{q}_{-i}] = S_i'(q) - \min\left\{\frac{F(\theta)}{f(\theta)}, \max\left\{S_i'(\overline{q}_{-i}(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\}, \text{ a.e..} \tag{3.5}$$

Observe that S_i being concave and $\overline{q}_{-i}$ non-increasing (due to incentive compatibility), the function $S_i'(\overline{q}_{-i}(\theta))$ is itself non-decreasing in θ. Moreover, $MHRC$ ensures that $\frac{F(\theta)}{f(\theta)}$ and $\frac{F(\theta)-1}{f(\theta)}$ are also non-decreasing in θ. Since the *min* and *max* operators preserve monotonicity, it immediately follows that $\mathcal{V}_i(\theta,q)[\overline{q}_{-i}]$ exhibits decreasing differences. That

[14] If $\overline{q}_{-i}(\theta)$ lies on the boundary of $\mathcal{Q}$, the precise statement of the virtual surplus needs to be amended. Lemma A.1 deals with the case where $0 \in \partial\overline{co}\left(S(\theta,\overline{q}_{-i}(\theta)) - \theta\overline{q}_{-i}(\theta)\right)$ and $\overline{q}_{-i}(\theta) \in \text{bd}\mathcal{Q}$.

[15] At points where $\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta))$ is a correspondence, the definition above should be intended as meaning a selection within this correspondence. From a remark in the text, the definition is almost everywhere non-ambiguous.

said, while decreasing differences is an important property, the difficulty remains that that principal i's virtual surplus depends on the action $\overline{q}_{-i}$ that is chosen in its absence, which in turn is an equilibrium object dependent on the tariffs offered by other principals.

In other applications (e.g., Examples 1-3), it is reasonable to assume that S_i is a linear function of q, say $S_i(q) = s_i q$. In this special case, $\mathcal{V}_i(\theta, q)$ is independent of $\overline{q}_{-i}$ and simplifies to

$$\mathcal{V}_i(\theta, q) = \begin{cases} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}, 0\right\} q, & \text{if } i \in \mathcal{A}, \\ \min\left\{s_i + \frac{1-F(\theta)}{f(\theta)}, 0\right\} q, & \text{if } i \in \mathcal{B}. \end{cases}$$

Here we can see that $MHRC$ implies principal $i \in \mathcal{A}$ will have non-increasing virtual surplus in θ, and therefore will choose to influence the agent for types in a lower interval; a symmetric argument applies to the case of $i \in \mathcal{B}$. It is this simple property that guarantees all equilibria to a common agency game with linear principal preferences must exhibit the monotonicity property discussed above.

Returning to our concave, differentiable S_i setting we can illustrate why (3.4) is the correct notion of virtual surplus for constructing best responses. Suppose that principal i values the agent's action (i.e., $i \in \mathcal{A}$, $S_i'(q) > 0$). Since $S_i'(\overline{q}_{-i}(\theta))$ is bounded below by some positive number, there always exists an interval of the form $[\underline{\theta}, \hat{\theta}_i)$, for which

$$\text{(3.6)} \qquad \mathcal{V}_{iq}(\theta, \overline{q}_{-i}(\theta))[\overline{q}_{-i}] = S_i'(\overline{q}_{-i}(\theta)) - \frac{F(\theta)}{f(\theta)} > 0.$$

In other words, principal i would like to expand the agent's output beyond $\overline{q}_{-i}(\theta)$ for those types. To foster intuition, suppose that principal i expands output above $\overline{q}_{-i}(\theta)$ over a small neighborhood $[\theta, \theta + d\theta]$ by a small amount dq. Principal i's expected marginal benefit of doing so would be $f(\theta)S_i'(\overline{q}_{-i}(\theta))dqd\theta$ while the expected extra information rent left to all inframarginal types so that they accept such deal would equal to $F(\theta)dqd\theta$. Condition (3.6) says that, for θ small enough, such a marginal change benefits principal i.[16] In contrast, suppose that $S_i'(\overline{q}_{-i}(\theta)) \leq \frac{F(\theta)}{f(\theta)}$. Inserting into (3.5) yields $\mathcal{V}_{iq}(\theta, \overline{q}_{-i}(\theta))[\overline{q}_{-i}] = 0$ and principal i would not like to marginally increase the agent's output in that case.

The intuition is even simpler in the case of linear principal preferences. Consider again Example 1. In that case, the marginal valuation s_i of principal i is independent of which

[16] Suppose now that principal i dislikes the agent's output, i.e., $S_i' < 0$. For θ in an interval of the form $\left[\hat{\theta}_i, \overline{\theta}\right]$, we thus have

$$\mathcal{V}_{iq}(\theta, \overline{q}_{-i}(\theta))[\overline{q}_{-i}] = S_i'(\overline{q}_{-i}(\theta)) + \frac{1-F(\theta)}{f(\theta)} < 0.$$

Principal i would like to reduce the agent's output below $\overline{q}_{-i}(\theta)$ for those types. Whether $i \in \mathcal{A}$ or $i \in \mathcal{B}$, principal i always influence the agent by increasing the latter's information rent $\overline{U}$ beyond his reservation payoff $\overline{U}_{-i}$ absent this principal and she does by rewarding the agent for changing his action in the direction she likes.

action is taken with the remaining $n-1$ other principals. To illustrate, consider the case $s_i > 0$.[17] Then the existence of an interval of the form $\left[\hat{\theta}_i, \overline{\theta}\right]$ over which $\mathcal{V}_{iq}(\theta, \overline{q}_{-i}(\theta)) = 0$ directly follows from $MHRC$. Because $MHRC$ holds, there is indeed a unique solution $\hat{\theta}_i$ to $s_i f(\hat{\theta}_i) = F(\hat{\theta}_i)$ provided that $1 > s_i f(\overline{\theta})$ and, moreover, $(\hat{\theta}_i, \overline{\theta}] = \{\theta \mid F(\theta) - s_i f(\theta) > 0\}$. $MHRC$ also implies $f'(\theta)/f(\theta) \leq f(\theta)/F(\theta)$, from which it follows that $F(\theta) - s_i f(\theta)$ is increasing if $F(\theta)/f(\theta) > s_i$. Hence, $F(\theta) - s_i f(\theta)$ is strictly increasing on $(\hat{\theta}_i, \overline{\theta}]$ and $\mathcal{V}_{iq}(\theta, \overline{q}_{-i}(\theta)) = 0$ on that interval.

3.4. *Best-Responses: Characterization*

With multiple principals, the equilibrium may depend upon the virtual surplus of every principal. One contribution of this paper is in determining the precise manner in which all $\mathcal{V}_i$s combine to determine the equilibrium allocation. We now present a key building block of our analysis.

Proposition 1 *Given the aggregate transfer function $\overline{T}_{-i}$ offered by other principals, and the agent's corresponding outside option $\overline{U}_{-i}$ and output $\overline{q}_{-i}$, the allocation $(\overline{U}, \overline{q})$ is a solution to principal i's program if and only if it satisfies (3.1)-(3.2)-(3.3), and*

$$\overline{q}(\theta) \in \arg\max_{q \in \mathcal{Q}} \; S(\theta, q) + \mathcal{V}_i(\theta, q)[\overline{q}_{-i}] + \overline{T}_{-i}(q), \;\; a.e. \tag{3.7}$$

where $\mathcal{V}_i(\theta, q)[\overline{q}_{-i}]$ satisfies (3.4).

Moreover, if $\mathring{\Omega}_i \neq \emptyset$, the following property holds:

$$\mathcal{V}_{iq}(\theta, \overline{q}(\theta))[\overline{q}_{-i}] = 0 \iff \overline{U}(\theta) = \overline{U}_{-i}(\theta) \text{ and } \overline{q}(\theta) = \overline{q}_{-i}(\theta), \quad \forall \theta \in \mathring{\Omega}_i. \tag{3.8}$$

If instead $\mathring{\Omega}_i = \emptyset$,

$$\begin{cases} \overline{t}_i(\overline{q}(\overline{\theta})) \geq 0 & \text{if } i \in \mathcal{A}, \\ \overline{t}_i(\overline{q}(\underline{\theta})) \geq 0 & \text{if } i \in \mathcal{B}. \end{cases} \tag{3.9}$$

Proposition 1 informs us that for any type for which $\mathcal{V}_{iq}(\theta, \overline{q}(\theta))[\overline{q}_{-i}] = 0$, principal i finds it optimal not to influence the agent's choice and thus $\overline{q}(\theta) = \overline{q}_{-i}(\theta)$ for such a type. The optimal transfer t_i which implements $(\overline{U}, \overline{q})$ above will have the property that $\overline{t}_i(\overline{q}(\theta)) = 0$ for all such θ. For these types, we say that principal i is *inactive* and (3.1) is binding over an interval Ω_i with non-empty interior. Instead, for any θ for which principal i is active, $\mathcal{V}_{iq}(\theta, \overline{q}(\theta))[\overline{q}_{-i}] \neq 0$, we have $\overline{U}(\theta) > \overline{U}_{-i}(\theta)$. Note principal i can be active and offer a positive payment to the agent even when (3.1) is binding if it arises at extreme points.

[17] The case $s_i < 0$ can be treated similarly.

Proposition 1 characterizes best-responses to any incentive compatible allocation $(\overline{U}_{-i}, \overline{q}_{-i})$. Its conclusions are particularly striking when we consider best-responses at equilibrium. Principal i will not induce a deviation away from such an equilibrium allocation $\overline{q}$, whenever $\mathcal{V}_{iq}(\theta, \overline{q}(\theta))[\overline{q}_{-i}] = 0$ where the virtual surplus is now relative to this equilibrium allocation. This condition can be written as

$$\partial\overline{co}(S_i)(\overline{q}(\theta)) \leq \frac{F(\theta)}{f(\theta)} \text{ for } i \in \mathcal{A} \text{ and } \partial\overline{co}(S_i)(\overline{q}(\theta)) \geq \frac{F(\theta)-1}{f(\theta)} \text{ for } i \in \mathcal{B}.$$

If principal i does not influence the agent in equilibrium at θ, then $\overline{q}(\theta) = \overline{q}_{-i}(\theta)$ and thus $\mathcal{V}_{iq}(\theta, \overline{q}(\theta))[\overline{q}] = 0$. If, at equilibrium, principal i chooses to influence the agent, it must instead be that the reverse of the inequalities above hold. In such a case, $\mathcal{V}_i(\theta, \overline{q}(\theta))[\overline{q}_{-i}]$ is independent of $\overline{q}_{-i}$, and we may again consider virtual surplus relative to $\overline{q}$ instead of $\overline{q}_{-i}$. We may therefore use Proposition 1 to characterize equilibrium behavior where the virtual surplus is computed relative to the equilibrium allocation itself; for $\overline{q}(\theta)$ interior, the virtual surplus is

$$\text{(3.10)} \quad \overline{\mathcal{V}}_i(\theta, \overline{q}(\theta))[\overline{q}] = S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{\partial\overline{co}(S_i)(\overline{q}(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q \text{ a.e..}$$

4. EQUILIBRIA: NECESSARY CONDITIONS

Our menu auction game is an *aggregative game* since the agent's choice depends on the aggregate payment T he receives, and principal i's preferences over strategy profiles can be reduced to preferences over her own tariff t_i and this aggregate payment T. Although the common agency game has infinite-dimensional strategies and incomplete information, it also has the convenient property that it is quasi-linear in strategies (i.e., payoffs are linear in payments). This allows us to aggregate the best response conditions given in Proposition 1. In the present context, the corresponding necessary conditions are obtained simply noting that $\overline{q}$ must solve (3.7) for each principal i. Hence, $\overline{q}$ must also maximize the sum of the objectives from these individual programs:

$$\text{(4.1)} \quad \overline{q}(\theta) \in \arg\max_{q \in \mathcal{Q}} \; S(\theta, q) + \overline{\mathcal{V}}(\theta, q)[\overline{q}] + (n-1)\left(S(\theta, q) + \overline{T}(q)\right),$$

where $\overline{T}$ implements $\overline{q}$ and $\overline{\mathcal{V}}(\theta, q)[\overline{q}] \equiv \sum_{i=1}^n \overline{\mathcal{V}}_i(\theta, q)[\overline{q}]$ is the aggregate virtual preferences of the principals relative to the equilibrium allocation itself.[18]

Because $\overline{T}$ appears in the objective in (4.1) and it must also implement $\overline{q}$, this necessary condition contains a fixed point. For a given equilibrium aggregate $\overline{T}$, there exists a $\overline{q}$, which in turn must be a solution to the program in (4.1). As we will demonstrate, there

[18] The second line follows from $\sum_{i=1}^n \overline{T}_{-i}(q) = \sum_{i=1}^n (\overline{T}(q) - \overline{t}_i(q)) = (n-1)\overline{T}(q)$.

might be an infinite number of solutions (equilibria) to this self-referencing program. A first theoretical contribution of this paper is to provide necessary conditions satisfied by all such allocations.

THEOREM 1 *Any equilibrium allocation $\overline{q}$ must satisfy the necessary conditions*[19]

$$\overline{q}(\theta) \in \arg\max_{q \in \overline{Q}} \; S(\theta, q) + \overline{\mathcal{V}}(\theta, q)[\overline{q}], \quad \forall \theta \in \Theta \tag{4.2}$$

where $\overline{Q} = \overline{q}(\Theta) \subseteq \mathcal{Q}$ is the equilibrium range.

Condition (4.2) represents a simplified, pointwise program that embeds the strategic interactions of the principals. Indeed, the fixed-point problem (4.1) has been transformed into what is apparently a simpler maximization problem. The comparison of (4.1) and (4.2) shows in fact a remarkable simplification. The extra term $(n-1)\left(S(\theta, q) + \overline{T}(q)\right)$ that corresponds to $n-1$ times the agent's payoff has now disappeared in the final formulation (4.2). Intuitively, $\overline{q}(\theta)$ is also a maximizer for this last term since it has to be the agent's equilibrium choice. Although no assumption on differentiability of the aggregate tariff $\overline{T}(q)$ is made in the first place, an *Envelope Condition* can be used to simplify the optimality requirement.

Everything happens is as if a *surrogate representative of the principals*, whose decisions reflect their non-cooperative behavior, is now optimizing on their behalf an objective function, namely $S(\theta, q) + \overline{\mathcal{V}}(\theta, q)[\overline{q}]$, which conflates the various influences of the principals. At any type θ, this surrogate principal should prefer to choose the equilibrium action $\overline{q}(\theta)$ rather than the action that would have been chosen by another type. This explains why in the maximand of (4.2), the maximization domain is over all possible actions that lie in the equilibrium range $\overline{Q}$. This maximization thus brings a set of incentive constraints for the surrogate principal that require a careful investigation provided in Theorem 4 below. Notice that the stand-alone action may lie in the equilibrium range $\overline{Q}$.

In the surrogate's objective, each principal's surplus function $S_i(q)$ is now replaced by its virtual surplus $\mathcal{V}_i(\theta, q)[\overline{q}]$ relative to the equilibrium allocation itself. Therefore, this maximization problem still contains a fixed-point requirement because the aggregate virtual surplus is relative to the equilibrium allocation itself. This aggregate virtual surplus reflects in which direction principals would like collectively to push the agent's action. Depending on whether the marginal aggregate $\overline{\mathcal{V}}_q(\theta, \overline{q}(\theta))[\overline{q}]$ at that equilibrium allocation is positive or negative, the overall influence of principals pushes action up (resp.

[19] From a technical point of view, the fact that $\overline{\mathcal{V}}(\theta, q)$ depends on the equilibrium allocation implies a careful use of Proposition 1. There, we showed that the best-response allocation $\overline{q}$ to a non-increasing allocation $\overline{q}_{-i}$ was itself non-increasing. When using this Proposition at equilibrium, and taking into account a fixed-point requirement, we thus have to make this monotonicity requirement explicit.

down) with respect to the stand-alone benchmark $\overline{q}_0(\theta)$.

5. MAXIMAL EQUILIBRIA

To make progresses on the characterization of the equilibrium set, we first observe that a solution to Condition (4.2) stands out for special consideration. Define thus the *maximal allocation* $q^m(\theta)$ as a solution to

$$q^m(\theta) \in \arg\max_{q\in\mathcal{Q}} \; S(\theta,q) + \mathcal{V}(\theta,q)[q^m]. \tag{5.1}$$

The allocation $q^m(\theta)$ in (5.1) is said to be *maximal* because, in contrast with the more general Condition (4.2), the optimization domain $\mathcal{Q}$ is now left unrestricted. Note that q^m as defined is a fixed point; it is immediate to show that such a fixed point always exists.[20] While there may be more than one solution in our most general setting, we now impose concavity and differentiability on our principals' and agent's preferences: preferences are continuously differentiable, S_i is concave for $i = 1,\ldots,n$, and S_0 is strictly concave. Under these minimal restrictions which are satisfied in all of our applications, a unique solution exists and we can speak of *the* maximal equilibrium allocation.

THEOREM 2 *Suppose that $S_i(q)$ (resp. $S_0(q)$) is (resp. strictly) concave and differentiable for all $i \in \mathcal{N}$. Any interior maximal allocation is uniquely defined as*

$$S_0'(q^m(\theta)) + \sum_{i\in\mathcal{A}} \max\left\{S_i'(q^m(\theta)) - \frac{F(\theta)}{f(\theta)}; 0\right\} + \sum_{i\in\mathcal{B}} \min\left\{S_i'(q^m(\theta)) + \frac{1-F(\theta)}{f(\theta)}; 0\right\} = \theta. \tag{5.2}$$

Furthermore, $q^m(\theta)$ so defined is non-increasing and continuous.

Section 8 below highlights how the conditions of Theorem 2 are relatively weak and readily satisfied in various economic settings.

For future reference, we define $\Omega_i^m = \left\{\theta \in \Theta \text{ s.t. } \frac{F(\theta)-1}{f(\theta)} \le S_i'(q^m(\theta)) \le \frac{F(\theta)}{f(\theta)}\right\}$. It is the subset of types for which principal i has no influence at the maximal allocation. In general, the influence area of this principal is thus determined by a joint condition on her own

[20] Consider the correspondence

$$\Psi(x) = \arg\max_{q\in\mathcal{Q}} \; S(\theta,q) + \sum_{i=1}^n S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{\partial\overline{co}(S_i)(x), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q.$$

Because $\mathcal{Q}$ is compact and the above maximand is continuous in (q,x), we can apply Berge's Theorem of the Maximum which states that the correspondence Ψ is upper hemi-continuous, non-empty and compact. Because $\sum_{i=0}^n S_i(q)$ is concave, Ψ is convex-valued. By Kakutani's Fixed-Point Theorem, Ψ admits a fixed-point $q^m(\theta)$.

preferences and the equilibrium action. This difficulty renders the characterization of influence areas rather difficult. Section 8 nevertheless shows that this difficulty can be overcome in structured economic environments.

5.1. *Sufficient Conditions*

Theorem 2 highlights conditions that ensure existence of a maximal allocation. To complete our analysis, Theorem 3 below now provides sufficient conditions for existence of a maximal equilibrium. The important step on that route is to construct equilibrium tariffs. To this end, we must define the assignment correspondence $\vartheta^m(q) = \{\theta \in \Theta | q = q^m(\theta)\}$. Under the assumptions of Theorem 2, Item 2., this correspondence is single-valued and continuous on $\mathcal{Q}^m \subseteq \mathcal{Q}$ since q^m is decreasing and continuous.[21]

THEOREM 3 *Suppose that $S_i(q)$ (resp. $S_0(q)$) is (resp. strictly) concave and differentiable for all $i \in \mathcal{N}$. The maximal allocation $q^m(\theta)$ satisfying (5.2) is an equilibrium allocation induced by the following equilibrium maximal tariffs t_i^m:*

- *When $\mathring{\Omega}_i^m \neq \emptyset$,*

$$t_i^m(q) = \int_{\hat{q}_i}^{q} \mathcal{V}_{iq}^m(\vartheta^m(x), x)dx \quad \forall q \in \mathcal{Q}^m, \quad \forall i \in \mathcal{N} \tag{5.3}$$

  where $\hat{q}_i \in q^m(\Omega_i^m)$ is arbitrary.

- *When $\mathring{\Omega}_i^m = \emptyset$,*

$$t_i^m(q) = \begin{cases} t_i^m(q^m(\overline{\theta})) + \int_{q^m(\overline{\theta})}^{q} \mathcal{V}_{iq}^m(\vartheta^m(x), x)dx & \forall i \in \mathcal{A}, \\ t_i^m(q^m(\underline{\theta})) + \int_{q^m(\underline{\theta})}^{q} \mathcal{V}_{iq}^m(\vartheta^m(x), x)dx & \forall i \in \mathcal{B}. \end{cases} \tag{5.4}$$

where $t_i^m(q^m(\overline{\theta})) \geq 0$ for $i \in \mathcal{A}$ and $t_i^m(q^m(\underline{\theta})) \geq 0$ for $i \in \mathcal{B}$.

The expressions of the tariffs in (5.3)-(5.4) are reminiscent of the *truthful tariffs* proposed by Bernheim and Whinston (1986) in their analysis of delegated common agency games under complete information. Remember that, in that setting, truthful tariffs are actually of the form

$$t_i(q) = \max\{S_i(q) - C_i; 0\}$$

[21] The correspondence would not be single-valued whenever $q^m(\theta)$ is constant. Thanks to $MHRC$, this scenario might only arise when $q^m(\theta)$ lies on the boundaries of $\mathcal{Q}$. This possibility of a corner solution is ruled out in the sequel by making extra assumptions (explicit in the economic examples under scrutiny) and we shall thus focus on interior solutions. Yet, modulo changes in the expression of the virtual surplus functions that handle that possibility of a corner solution, these technicalities can easily be dealt with. (See Lemma A.1 for details.)

for some constants C_i.[22] When positive, these schedules reflect the preferences of principals between alternatives. Under asymmetric information, informational distortions reduce (resp. increase) the marginal contribution of a principal $i \in \mathcal{A}$ (resp. $i \in \mathcal{B}$) below (resp. above) her marginal valuation. The *maximal contribution schedules* (5.3)-(5.4) reflect the virtual surplus of principals between alternatives.

6. DISCONTINUOUS EQUILIBRIA

We now demonstrate that there are (candidate) equilibrium allocations which satisfy the necessary conditions (4.2) but do not satisfy (5.1) when $\overline{Q} \subsetneq \mathcal{Q}^m$, and so Condition (5.1) implicitly refines the equilibrium set. To investigate the possibility of such equilibria and get a clear characterization, we now adopt the set of assumptions made in Theorem 2. Recall that under these circumstances, an interior maximal allocation exists, uniquely defined as (5.2), and is continuous and decreasing.

The restriction $\overline{Q} \subsetneq \mathcal{Q}^m$ on the equilibrium range of actions of course only matters when binding. When not so, the equilibrium action is necessarily the maximal allocation. Candidate equilibrium allocations are thus identical to the maximal allocation on $\mathring{\overline{Q}}$. Accordingly, we now rewrite (4.2) as

$$(6.1) \qquad \overline{q}(\theta) \in \arg\max_{q \in \overline{Q}} \; S(\theta, q) + \mathcal{V}^m(\theta, q), \quad \forall \theta \in \Theta.$$

Instead, Condition (4.2) is a priori compatible with the existence of a countable number of downward discontinuities in the action profile when the range of equilibrium values $\overline{Q}$ is not connected. As already noticed in the intrinsic common agency scenario by Martimort, Semenov and Stole (2018), the characterization of equilibrium allocations by means of the surrogate principal's incentive constraints (4.2) bears strong similarities with the characterization of implementable allocations found in the mechanism design literature on delegation as in Holmström (1984), Melumad and Shibano (1991), Martimort and Semenov (2006), Alonso and Matouschek (2008) and Amador and Bagwell (2013). Borrowing techniques that were developed in the aforementioned literature provides a sharp requirement for all equilibrium allocations.

THEOREM 4 *Suppose that $S_i(q)$ (resp. $S_0(q)$) is (resp. strictly) concave and differentiable for all $i \in \mathcal{N}$.*

1. *Any candidate equilibrium allocation $\overline{q}$ satisfying (6.1) is non-increasing.*

[22]The possible values of those constants are found at equilibrium from the binding participation constraints of the agent's in each principal's best-response problem. In contrast with the case of incomplete information analyzed here, those participation constraints are always binding.

2. At any point of differentiability, the following condition holds:

$$\dot{\overline{q}}(\theta)\left(S_0'(\overline{q}(\theta)) - \theta + \mathcal{V}_q^m(\theta, \overline{q}(\theta))\right) = 0. \tag{6.2}$$

3. At any isolated point of discontinuity, $\theta_0 \in (\underline{\theta}, \overline{\theta})$, bunching arises on both sides of θ_0 with

$$\overline{q}(\theta) = \overline{q}^m(\theta_1) \quad \forall \theta \in [\theta_1, \theta_0) \text{ and } \overline{q}(\theta) = \overline{q}^m(\theta_2) \quad \forall \theta \in (\theta_0, \theta_2]$$

for some θ_1 and θ_2 such that $\theta_1 < \theta_0 < \theta_2$.[23] *The surrogate surplus is continuous at θ_0:*

$$[S_0(\theta_0, q) + \mathcal{V}^m(\theta_0, q)]_{\overline{q}^m(\theta_2)}^{\overline{q}^m(\theta_1)} = 0. \tag{6.3}$$

To prove that allocations that satisfy the above necessary properties are actually part of an equilibrium, an important step, much like in Theorem 1 above, consists in constructing tariffs that implement these allocations. The difficulty is that, the delegated common agency game under consideration being an aggregate game which is not bijective, not all discontinuities can be given consideration. Of course, an easy way to handle discontinuity would be to have principals coordinate on large punishments if the agent would choose actions in any discontinuity gap. This coordination is feasible under intrinsic common agency as shown in Martimort, Semenov and Stole (2018). In a delegated common agency game, the so constructed tariffs must remain non-negative for all principals. This requirement constrains what sort of discontinuities are sustainable in equilibrium. Our next theorem provides one such construction and exhibits an important class of equilibria whose tariffs are simply truncated version of maximal contribution schedules. For simplicity, we consider the case where all principals have congruent interests.

THEOREM 5 *Suppose that, $\mathcal{A} = \mathcal{N}$, and that $S_i(q)$ (resp. $S_0(q)$) is (resp. strictly) concave and differentiable for all $i \in \mathcal{N}$. Consider a triplet $(\theta_0, \theta_1, \theta_2) \in \Theta^3$ with $\theta_1 < \theta_0 < \theta_2$,*

$$[S_0(\theta_0, q) + \mathcal{V}^m(\theta_0, q)]_{q^m(\theta_2)}^{q^m(\theta_1)} = 0, \tag{6.4}$$

$$\mathcal{V}_{iq}^m(\theta, q^m(\theta) > 0 \quad \forall \theta \in [\theta_1, \theta_2], \forall i \in \mathcal{N}, \tag{6.5}$$

and

$$\frac{F(\theta_0)}{f(\theta_0)} = \frac{1}{q^m(\theta_1) - q^m(\theta_2)} \int_{q^m(\theta_2)}^{q^m(\theta_1)} \frac{F(\vartheta^m(x))}{f(\vartheta^m(x))} dx. \tag{6.6}$$

[23] $\overline{q}$ can be made either right-continuous ($\overline{q}(\theta_0) = \overline{q}^m(\theta_2)$) or left-continuous ($\overline{q}(\theta_0) = \overline{q}^m(\theta_1)$) with, of course, no consequences on payoffs for any player.

The allocation $\overline{q}$ *defined as*

$$(6.7)\qquad \overline{q}(\theta)=\begin{cases} q^m(\theta) & \text{if } \theta\in[\underline{\theta},\theta_1]\bigcup[\theta_2,\overline{\theta}],\\ q^m(\theta_1) & \text{if } \theta\in[\theta_1,\theta_0],\\ q^m(\theta_2) & \text{if } \theta\in(\theta_0,\overline{\theta}].\end{cases}$$

is an equilibrium allocation with θ_0 *as a downward jump discontinuity. It is implemented by means of truncated maximal tariffs defined as*

$$(6.8)\qquad \bar{t}_i(q)=\begin{cases} t_i^m(q) & \text{if } q\in\mathcal{Q}^m/\,[q^m(\theta_2),q^m(\theta_1)],\\ 0 & \text{otherwise.}\end{cases}$$

Collective Action (Continued). To exhibit such discontinuous equilibria, consider the simple case where S_0 is quadratic and, more specifically $S_0(q)=\lambda q-\frac{q^2}{2}$ where λ is a non-negative parameter which is large enough such that output remains positive under all circumstances below. We also assume that θ is uniformly distributed over $[0,1]$. For simplicity, we also take $n=2$ and $s_1=s_2>0$. Inserting into (8.6), the maximal allocation is shown to satisfy

$$(6.9)\qquad q^m(\theta)=\lambda-\theta+2\max\{s-\theta;0\}.$$

Fix now $\theta_0\in[0,1]$ and take $\theta_2=\theta_0+\frac{\Delta}{2}<s$ and $\theta_1=\theta_0-\frac{\Delta}{2}>0$. It is straightforward to check that conditions (6.4), (6.5) and (6.6) hold altogether for any such Δ. In other words, there are a continuum of discontinuous equilibria. The discontinuity is always at θ_0 and the downward jump discontinuity is from $q^m(\theta_1)$ to $q^m(\theta_2)$. ■

Equilibrium Selection. Because equilibrium tariffs (6.8) are now truncated versions of maximal contribution schedules, the agent's possible choices are *de facto* restricted. The next proposition, whose proof is thus immediate, provides thus a strong reason to focus on maximal allocations nevertheless.

Proposition 2 *Compared to the class of discontinuous equilibria characterized in Theorem 5, the agent's payoff is greater in the maximal equilibrium.*

More generally, when the maximal allocation is unique (as in the concave-differentiable setting), we have an immediate corollary to Theorem 2:

Corollary 1 *Let* $\overline{q}(\cdot)$ *be an equilibrium allocation that is fully separating over the open interval* (θ_1,θ_2)*. Then*

$$\overline{q}(\theta)=q^m(\theta)\ \text{ for all } \theta\in(\theta_1,\theta_2).$$

Thus even discontinuous equilibrium allocations correspond to the maximal allocation over regions where there is full separation, suggesting the economic forces operating in the maximal equilibrium are more universally relevant.

7. ALTERNATIVE SCENARIOS

This section discusses alternative scenarios for the collective actions of principals. It shows how our results might be modified.

7.1. *Cooperative Principals*

This first benchmark allows us to better understand the nature of the distortions induced by the principals' non-cooperative behavior. Suppose that there is a single principal who has preferences given by $\sum_{i=1}^{n} S_i(q)$. Alternatively, one can think of a cooperative formed with all principals designing their compensation schedule to maximize their collective surplus. In this case, the agent's sole outside option is the stand-alone solution. To fix ideas, suppose that $S_i(q)$ is strictly concave, differentiable for all $i \in \mathcal{N}$, and that all principals enjoy the good ($\mathcal{A} = \mathcal{N}$). The virtual surplus of that cooperative principal relative to the agent's stand-alone allocation (assuming its interiority) is simply expressed as

$$\mathcal{V}(\theta, q)[\overline{q}_0] = \sum_{i=1}^{n} S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{\sum_{i=1}^{n} S_i'(\overline{q}_0(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q. \tag{7.1}$$

The optimal cooperative solution then solves

$$q^{coop}(\theta) = \arg\max_{q \in \mathcal{Q}} \; S(\theta, q) + \mathcal{V}(\theta, q)[\overline{q}_0].$$

The comparison of $\mathcal{V}(\theta, q)[\overline{q}_0]$ and $\overline{\mathcal{V}}(\theta, q)[\overline{q}]$ highlights two effects. On the one hand, by acting cooperatively, principals are able to lessen the agent's participation constraint which is now given by the stand-alone action rather than, for each principal, the next-best option absent that principal. This effect tends to make principals more willing to modify the agent's action than in the non-cooperative scenario. On the other hand, when non-cooperating, each principal is now concerned with rent extraction; an effect that appears as a lower impact of information distortions on the right-hand side of (7.1). Depending on whether this principal likes or dislikes the agent's action, this might end up in more or less distortion compared with the cooperative scenario. Section 8 below will show how those different effects interact and will provide more complete comparative statics between the cooperative and the non-cooperative settings.

7.2. *Intrinsic versus Delegated Common Agency*

When the agent must either accept of reject the entire set of the n offers, common agency is *intrinsic*. The set of equilibria for this simpler setting is explored in Martimort, Semenov and Stole (2018). Intrinsic common agency is the appropriate setting if the principals have some control of the agent's choice as in the case of public regulation by different government agencies. When common agency is intrinsic, the principals' activity sets always coincide, so the equilibrium analysis of these games avoids the difficulties in the present paper. Nonetheless, intrinsic common agency with public contracts provides an interesting comparison for the influence games in the current paper, which we discuss in Section 4 below.

Martimort, Semenov and Stole (2018) have studied intrinsic common agency games and showed that all equilibria of such games can also be expressed as optimization problems for a surrogate principal. There are three noticeable differences between the intrinsic and delegate scenarios and these differences significantly complicate the analysis of this paper. First, under intrinsic common agency, all principals consider the same participation constraint for the agent with the latter's sole outside option being now his stand-alone payoff:

$$(7.2) \qquad U(\theta) \geq \overline{U}_0(\theta).$$

Second, and because all principals consider the same set of incentive-feasible allocations, intrinsic common agency games are bijective aggregate games in the vocabulary of Martimort and Stole (2012). Any incentive-compatible allocation can be achieved by a given principal provided that she undoes the aggregate offers made by her rivals, possibly with negative payments. This property aligns the preferences of principals who all achieve the same equilibrium net payoff. Under these circumstances, it is straightforward to demonstrate that the necessary conditions that pertain to a solution to the surrogate principal's problem are also sufficient. Our delegated common agency game is not bijective for the simple reason that a given principal might not be able to undo others' offers when restricted to offer positive payments.

Third, and again because all principals consider the same participation constraint, they all agree on the identity of the worst type. All informational distortions due to their non-cooperative behavior thus go in the same direction. Provided that $S_i(q)$ is concave and differentiable for all $i \in \mathcal{N}$ and $\overline{q}_0(\theta)$ is interior, it can be shown that the aggregate virtual surplus can now be expressed as[24]

$$\mathcal{V}^I(\theta, q) = \sum_{i=1}^{n} S_i(q) - \sum_{i=1}^{n} \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{S_i'(\overline{q}_0(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q. \tag{7.3}$$

In comparison with the cooperative scenario, this expression leads to an overall n-fold informational distortion whose consequences are studied in more details in Martimort, Semenov and Stole (2018). For the sake of the present paper, we may just observe that, when evaluated at the stand-alone action, the marginal aggregate virtual surplus under intrinsic agency is worth

$$\mathcal{V}_q^I(\theta, \overline{q}_0(\theta)) = \sum_{i=1}^{n} \max\left\{S_i'(\overline{q}_0(\theta)) - \frac{F(\theta)}{f(\theta)}; \min\left\{S_i'(\overline{q}_0(\theta)) + \frac{1-F(\theta)}{f(\theta)}; 0\right\}\right\}.$$

while, under delegated common agency, the marginal aggregate virtual surplus evaluated at the same point is

$$\overline{\mathcal{V}}_q(\theta, \overline{q}_0(\theta)) = \sum_{i=1}^{n} \max\left\{S_i'(\overline{q}_0(\theta)) - \frac{F(\theta)}{f(\theta)};\right.$$
$$\left.\min\left\{S_i'(\overline{q}_0(\theta)) + \frac{1-F(\theta)}{f(\theta)}; S_i'(\overline{q}_0(\theta)) - S_i'(\overline{q}(\theta))\right\}\right\}.$$

Whenever the collective action of principals pushes the equilibrium action $\overline{q}(\theta)$ above the stand-alone action $\overline{q}_0(\theta)$, we have by concavity of S_i, $S_i'(\overline{q}_0(\theta)) \geq S_i'(\overline{q}(\theta))$. It immediately follows that $\overline{\mathcal{V}}_q(\theta, \overline{q}_0(\theta)) \geq \mathcal{V}_q^I(\theta, \overline{q}_0(\theta))$ and all principals have more incentives to expand output beyond that stand-alone action under delegated common agency than under intrinsic common agency. The reverse holds when the equilibrium action $\overline{q}(\theta)$ lies below the stand-alone action $\overline{q}_0(\theta)$. Again, Section 8 below will provide more precise comparisons in structured environments.

7.3. *Ex Ante Contracting*

Consider a scenario where the principals and their common agent contract under symmetric but incomplete information at the *ex ante* stage; i.e., before the agent learns his type. With this timing, the agent's *ex post* participation constraint (3.1) is now replaced by

$$\int_{\underline{\theta}}^{\overline{\theta}} U(\theta) f(\theta) d\theta \geq \int_{\underline{\theta}}^{\overline{\theta}} \overline{U}_{-i}(\theta) f(\theta) d\theta. \tag{7.4}$$

[24] The proof is similar to those in Lemma A.1 and are thus omitted.

Of course, this constraint is binding at the best response for each principal. Because contracting takes place *ex ante*, there is no longer any friction coming from asymmetric information and each principal maximizes the bilateral surplus of her relationship with the common agent, shifting surplus within this bilateral coalition with a fee in order to ensure the agent's participation. Virtual surpluses are now replaced by true surplus functions in expressing best responses for each principal. We could further proceed as above and aggregate those best-response conditions to get

$$(7.5)\qquad \overline{q}^{ea}(\theta) \in \arg\max_{q\in\overline{Q}}\; S(\theta,q) + \sum_{i=1}^{n} S_i(q), \quad \forall\theta\in\Theta$$

When the domain of maximization $\overline{Q}$ is left unrestricted, this optimality condition just states that $\overline{q}^{ea}$ should be efficient. This scenario has already been studied by Laussel and Le Breton (1998) who study efficient equilibria. As they show, this efficient allocation is implemented by means of *truthful schedules* which, under *ex ante* contracting takes the form

$$(7.6)\qquad T_i(q) = S_i(q) - C_i, \quad \forall i\in\mathcal{N}$$

where C_i is a constant which is actually principal i's payoff. These constants $(C_i)_{1\leq i\leq n}$ are then obtained from the binding participation constraints (7.4) as the solutions to

$$(7.7)\qquad \max_{q\in\overline{Q}}\; \mathbb{E}_\theta\left(S(\theta,q) + \sum_{i=1}^{n} S_i(q)\right) - C_i$$
$$= \max_{\mathcal{J}\bigcup\{0\}}\left(\max_{q\in\overline{Q}}\; \mathbb{E}_\theta\left(S(\theta,q) + \sum_{i\in\mathcal{J}} S_i(q)\right) - C_i\right), \quad \forall i\in\mathcal{N}.$$

It should be stressed that here also $\overline{Q}$ may be a strict subset of $\mathcal{Q}$ and efficiency may not be reached everywhere. The techniques we develop in Section 6 below could be used here also to describe the whole class of such discontinuous equilibria. The one that maximizes aggregate surplus, of course, has $\overline{Q} = \mathcal{Q}$ and output is first-best.

8. MAXIMAL EQUILIBRIA AT WORK

We now show how our characterization of maximal equilibria helps to derive important insights for structured economic environments that are of much interest for applications.

8.1. *Public Good Games*

EXAMPLE 1 offers a particularly striking example of our general approach. Using (3.4), we first observe that principal i's virtual surplus at the maximal allocation is linear:

$$\mathcal{V}_i^m(\theta, q) = \begin{cases} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}; 0\right\} q & \text{for } i \in \mathcal{A}, \\ \min\left\{s_i + \frac{1-F(\theta)}{f(\theta)}; 0\right\} q & \text{for } i \in \mathcal{B}, \end{cases} \tag{8.1}$$

and the marginal virtual surplus for a given principal does not depend on actions that might be taken in her absence. As a result, the surrogate principal's problem becomes a simple optimization problem. This property is the source of many sharp results in what follows. In particular, the influence area of each principal is now entirely determined by her own preferences.

PROPOSITION 3 *Suppose that principals have linear surplus functions, i.e., $S_i(q) = s_i q$ for all $i \in \mathcal{N}$, and that $C(q)$ is strictly convex with*

$$\sum_{i \in \mathcal{N}} s_i \geq \overline{\theta} + C'(0) + \frac{|\mathcal{A}|}{f(\overline{\theta})}. \tag{8.2}$$

Suppose also that

$$1 \geq f(\overline{\theta}) s_i \quad \text{if } i \in \mathcal{A} \text{ and } 1 \leq -f(\underline{\theta}) s_i \quad \text{if } i \in \mathcal{B}. \tag{8.3}$$

An interior maximal equilibrium exists and is unique.

This maximal allocation q^m solves

$$\sum_{i \in \mathcal{A}} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}; 0\right\} + \sum_{i \in \mathcal{B}} \min\left\{s_i + \frac{1 - F(\theta)}{f(\theta)}; 0\right\} = \theta + C'(q^m(\theta)). \tag{8.4}$$

VIRTUAL LINDAHL-SAMUELSON CONDITIONS. Condition (8.4) is a virtual version of Lindahl-Samuelson conditions. The sum of the principals' marginal virtual surplus balances the agent's marginal cost of producing the public good at a maximal equilibrium allocation. A principal i who enjoys (resp. dislikes) the public good, $i \in \mathcal{A}$ (resp. $i \in \mathcal{B}$), influences the agent with a type $\theta \leq \hat{\theta}_i$ (resp. $\theta \geq \hat{\theta}_i$). Thanks to (8.3), the cut-off $\hat{\theta}_i$ is defined as $s_i = \frac{F(\hat{\theta}_i)}{f(\hat{\theta}_i)}$ for $i \in \mathcal{A}$ (resp. $s_i = \frac{F(\hat{\theta}_i)-1}{f(\hat{\theta}_i)}$ for $i \in \mathcal{B}$).

MAXIMAL CONTRIBUTIONS. Turning to the expression of these payments and using (5.3), the maximal allocation is implemented by means of the following maximal contributions:

$$(8.5)\qquad t_i^m(q) = \begin{cases} \int_{q^m(\hat{\theta}_i)}^q \max\left\{s_i - \frac{F(\vartheta^m(x))}{f(\vartheta^m(x))}; 0\right\} dx & \text{for } i \in \mathcal{A}, \\ \int_{q^m(\hat{\theta}_i)}^q \min\left\{s_i + \frac{1-F(\vartheta^m(x))}{f(\vartheta^m(x))}; 0\right\} dx & \text{for } i \in \mathcal{B} \end{cases}$$

where $\vartheta^m(q)$ is the assignment rule for the maximal allocation. Confirming our earlier findings, it can be readily checked that a principal $i \in \mathcal{A}$ (resp. $i \in \mathcal{B}$) who enjoys (resp. dislikes) the public good wants to increase (resp. decrease) its level, and thus $t_i^{m'}(q) \geq 0$ (resp. $t_i^{m'}(q) \leq 0$) at all $q \in \mathcal{Q}$.

Comparative Statics. An interesting comparative static is to ask how a constant-sum redistribution of the principals' marginal payoffs impacts the maximal equilibrium allocation even though such a redistribution would have no impact on the efficient allocation. To this end, arrange the principals from highest to lowest marginal payoff, $s_1 \geq \cdots \geq s_n$ and denote the payoff vector as $\mathbf{s} = (s_1, \ldots, s_n)$. If principals are not congruent and $\mathcal{A} = \{1, \ldots, j\}$, define $\mathbf{s}_A = (s_1, \ldots, s_j)$ and $s_B = (s_{j+1}, \ldots, s_n)$, allowing us to write with some abuse of notation the payoff vector as $\mathbf{s} = (\mathbf{s}_A, \mathbf{s}_b)$. We have the following proposition.

Proposition 4 *Consider two configurations of principal preferences,* $\mathbf{s} = (\mathbf{s}_A, \mathbf{s}_B)$ *and* $\tilde{\mathbf{s}} = (\tilde{\mathbf{s}}_A, \tilde{\mathbf{s}}_B)$. *If* $\tilde{\mathbf{s}}_A$ *is a mean-preserving spread*[25] *of* $\mathbf{s}_A$ *and* $\tilde{\mathbf{s}}_B = \mathbf{s}_B$, *then the associated maximal allocations in each game have the property that for all* θ

$$\overline{q}^m_{\tilde{\mathbf{s}}}(\theta) \geq \overline{q}^m_{\mathbf{s}}(\theta),$$

with a strict inequality for some positive measure. Similarly, if $\tilde{\mathbf{s}}_A = \mathbf{s}_A$ *and* $\tilde{\mathbf{s}}_B$ *is a mean-preserving spread of* $\mathbf{s}_B$, *then*

$$\overline{q}^m_{\tilde{\mathbf{s}}}(\theta) \leq \overline{q}^m_{\mathbf{s}}(\theta),$$

with a strict inequality for some positive measure.

The result follows from noting that the lefthand side of (8.4) is convex in s_i for $i \in \mathcal{A}$ and concave in s_i for $i \in \mathcal{B}$. Applying Jensen's inequality finishes the proof. Several remarks follow from this simple comparative static.

Cooperative Principals. As a first illustration of Proposition 4, consider the case of cooperating principals in a setting where preferences are congruent, say $s_i > 0$ for all

[25] Given two configurations $\mathbf{s}$ and $\tilde{\mathbf{s}}$ with the same mean (i.e., $\sum_{i=1}^n s_i = \sum_{i=1}^n \tilde{s}_i$), we define the associated discrete distributions on the combined domain $\cup_i s_i \ \cup_j \tilde{s}_j$. If the distribution for $\mathbf{s}$ second-order stochastically dominates the distribution for $\tilde{\mathbf{s}}$, then we say that $\tilde{\mathbf{s}}$ is a mean-preserving spread of $\mathbf{s}$.

$i \in \mathcal{N}$. This is equivalent to one principal having now preferences $\tilde{s}_1 = \sum_{i=1}^{n} s_i > 0$ and the other $n-1$ principals having now preferences $\tilde{s}_j = 0$ (for $j \neq 1$) with contributions as defined in (8.5) identically zero. It follows that $\tilde{\mathbf{s}}$ is more disperse than $\mathbf{s}$. From Proposition 4, we deduce that the Baron and Myerson (1982)'s outcome with principals are cooperating entails a higher allocation in comparison with the noncooperative scenario:

$$q^{coop}(\theta) \geq \overline{q}^m(\theta) \quad \forall \theta \in \Theta.$$

Output is inefficiently low in the noncooperative setting relative to the cooperative Baron and Myerson (1982). We should emphasize that the source of this free-riding problem among principals is asymmetric information. If information were complete or if contracting takes place *ex ante* (which is tantamount to eliminating the inverse-hazard terms from the equation as discussed in Section 7.3), the maximal equilibrium leads to full efficiency. Each principal would offer the marginal truthful tariff $t_i'(q) = s_i$ as in Bernheim and Whinston's (1986). Thus, free riding does not arise in complete-information public good games if principals have the ability to offer nonlinear tariffs to a common agent rather than making direct, one-dimensional contributions to the public good. With asymmetric information, however, each principal has private incentives to distort the agent's output choice to extract additional information rent. Because each principal ignores the negative externality that doing so imposes on others, from a collective viewpoint, the principals inefficiently extract too much rent. The free-riding problem present in our setting more closely fits the narrative of a *"tragedy of the commons"* in which each principal over-harvests a common resource – here the agent's information rent.[26]

Non-Neutrality. As a second illustration of Proposition 4, consider now the case of two principals. A unit tax on principal 1's use of the public good that is exactly offset by a unit subsidy on principal 2's use could have a real impact on the equilibrium allocation of public goods if this policy changes the set of active principals for some types.[27] The fact that mean-preserving variations in the principals' preferences can have real impacts in the final allocation is reminiscent of findings in the public finance literature on voluntary contribution games (see, e.g., Bergstrom, Blume and Varian, 1986). This literature, which has focused on complete information games in which players' strategies are scalar contributions (as opposed to nonlinear schedules under asymmetric information as considered here), demonstrates that neutrality arises in simple public goods games precisely

[26]Similar findings arise in the private common agency settings analyzed in Stole (1991), Martimort (1992) and Martimort and Stole (2009) where the former papers consider intrinsic common agency games while the later discusses also the scenario of delegated common agency. When different principals control different activities undertaken by their agent which are complements, each of them extracts too much of the agent's information rent; inducing excessively low levels of activities.

[27]This is not the case in models of intrinsic common agency, as shown in Martimort and Stole (2012), because in such games all principals are active on the same type set and the allocation is unchanged by mean-preserving variations in the principals' preferences.

when the set of contributors is unaffected by a variation in preferences or incomes. When the set of contributors changes, however, the level of public good provision is typically altered. Similarly, we find in our richer asymmetric-information setting that the key source of non-neutrality is that an underlying variation can impact the set of active principals.

8.2. *Collective Action*

The analysis for EXAMPLE 2 follows *mutatis mutandis* from Proposition 3. The only change being that the maximal allocation $q^m(\theta)$, when interior, now solves

$$S_0'(q^m(\theta)) + \sum_{i\in\mathcal{A}} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}; 0\right\} + \sum_{i\in\mathcal{B}} \min\left\{s_i + \frac{1-F(\theta)}{f(\theta)}; 0\right\} = \theta.^{28} \quad (8.6)$$

Several sharp economic insights emerge for this specific political economy context.

THE FREE-RIDING PROBLEM. Consider the case in which n symmetric principals have the same marginal benefit $s_1 = S/n$ so that the aggregate principal benefit Sq taken as a group is fixed independent of n. Using (8.5), it follows that an increase in the number of principals, holding S fixed, reduces collective action. Specifically, we have

$$S_0'(q_n^m(\theta)) + n\max\left\{\frac{S}{n} - \frac{F(\theta)}{f(\theta)}; 0\right\} = \theta.$$

For $n \to \infty$, $q_n^m(\theta)$ now converges pointwise towards the stand-alone action $\overline{q}_0(\theta)$. This asymptotic inefficiency result is reminiscent of the asymptotic inefficiency result found in public good games by Mailath and Postlewaite (1990) but its source is quite different. In Mailath and Postlewaite (1990)'s setting, the agent's cost function is common knowledge, the agent's decision is binary, and each contributing principal has private information about his own willingness to pay. Their result follows because the probability that any contributing pricinpal is pivotal goes to zero as the number of players increases. In contrast, in our setting inefficiency arises because each (uninformed) principal attempts to extract the (privately informed) agent's marginal rent, ignoring the externality she exerts on others when doing so.

HETEROGENEITY. In the context of group formation, the finding in Proposition 4 formalizes the ideas put forward by Olson (1965) and Stigler (1974) that a group is more likely to be influential if members' preferences are heterogeneous (e.g., a combination of small and large stakeholders, rather than a group of equal stakeholders). This idea has also been formalized in a simple setting of binary actions and preferences by Le Breton and Salanié (2003). The present paper shows that this result remains prominent in a richer setting.

[28] Remember that, for the sake of EXAMPLE 2, we assumed that S_0 is strictly concave, differentiable and that the stand-alone action $\overline{q}_0(\theta)$ satisfies (2.4).

Crowding-Out. Another political effect noted by Olson (1965) is that an increase in the stake of one interest group member raises that person's contribution, possibly lowers the contribution of others, but on net raises the total contribution (i.e., crowding out may arise, but it is never complete). We can find a similar result in the case of public good provision where the increase in stake is modeled by an increase in s_i, and we can ask what happens to the maximal equilibrium allocation (and the marginal transfers of all principals) in this case.

Corollary 2 *Suppose that preferences are congruent, $\mathcal{A} = \mathcal{N}$. Consider two configurations of the principals' preferences, $\mathbf{s}$ and $\tilde{\mathbf{s}}$, in which $\tilde{s}_i = s_i + u_i,\ u_i > 0$, but $\tilde{s}_j = s_j$ for $j \neq i$. Then the associated maximal equilibrium allocations satisfy*

$$q^m_{\tilde{\mathbf{s}}}(\theta) \geq q^m_{\mathbf{s}}(\theta) \quad \forall \theta \in \Theta$$

with strict inequality for some positive measure of types.

Furthermore, both the marginal aggregate payment function and the marginal payment function of principal i weakly increase over the set of equilibrium choices (and strictly so for a subset of outputs), while the marginal payment functions of the other principals, $j \neq i$, weakly decrease over the set of equilibrium choices (and strictly so for a subset of outputs). Crowding-out is less than perfect.

Intensive and Extensive Margins. To unveil the nature of distortions in delegated common agency games, we consider now the simple case $n = 2$ with both principals being congruent, with $s_1 > s_2 > 0$. The optimality condition (8.6) now rewrites as

$$(8.7) \qquad S_0'(q^m(\theta)) + \sum_{i=1}^{2} \max\left\{ s_i - \frac{F(\theta)}{f(\theta)}; 0 \right\} = \theta.$$

Suppose also that $1 \geq f(\overline{\theta}) s_1$. Together with $MHRC$, this condition implies that there always exists a unique interior solution $\hat{\theta}_i$ to

$$s_i = \frac{F(\hat{\theta}_i)}{f(\hat{\theta}_i)} \quad i = 1, 2.$$

Observe that $\hat{\theta}_1 > \hat{\theta}_2$, so that the stronger principal 1 has a greater activity set than the weaker principal 2. (i.e., $\left[\underline{\theta}, \hat{\theta}_2\right] \subset \left[\underline{\theta}, \hat{\theta}_1\right]$).

The optimality condition (8.7) clearly shows how distortions manifest themselves along two dimensions. First, because each active principal contributes less than her marginal valuation, inefficient provision arises at the intensive margin. The equilibrium action is lower than the cooperative solution and, eventually, features the same two-fold distortion that is present in intrinsic common-agency games. It is the case when both principals are active, i.e., for $\theta \in \left[\underline{\theta}, \hat{\theta}_2\right]$. A second distortion, novel to delegated common agency games, emerges from limited participation by the weaker principal 2. The agent's action is now also distorted at the extensive margin.

Exclusive Contracting. Another interpretation of the limited participation that may arise under asymmetric information is that some form of exclusive contracting emerges endogenously even if exclusivity clauses cannot be enforced at the outset. This is so even if both principals would otherwise have contracted with the agent under complete information. This finding is reminiscent of an important insight developed by Bernheim and Whinston (1998) in their study of vertical relationships between manufacturers and retailers. They showed that exclusive dealing in marketing practices arises when the agency costs of a common representation are too large compared with those under exclusive dealing. There is, however, an important difference between their result and ours. They assume that the possibility of exclusive representation arises *ex ante*, i.e., before the realization of uncertainty. Although their general contracting model is thus consistent with hidden actions or hidden information, it cannot account for the possibility of exclusivity arising for some realizations of shocks and not for others. In this regard, our model, where contracting takes place *ex post*, i.e., once the agent is already informed, generates richer patterns of behavior.[29]

8.3. *Pork Barrel Politics*

As argued above, Example 3 is an instance where principals have conflicting interests. The marginal virtual surplus for principals 1 and 2 have actually opposite signs, namely

$$\mathcal{V}_{1q}(\theta, q) = \max\left\{b - \theta - \frac{1}{2}; 0\right\} \geq 0 \geq \min\left\{-b - \theta + \frac{1}{2}; 0\right\} = \mathcal{V}_{2q}(\theta, q).$$

For simplicity, suppose that $b > 1$, the optimality condition (8.6) then yields

$$(8.8) \qquad q^m(\theta) = \begin{cases} 1 & \text{if } \theta \in \left[-\frac{1}{2}, -\frac{1}{6}\right], \\ \frac{1}{2} - 3\theta & \theta \in \left[-\frac{1}{6}, \frac{1}{6}, \right], \\ 0 & \text{if } \theta \in \left[\frac{1}{6}, \frac{1}{2}\right]. \end{cases}$$

[29]Calzolari and Denicolo (2015) develop a theory of exclusive dealing in manufacturer-retailer relationships based on the premise that, under asymmetric information, a dominant firm may exclude less efficient upstream competitors. One key difference with our framework is that their model has competing principals contracting on different outputs available to a common retailer. In other words, their model is a model of *private common agency* while ours has principals contracting on the same variable (public common agency).

While the complete information benchmark offers a rather balanced distribution of budget across groups, the maximal equilibrium above is more sensitive to the decision-maker's preferences and may end up in extreme allocations with one group getting the entire budget while the other obtains nothing. In other words, there is more polarization at that equilibrium than under complete information. The intuition is straightforward. Each principal wants to influence types who are more inclined to grant favors while, for incentive compatibility reasons, she also eschews contributions to types less willing to do so.

Note that these allocations lie on the boundaries of the feasible set for the most extreme types and, as such, fail to be strictly monotonic. We may nevertheless still define an assignment function that applies on the interior of the feasible set as

$$\vartheta^m(q) = \frac{1}{6} - \frac{q}{3} \quad \forall q \in [0,1]\,. \tag{8.9}$$

Using the general formula (8.5) yields then the expression of the maximal contribution of each principal respectively as

$$t_1^m(q) = \int_0^q \max\left\{b - \frac{1}{2} - \vartheta^m(\tilde{q}), 0\right\} d\tilde{q} = \left(b - \frac{2}{3}\right) q + \frac{q^2}{6}$$

and

$$t_2^m(q) = \int_1^q \min\left\{-b + \frac{1}{2} - \vartheta^m(\tilde{q}), 0\right\} d\tilde{q} = t_1^m(1-q).$$

With such maximal contributions, types whose choice lies on the boundaries of the feasible set receive payments from only one principal. For instance, if $\theta \in \left[-\frac{1}{2}, -\frac{1}{6}\right]$, the agent only receives a positive payment from principal 1.

It is interesting to investigate whether *ex post* contracting makes head-to-head competition for the agent's services fiercer than the scenario where principals have more commitment ability and can design contribution schedules *ex ante*, i.e, before the agent learns his type, and the efficient allocation $q^{ea}(\theta) = q^{fb}(\theta)$ is implemented by means of truthful tariffs of the form (7.6). To this end, we compare the agent's expected payoff under both scenarios.

PROPOSITION 5 *Suppose that $b > 1$ and θ is uniformly distributed on $\Theta = \left[-\frac{1}{2}, \frac{1}{2}\right]$, then the agent's expected payoff is less in the maximal equilibrium reached under ex post contracting than in the truthful equilibrium achieved under ex ante contracting.*

To understand this result, it is useful to decompose the impact of moving from *ex ante* contracting to *ex post* contracting first on the agent's gross surplus $S_0(\theta, q) = \left(\frac{1}{2} - \theta\right) q - \frac{q^2}{2}$ and, second on his payments under both scenarios. In terms of gross surplus, since $q^m(\theta)$

fluctuates more than $q^{ea}(\theta)$ and the gross surplus function is strictly concave, the expected gross surplus diminishes with *ex post* contracting. In terms of payments, it can be proved that, under *ex ante* contracting,[30] the total contribution received the agent at the (unique) truthful equilibrium is higher than that reached at the maximal equilibrium under *ex post* contracting.[31] Intuitively, under *ex post* contracting, both principals reduce payments to extract the agent's rent; this also makes this scenario less attractive for the agent.

8.4. *Protection for Sale*

Our international trade Example 5 features a case with two congruent principals, the producers in group 1 and 2 respectively, who both want to push for an import tariff/export subsidy. Next proposition summarizes a few findings of our analysis.

Proposition 6 *The import tariff/export subsidy at the maximal equilibrium* $\gamma^m(\theta) = \Gamma(q^m(\theta))$ *reflects the influence of producers.*

1. *Both groups are active when θ is close enough to zero. The import tariff/export subsidy then satisfies*

$$\frac{\gamma^m(\theta)}{p+\gamma^m(\theta)} = \frac{1}{(1+\lambda)\left(2\frac{1-F(\theta)}{f(\theta)} - \theta\right)} \frac{\frac{\mathcal{S}(p+\gamma^m(\theta))}{\mathcal{M}(p+\gamma^m(\theta))}}{\varepsilon(p+\gamma^m(\theta))} \tag{8.10}$$

where $\varepsilon(p) = -\frac{p\mathcal{M}'(p)}{\mathcal{M}(p)}$ is the elasticity of imports, $\mathcal{M}(p) = \mathcal{D}(p) - \mathcal{S}(p)$.

2. *Activity sets are nested with principal 1, the most efficient producers, being always active. If only principal 1 is active, the import tariff/export subsidy satisfies*

$$\frac{\gamma^m(\theta)}{p+\gamma^m(\theta)} = \frac{1}{(1+\lambda)\left(\frac{1-F(\theta)}{f(\theta)} - \theta\right)} \frac{\frac{\mathcal{S}_1(p+\gamma^m(\theta))}{\mathcal{M}(p+\gamma^m(\theta))}}{\varepsilon(p+\gamma^m(\theta))}. \tag{8.11}$$

Strikingly, our model generates simple comparative statics much alike those in Grossman and Helpman (1994). This is the reason why we chose to express optimality conditions in a very similar way. Here also, it is true that, as groups find it easier to self-organize (i.e., λ lower) or as the policy-maker cares less about social welfare (i.e., θ lower), the

[30] See the Proof of Proposition 5 in the Appendix.

[31] Formally, we have

$$\mathbb{E}_\theta\left(t_1^{ea}(q^{ea}(\theta)) + t_2^{ea}(q^{ea}(\theta))\right) = b - \frac{1}{3}.$$

Under the maximal equilibrium reached under *ex post* contracting, we instead have

$$\mathbb{E}_\theta\left(t_1^{m}(q^{m}(\theta)) + t_2^{m}(q^{m}(\theta))\right) = b - \frac{14}{27}.$$

import tariff/export subsidy increases (decreases) to reflect the groups' greater influence. At the extreme, if the policy-maker only cares about lobbying contributions (i.e., $\theta = 0$), he sets an infinite import tariff/export subsidy.[32]

The import tariff (resp. export subsidy) is also greater as influencing groups are producing a greater fraction of imports (resp. exports). Interestingly, a positive productivity shock that would allow producers to expand output would increase their joint influence, up to the point where even low-productivity producers might find it optimal to influence the decision-makers. As a result, greater trade barriers are erected.

In Grossman and Helpman (1994), the structure of interested groups susceptible to influence the decision-maker is given at the outset. Since the influence game takes place under complete information, a Coasian outcome always arises. Whether principals cooperate or not does not change the implemented policy that maximizes the overall payoff or the coalition made of those groups and the policy-maker. This neutrality results fails here. Had the two groups merge, they would induce a cooperative trade policy $\gamma^{coop}(\theta)$ that would satisfy

$$(8.12) \qquad \frac{\gamma^{coop}(\theta)}{p+\gamma^{coop}(\theta)} = \frac{1}{(1+\lambda)\left(\frac{1-F(\theta)}{f(\theta)} - \theta\right)} \frac{\frac{\mathcal{S}(p+\gamma^{coop}(\theta))}{\mathcal{M}(p+\gamma^{coop}(\theta))}}{\varepsilon(p+\gamma^{coop}(\theta))}.$$

The difference with (8.10) is that, because cooperative principals now harvest the policy-maker's rent only once, the choice of the trade instrument is less tilted towards free trade. Cooperative producers are better able to push for an import tariff (resp. export subsidy) when the country is a net importer (resp. exporter).

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[32] This extreme possibility can be easily ruled out if we assume that $\overline{\theta} < 0$; i.e., the policy-maker always gives a positive weight to social welfare.

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ONLINE APPENDIX A: PROOF OF PROPOSITION 1

The proof of Proposition 1 proceeds in three steps. First, using a result in Martimort and Stole (2022), we provide a set of conditions that are necessary and sufficient for the solution to principal i's relaxed program (ignoring the convexity constraint on U). Second, we demonstrate the adjoint equations in these conditions can be further simplified given that the principal's preferences are linear in q. Third, we show that the solution to the relaxed and simplified program is a solution to the original program.

STEP 0: STATEMENT OF THE PROBLEM. For the sake of completeness, we now briefly present Theorem 1 in Martimort and Stole (2022). This latter paper considers general control problems (beyond the class of principal-agent models) in which the state variable, u, is restricted to be an absolutely continuous function on the interval $\Theta = [\underline{\theta}, \overline{\theta}]$. Let $AC(\Theta, \mathbb{R})$ denote the set of such functions. In the present context, the state variable is the agent's information rent as a function of his type, absolute continuity then follows from incentive compatibility.[33] Martimort and Stole (2022) focus attention on problems in which that state variable must satisfy a non-negativity participation constraint constraint:

$$(A.1) \qquad u(\theta) \geq 0 \quad \forall \theta \in \Theta.$$

When the state variable u is both absolutely continuous and non-negative, it is said *admissible*. We are interested in the following pure-state control program:

$$(\mathcal{P}) : \text{Maximize}_{u \in AC(\Theta, \mathbb{R})} \ \mathbb{R} \int_{\underline{\theta}}^{\overline{\theta}} \left(s(\theta, -\dot{u}(\theta)) - u(\theta)\right) f(\theta) d\theta \text{ s.t. (A.1).}$$

STEP 1 below shows how this general formalism applies to our common agency context. Readers already familiar with the work of Jullien (2001) have certainly recognized the well-known framework developed with type-dependent participation constraints. The key novelty in Martimort and Stole (2022) is that similar results are obtained with substantially weaker assumptions on the primitive function s. In particular, $s(\theta, v)$ is not necessarily concave nor continuously differentiable. Accordingly, let $\overline{co}(s)(\theta, v)$ denote the v-concave envelope of $s(\theta, v)$. We denote the sup-differential of $\overline{co}(s)$ as $\partial_v \overline{co}(s)(\theta, v)$[34] Because $\overline{co}(s)$ is concave, it is a.e. differentiable (Rockafellar, 1997, Theorem 25.5.). Henceforth, the correspondence $\partial_v \overline{co}(s)$ is a.e. single-valued.

Theorem 1 in Martimort and Stole (2022) is the main result for this class of problems. Necessary and sufficient conditions are stated in terms of a probability measure which serves to express a complementary slackness condition (A.2) and a first-order optimality condition (A.4).

THEOREM A.1 *(Martimort and Stole, 2022): $\overline{u}$ is a solution to program $(\mathcal{P})$ if and only if $\overline{u}$ is admissible and there exists a probability measure μ defined over the Borel subsets of Θ with*

[33] See Milgrom and Segal (2002).

[34] Remember that $\partial_v \overline{co}(s)(\theta, v) = \{p \text{ s.t. } \overline{co}(s)(\theta, w) \leq \overline{co}(s)(\theta, v) + p(w - v) \quad \forall w\}$.

an associated adjoint function, $\overline{M} : \Theta \to [0,1]$*, defined by* $\overline{M}(\underline{\theta}) = 0$ *and*

$$\overline{M}(\theta) = \int_{[\underline{\theta},\theta)} \mu(d\tilde{\theta}) \quad \text{for } \theta > \underline{\theta},$$

such that the following conditions are satisfied:

$$\text{(A.2)} \qquad \int_{\underline{\theta}}^{\overline{\theta}} \overline{u}(\tilde{\theta})\mu(d\tilde{\theta}) = 0,$$

$$\text{(A.3)} \qquad \overline{co}(s)(\theta, -\dot{\overline{u}}(\theta)) = s(\theta, -\dot{\overline{u}}(\theta)) \text{ for a.e. } \theta \in \Theta,$$

$$\text{(A.4)} \qquad \overline{M}(\theta) \in F(\theta) - f(\theta)\partial_v \overline{co}(s)(\theta, -\dot{\overline{u}}(\theta)) \text{ for a.e. } \theta \in \Theta.$$

Heuristic Proof. Before proceeding, it is useful to give an heuristic proof of this Theorem.[35] First, observe that the cone $u \geq \overline{u}$ with $v \in AC(\Theta, \mathbb{R})$ and $\dot{u}(\theta) = -v(\theta)$ a.e. defines a set of allocations which are admissible deviations. Second, for any such deviation, we necessarily have

$$\int_{\underline{\theta}}^{\overline{\theta}} (u(\theta) - \overline{u}(\theta))d\mu(\theta) \geq 0$$

since μ is positive. Integrating by parts the left-hand side yields

$$0 \leq \left[(u(\theta) - \overline{u}(\theta))\overline{M}(\theta)\right]_{\underline{\theta}}^{\overline{\theta}} + \int_{\underline{\theta}}^{\overline{\theta}} \overline{M}(\theta)(v(\theta) - \overline{v}(\theta))d\theta = u(\overline{\theta}) - \overline{u}(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} M(\theta)(v(\theta) - \overline{v}(\theta))d\theta$$

where the last equality follows from $\overline{M}(\underline{\theta}) = 0$ and $\overline{M}(\overline{\theta}) = 1$. Similarly, another integration by parts yields

$$\int_{\underline{\theta}}^{\overline{\theta}} (u(\theta) - \overline{u}(\theta))f(\theta)d\theta = u(\overline{\theta}) - \overline{u}(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} F(\theta)(v(\theta) - \overline{v}(\theta))d\theta$$

Inserting above yields

$$0 \leq \int_{\underline{\theta}}^{\overline{\theta}} (u(\theta) - \overline{u}(\theta))f(\theta)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} (\overline{M}(\theta) - F(\theta))(v(\theta) - \overline{v}(\theta))d\theta$$

It immediately follows from a simple convexity argument that, if $\overline{M}(\theta)$ satisfies (A.4),

$$0 \leq \int_{\underline{\theta}}^{\overline{\theta}} (u(\theta) - \overline{u}(\theta))f(\theta)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} f(\theta)(\overline{co}(s)(\theta, -\dot{\overline{u}}(\theta)) - \overline{co}(s)(\theta, -\dot{u}(\theta)))d\theta$$

or

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\overline{co}(s)(\theta, \overline{v}(\theta)) - \overline{u}(\theta)\right) f(\theta)d\theta \geq \int_{\underline{\theta}}^{\overline{\theta}} \left(\overline{co}(s)(\theta, v(\theta)) - u(\theta)\right) f(\theta)d\theta$$

[35] The complete proof is omitted for the sake of keeping the present paper at reasonable length.

Using that $\overline{co}(s) \geq s$, the right-hand side above is greater than

$$\int_{\underline{\theta}}^{\overline{\theta}} (s(\theta, v(\theta)) - u(\theta)) f(\theta) d\theta$$

for any admissible pair (u, v). Using (A.3) then yields

$$\int_{\underline{\theta}}^{\overline{\theta}} (s(\theta, \overline{v}(\theta)) - \overline{u}(\theta)) f(\theta) d\theta \geq \int_{\underline{\theta}}^{\overline{\theta}} (s(\theta, v(\theta)) - u(\theta)) f(\theta) d\theta$$

i.e., $(\overline{u}, \overline{q})$ is an optimal allocation, as requested. *Q.E.D.*

To prepare for the rest of our analysis, it is also useful to consider the case where admissible profiles $u(\theta)$ are either monotonically increasing or decreasing respectively, and draw from this assumption further properties for the adjoint function $\overline{M}(\theta)$. First, notice that the support of the probability measure μ, i.e., the set of points θ such that $\overline{u}(\theta) = 0$, is necessarily non-empty, and closed, and thus either of the form $[\hat{\theta}, \overline{\theta}]$ or of the form $[\underline{\theta}, \hat{\theta}]$. From this observation, we get, for $\overline{u}$ non-increasing, that

$$\text{(A.5)} \qquad \overline{M}(\theta) \begin{cases} \in F(\theta) - f(\theta)\partial_v \overline{co}(s)(\theta, 0) & \text{a.e., if } \theta \in [\hat{\theta}, \overline{\theta}], \\ = 0 & \text{if } \theta \in [\underline{\theta}, \hat{\theta}). \end{cases}$$

In the case where $\overline{u}$ is non-decreasing, we instead have

$$\text{(A.6)} \qquad \overline{M}(\theta) \begin{cases} \in F(\theta) - f(\theta)\partial_v \overline{co}(s)(\theta, 0) & \text{a.e, if } \theta \in [\underline{\theta}, \hat{\theta}) \\ = 1 & \text{if } \theta \in [\hat{\theta}, \overline{\theta}]. \end{cases}$$

Some remarks are in order. First, remember that $\partial_v \overline{co}(s)(\theta, 0)$ is a.e. single-valued so that $\overline{M}$ is a.e. defined without any ambiguity. At a point θ where $\partial_v \overline{co}(s)(\theta, 0)$ is multivalued, $\overline{M}(\theta)$ is a selection within the correspondence $F(\theta) - f(\theta)\partial_v \overline{co}(s)(\theta, 0)$. Second, $\overline{M}(\theta)$ must necessarily be non-decreasing. From this, we deduce that $F(\theta) - f(\theta)\partial_v \overline{co}(s)(\theta, 0)$ is non-decreasing on $supp\,\{\mu\}$. This condition implicitly puts some restriction on the support $supp\,\{\mu\}$ where $\overline{u}(\theta) = 0$. Third, when $\overline{u}$ is non-increasing and $supp\,\{\mu\} = \{\overline{\theta}\}$, μ puts mass one at $\overline{\theta}$. Similarly, when $\overline{u}$ is non-decreasing and $supp\,\{\mu\} = \{\underline{\theta}\}$, μ puts mass one at $\underline{\theta}$.

Step 1: The relaxed program. Let us now come back to our more specific optimization program $(\mathcal{P}_i^r)$. Because the domain of (U, q) is the set of incentive compatible, individually rational allocations, U is convex on a compact set and q is non-increasing. It follows that q is measurable, U is absolutely continuous and thus a.e. differentiable. The same applies to the pair $(\overline{U}_{-i}, \overline{q}_{-i})$. As such, we may focus our attention on the domain $\mathcal{D} = \{(U, q) \text{ satisfying (3.1)-(3.2) with } q \text{ non-increasing }\}$. Consider thus the relaxed program $(\mathcal{P}_i^r)$ taken over this set of admissible allocations, but that ignores the convexity constraint (3.3):

$$(\mathcal{P}_i^r): \quad \max_{(U,q) \in \mathcal{D}} \int_{\underline{\theta}}^{\overline{\theta}} \big(S_i(\theta, q(\theta)) + S(\theta, q(\theta)) + \overline{T}_{-i}(q(\theta)) - U(\theta)\big) f(\theta) d\theta \text{ s.t. (3.1)-(3.2).}$$

Note that, in the above description of $(\mathcal{P}_i^r)$, we have implicitly allowed principal i to resolve the agent's indifference in her favor if the agent's best-response set is multi-valued. Because incentive compatibility requires that the agent's indirect utility function is convex, and because a convex function has at most a countable number of kinks, the set of types who do not have a unique optimal choice is necessarily of measure zero. Thus, we may arbitrarily assign the agent's choice in case of indifference (i.e., we may take any selection satisfying (2.2)) without any impact on the best-responses of the players in (2.3). By the same token, (3.2) can be replaced by the requirement

$$\text{(A.7)} \qquad \dot{U}(\theta) = -q(\theta), \qquad \text{a.e.}$$

without changing the value of the program $(\mathcal{P}_i^r)$.

We now rewrite $(\mathcal{P}_i^r)$ using a change of variables in order to get a more useful format amenable to applying Theorem A.1 above. Specifically, define the net utility that principal i's contract provides to the agent as $u_i = U - \overline{U}_{-i}$. We use u_i as the state variable and $v_i = q - \overline{q}_{-i}$ as the control variable in our new optimal control problem. It follows that (3.1) rewrites in this context as

$$\text{(A.8)} \qquad u_i(\theta) \geq 0.$$

It also follows from (A.7) that $u_i(\theta)$ is absolutely continuous and a.e. differentiable with

$$\text{(A.9)} \qquad \dot{u}_i(\theta) = -v_i(\theta) \quad \text{a.e..}$$

Now define principal i's incremental surplus as

$$s_i(\theta, v_i) = S_i(v_i + \overline{q}_{-i}(\theta)) - S_i(\overline{q}_{-i}(\theta)) + S(\theta, v_i + \overline{q}_{-i}(\theta)) + \overline{T}_{-i}(v_i + \overline{q}_{-i}(\theta)) - \overline{U}_{-i}(\theta)$$

or, using the definition of $\overline{U}_{-i}(\theta)$ as $\overline{U}_{-i}(\theta) = S(\theta, \overline{q}_{-i}(\theta)) + \overline{T}_{-i}(\overline{q}_{-i}(\theta))$,

$$\text{(A.10)} \quad s_i(\theta, v_i) = \left[S_i(x + \overline{q}_{-i}(\theta)) + S(\theta, x + \overline{q}_{-i}(\theta)) + \overline{T}_{-i}(x + \overline{q}_{-i}(\theta))\right]_0^{v_i}.$$ [36]

Using (A.9), we can now state principal i's *relaxed* program in terms of net payoffs in a form which is comparable to the generic form $(\mathcal{P})$ above as

$$(\mathcal{P}_i^r): \qquad \max_{u_i \in AC(\Theta, \mathbb{R})} \int_{\underline{\theta}}^{\overline{\theta}} \left(s_i(\theta, -\dot{u}_i(\theta)) - u_i(\theta)\right) f(\theta) d\theta \text{ s.t. (A.8).}$$

We now apply Theorem A.1 and conclude that for any transfer $\overline{T}_{-i}$ offered by rival principals, the rent-output profile $(\overline{U}, \overline{q})$ is a solution to $(\mathcal{P}_i^r)$ if and only if $(\overline{u}_i, \overline{v}_i)$ satisfies (A.8) and (A.9) and there exists a probability measure μ_i defined over the Borel subsets of θ with an associated

[36] Here, we use the notation $[f(x)]_{x_2}^{x_1} = f(x_1) - f(x_2)$.

adjoint function, $\overline{M}_i : \theta \to [0,1]$, defined by $\overline{M}_i(\underline{\theta}) = 0$ and for $\theta > \underline{\theta}$,

$$\overline{M}_i(\theta) \equiv \int_{[\underline{\theta},\theta)} \mu_i(d\theta),$$

such that the following two conditions are satisfied:

$$\text{(A.11)} \quad supp\,\{\mu_i\} \subseteq \{\theta \,|\, \overline{u}_i(\theta) = 0\}\,,$$

$$\text{(A.12)} \quad \overline{co}(s_i)(\theta, \overline{v}_i(\theta)) = s_i(\theta, \overline{v}_i(\theta)) \text{ for a.e. } \theta \in \Theta,$$

$$\text{(A.13)} \quad \overline{M}_i(\theta) \in F(\theta) - f(\theta)\partial_{v_i}\overline{co}(s_i)(\theta, \overline{v}_i(\theta)) \text{ for a.e. } \theta \in \Theta.$$

Step 2: Characterization of the adjoint function, $\overline{M}_i$, for monotonic common-agency games. We prove the following simplifying lemma that characterizes adjoint functions such that the optimality conditions (A.11)-(A.12)-(A.13) hold.

Lemma A.1 *Consider a monotonic common-agency game, and let $(\overline{q}, \overline{U})$ be an equilibrium allocation which solves $(\mathcal{P}_i^r)$ for each principal i. The adjoint function $\overline{M}_i$ for this problem satisfies first $\overline{M}_i(\underline{\theta}) = 0$ and second the following properties.*

1. *When $\overline{q}_{-i}(\theta)$ is interior*

$$\text{(A.14)} \quad \overline{M}_i(\theta) = \begin{cases} \max\left\{0, F(\theta) - f(\theta)\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta))\right\} & a.e. for\ i \in \mathcal{A}, \\ \min\left\{1, F(\theta) - f(\theta)\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta))\right\} & a.e.\ or\ i \in \mathcal{B}. \end{cases}$$

In particular, when S_i is concave and differentiable, (A.14) becomes

$$\text{(A.15)} \quad \overline{M}_i(\theta) = \begin{cases} \max\left\{0, F(\theta) - f(\theta)S_i'(\overline{q}_{-i}(\theta))\right\} & a.e.\ for\ i \in \mathcal{A}, \\ \min\left\{1, F(\theta) - f(\theta)S_i'(\overline{q}_{-i}(\theta))\right\} & a.e.\ for\ i \in \mathcal{B}. \end{cases}$$

2. *When $\overline{q}_{-i}(\theta)$ lies on the boundary of $\mathcal{Q}$,*

(A.16)

$$\overline{M}_i(\theta) = \begin{cases} \max\left\{0, F(\theta) - f(\theta)(\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)) + \partial\overline{co}(\overline{T}_{-i} + S_0)(\overline{q}_{-i}(\theta)) - \theta)\right\} & a.e.\ for\ i \in \mathcal{A}, \\ \min\left\{1, F(\theta) - f(\theta)(\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)) + \partial\overline{co}(\overline{T}_{-i} + S_0)(\overline{q}_{-i}(\theta)) - \theta)\right\} & a.e.\ for\ i \in \mathcal{B}. \end{cases}$$

In particular, if $\overline{q}_{-i}(\theta) \equiv \overline{q}_0(\theta)$ on the boundary of $\mathcal{Q}$, (A.16) becomes

(A.17)

$$\overline{M}_i(\theta) = \begin{cases} \max\{0, F(\theta) - f(\theta)(\partial\overline{co}(S_i)(\overline{q}_0(\theta)) + \partial\overline{co}(S_0)(\overline{q}_0(\theta)) - \theta)\} & a.e.\ for\ i \in \mathcal{A}, \\ \min\{1, F(\theta) - f(\theta)(\partial\overline{co}(S_i)(\overline{q}_0(\theta)) + \partial\overline{co}(S_0)(\overline{q}_0(\theta)) - \theta)\} & a.e.\ for\ i \in \mathcal{B}. \end{cases}$$

PROOF OF LEMMA A.1: For a monotonic common-agency game, $u_i = U - \overline{U}_{-i}$ is decreasing (resp. increasing) when $i \in \mathcal{A}$ (resp. when $i \in \mathcal{B}$) and the characterization of $\overline{M}_i$ follows from (A.5) and (A.6) respectively. We obtain, for $i \in \mathcal{A}$

$$\text{(A.18)} \quad \overline{M}_i(\theta) \begin{cases} \in F(\theta) - f(\theta)\partial_v \overline{co}(s_i)(\theta, 0) & \text{a.e., if } \theta \in supp\,\{\mu_i\} = [\hat{\theta}_i, \overline{\theta}], \\ = 0 \text{ if } \theta \in [\underline{\theta}, \hat{\theta}_i) \end{cases}$$

and, for $i \in \mathcal{B}$,

$$\text{(A.19)} \quad \overline{M}_i(\theta) \begin{cases} = 1 \text{ if } \theta \in (\hat{\theta}_i, \overline{\theta}], \\ \in F(\theta) - f(\theta)\partial_v \overline{co}(s_i)(\theta, 0) & \text{a.e., if } \theta \in supp\,\{\mu_i\} = [\underline{\theta}, \hat{\theta}_i]. \end{cases}$$

We can further refine this characterization. Denote the set of perturbations v_i which are admissible in the relaxed problem $(\mathcal{P}_i^r)$ as $\mathcal{D}_i = \left\{v_i \text{ s.t. } v_i + \overline{q}_{-i} \in \mathcal{Q}\right\}$. Observe that both sides of (A.10) are equal for $v_i = 0$, i.e., $s_i(\theta, 0) = 0$. Taking concave envelopes of both sides of (A.10) and using the fact that the concavification operator is subadditive yields, for all $v_i \in \mathcal{D}_i$,

$$\text{(A.20)} \quad \overline{co}(s_i)(\theta, v_i) \leq \overline{co}(S_i(v_i + \overline{q}_{-i}(\theta))) - S_i(\overline{q}_{-i}(\theta))$$

$$+\overline{co}(\overline{T}_{-i}(v_i + \overline{q}_{-i}(\theta)) - \overline{T}_{-i}(\overline{q}_{-i}(\theta)) + S_0(v_i + \overline{q}_{-i}(\theta)) - S_0(\overline{q}_{-i}(\theta)) - \theta v_i).$$

Since, $s_i(\theta, 0) = \overline{co}(s_i)(\theta, 0) = 0$, we immediately deduce from the inequality (A.20) between two concave functions that take the same value at $v_i = 0$ the following inclusion for their sup-differentials,

$$\text{(A.21)} \quad \partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)) + \partial\overline{co}(\overline{T}_{-i} + S_0)(\overline{q}_{-i}(\theta)) - \theta \subseteq \partial_{v_i}\overline{co}(s_i)(\theta, 0).$$

By definition, we have

$$\text{(A.22)} \quad \overline{q}_{-i}(\theta) \in \arg\max_{q \in \mathcal{Q}} \overline{T}_{-i}(q) + S_0(q) - \theta q.$$

We know distinguish two cases:

1. When $\overline{q}_{-i}(\theta)$ is an interior solution to the agent's problem, we have

   $$\text{(A.23)} \quad 0 \in \partial\overline{co}(\overline{T}_{-i} + S_0)(\overline{q}_{-i}(\theta)) - \theta.$$

   Inserting into (A.21) implies that

   $$\text{(A.24)} \quad \partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)) \subseteq \partial_{v_i}\overline{co}(s_i)(\theta, 0), \text{ a.e.}$$

   Because $\partial_{v_i}\overline{co}(s_i)(\theta, 0)$ is a.e. single valued, we thus have

   $$\text{(A.25)} \quad \partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)) = \partial_{v_i}\overline{co}(s_i)(\theta, 0) \text{ a.e..}$$

Inserting into (A.18) (resp. (A.19)) gives (A.14).

Suppose now that S_i is concave and differentiable. Concavity implies $\overline{co}(S_i) = S_i$. Differentiability thus implies $\partial\overline{co}(S_i) = S_i'$. Observe that $\overline{q}_{-i}(\theta)$ is non-decreasing and thus a.e. differentiable and continuous. Therefore, $S_i'(\overline{q}_{-i}(\theta))$ is defined a.e.. [37] Then, (A.14) becomes (A.15).

2. When $\overline{q}_{-i}(\theta)$ lies on the boundary of $\mathcal{Q}$, i.e., $\overline{q}_{-i}(\theta) = q_{min}$ or $\overline{q}_{-i}(\theta) = q_{max}$, we directly insert (A.21) into (A.18) to get (A.16). In particular, if $\overline{q}_{-i}(\theta) \equiv \overline{q}_0(\theta)$ on the boundary of $\mathcal{Q}$, $\overline{T}_{-i} = 0$ and (A.16) writes as (A.17).

Q.E.D.

Step 3: Transformation by means of $\mathcal{V}_i(\theta, q)[\overline{q}_{-i}]$. From (A.13), we deduce

$$\overline{v}_i(\theta) \in \arg\max_v \ \overline{co}(s_i)(\theta, v) - \frac{F(\theta) - \overline{M}_i(\theta)}{f(\theta)} v.$$

From (A.12) and the fact that $\overline{co}(s_i) \geq s_i$, we can rewrite

$$\overline{v}_i(\theta) \in \arg\max_v \ s_i(\theta, v) - \frac{F(\theta) - \overline{M}_i(\theta)}{f(\theta)} v$$

or, expressed in terms of q,

$$\overline{q}(\theta) \in \arg\max_{q\in\mathcal{Q}} \left[S_i(x) + S_0(\theta, x) + \overline{T}_{-i}(x) - \frac{F(\theta) - \overline{M}_i(\theta)}{f(\theta)} x \right]_{\overline{q}_{-i}(\theta)}^{q}. \tag{A.26}$$

Consider thus $\mathcal{V}_i(\theta, q)[\overline{q}_{-i}]$ defined as

$$\mathcal{V}_i(\theta, q)[\overline{q}_{-i}] = S_i(q) - \frac{F(\theta) - \overline{M}_i(\theta)}{f(\theta)} q \tag{A.27}$$

where $\overline{M}_i(\theta)$ is given by (A.14). The optimality condition (A.26) finally rewrites as (3.7).

Since $\overline{U}(\theta) = \overline{U}_{-i}(\theta)$ for $\theta \in \Omega_i$ and both $\overline{U}(\theta)$ and $\overline{U}_{-i}(\theta)$ are a.e. differentiable, we have $\overline{q}(\theta) = \overline{q}_{-i}(\theta))$ a.e. $\theta \in \text{int}\Omega_i$. Therefore, (3.8) follows.

Step 4: The solution to the relaxed program $(\mathcal{P}_i^r)$ is convex. Simple revealed preference arguments show that $\overline{q}(\theta)$ is necessarily non-decreasing since $\mathcal{V}_i(\theta, q)[\overline{q}_{-i}]$ and $S(\theta, q)$ both have decreasing differences. *Q.E.D.*

[37] At a point θ where $\overline{q}_{-i}(\theta)$ has a downward-jump discontinuity, we take the convention that $\overline{q}_{-i}(\theta) = \lim_{\tilde{\theta}\to\theta^+} \overline{q}_{-i}(\tilde{\theta})$.

ONLINE APPENDIX B: PROOFS OF THE THEOREMS

PROOF OF THEOREM 1: Proposition 1 must hold for any equilibrium allocation. Adding up (3.7) across all n principals, we obtain the following condition:

$$\text{(B.1)}\qquad \overline{q}(\theta) \in \arg\max_{q\in\mathcal{Q}}\ S(\theta,q)+\overline{\mathcal{V}}(\theta,q)+(n-1)(S(\theta,q)+\overline{T}(q)),\ \text{a.e. } \theta$$

where $\overline{T}$ implements $(\overline{U},\overline{q})$. Simple revealed preference arguments show that $\overline{q}(\theta)$ is necessarily non-decreasing since $\overline{\mathcal{V}}(\theta,q)$ and $S(\theta,q)$ both have decreasing differences.

Define the value function for the above program as

$$\text{(B.2)}\qquad \overline{V}(\theta)\equiv\max_{q\in\mathcal{Q}}\ S(\theta,q)+\overline{\mathcal{V}}(\theta,q)+(n-1)(S(\theta,q)+\overline{T}(q)).$$

Remember that $\overline{M}_i$, as a distribution function, has bounded variation. Therefore, $\mathcal{V}_i(\theta,q)[\overline{q}_{-i}]$ and thus $\overline{\mathcal{V}}(\theta,q)$ have also bounded variation. From that, and the fact that the above maximand is upper semi-continuous in q and $\mathcal{Q}$ is compact, it follows that $\overline{V}$ is itself absolutely continuous (Milgrom and Segal, 2002). Moreover, given that $(\overline{U},\overline{q})$ is an incentive-compatible allocation which solves this program, we have

$$\text{(B.3)}\qquad \overline{V}(\theta)=S(\theta,\overline{q}(\theta))+\overline{\mathcal{V}}(\theta,\overline{q}(\theta))+(n-1)\overline{U}(\theta).$$

Because $\overline{V}$ is absolutely continuous, it is almost everywhere differentiable. Applying the Envelope Theorem (Milgrom and Segal, 2002), we get

$$\dot{\overline{V}}(\theta)=\overline{\mathcal{V}}_\theta(\theta,\overline{q}(\theta))-n\overline{q}(\theta),\qquad \text{a.e.}$$

From absolute continuity, we then deduce the integral representation

$$\overline{V}(\theta)-\overline{V}(\theta')=\int_{\theta'}^{\theta}(\overline{\mathcal{V}}_\theta(x,\overline{q}(x))-n\overline{q}(x))dx\quad \forall(\theta,\theta')\in\Theta^2.$$

Because $\overline{U}$ is also absolutely continuous, we thus have for any pair (θ,θ')

$$\overline{U}(\theta)-\overline{U}(\theta')=-\int_{\theta'}^{\theta}\overline{q}(x)dx$$

Note that

$$\left[S_0(\tilde\theta,\overline{q}(\tilde\theta))+\overline{\mathcal{V}}(\tilde\theta,\overline{q}(\tilde\theta))\right]_{\theta'}^{\theta}=\left[\overline{V}(\tilde\theta)-(n-1)\overline{U}(\tilde\theta)\right]_{\theta'}^{\theta},$$

or more simply

$$\text{(B.4)}\qquad \left[S_0(\tilde\theta,\overline{q}(\tilde\theta))+\overline{\mathcal{V}}(\tilde\theta,\overline{q}(\tilde\theta))\right]_{\theta'}^{\theta}=\int_{\theta'}^{\theta}(\mathcal{V}_\theta(x,\overline{q}(x))-\overline{q}(x))dx.$$

Using the relationship

$$\left[S_0(\tilde{\theta},\overline{q}(\theta')) + \mathcal{V}(\tilde{\theta},\overline{q}(\theta'))\right]_{\theta'}^{\theta} = \int_{\theta'}^{\theta} (\mathcal{V}_\theta(x,\overline{q}(\theta')) - \overline{q}(\theta'))dx$$

that $\mathcal{V}_i(\theta,q)[\overline{q}_{-i}]$ has decreasing differences and that $\overline{q}$ is non-increasing, we obtain:

$$\left[S_0(\theta,\overline{q}(\tilde{\theta}) + \mathcal{V}(\theta,\overline{q}(\tilde{\theta}))\right]_{\theta'}^{\theta} = \int_{\theta'}^{\theta}\int_{\overline{q}(\theta')}^{\overline{q}(x)} (\mathcal{V}_{\theta q}(x,\tilde{q}) - 1)\, d\tilde{q}dx \geq 0.$$

Because any $q' \in \overline{Q}(\Theta)$ can be identified with some $\theta' \in \theta$ such that $q' = \overline{q}(\theta')$, the inequality implies that $\overline{q}(\theta)$ satisfies (4.2) pointwise in θ.

By definition, the maximal allocation $q^m(\theta)$ defined as (5.1) also satisfies (4.2). Moreover, any putative equilibrium with range $\overline{q} = \overline{q}(\Theta)$ is such that $\overline{q} \subseteq q^m(\Theta)$.

Q.E.D.

Proof of Theorem 2: When S_i is concave and differentiable and the maximal allocation is interior, the virtual surplus as defined $\mathcal{V}(\theta,q)[q^m]$ can be expressed as

$$\text{(B.5)} \qquad \mathcal{V}_i^m(\theta,q) = S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{S_i'(q^m(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q.$$

The maximand in (5.1) is strictly concave when S_0 is strictly so and any interior solution $q^m(\theta)$ is thus given by the first-order condition

$$\text{(B.6)} \qquad \sum_{i=0}^{n} S_i'(q^m(\theta)) = \theta + \sum_{i=1}^{n} \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{S_i'(q^m(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\}.$$

Decomposing for $i \in \mathcal{A}$ and $i \in \mathcal{B}$ yields (5.2). Because of strict concavity of S_0 and concavity of S_i,

$$\text{(B.7)} \qquad S_0'(q) + \sum_{i\in\mathcal{A}} \max\left\{S_i'(q) - \frac{F(\theta)}{f(\theta)}; 0\right\} + \sum_{i\in\mathcal{B}} \min\left\{S_i'(q) + \frac{1-F(\theta)}{f(\theta)}; 0\right\}$$

is a decreasing function of q and thus $q^m(\theta)$ as defined in (5.2) is unique. Because $MHRC$ holds, (B.7) is a non-increasing function of θ. It is then routine to check that $q^m(\theta)$ is itself non-increasing. Because of that, q^m is almost everywhere differentiable and thus almost everywhere continuous. Moreover, it cannot have a jump discontinuity at any point since then (5.2) would have two solutions at this point. Hence, q^m is continuous. *Q.E.D.*

Proof of Theorem 3: To prove that the necessary conditions (5.1) satisfied by q^m are also sufficient, we construct individual tariffs that implement this allocation at equilibrium.

Preliminaries. For a maximal allocation, the marginal virtual surplus (3.5) relative to that allocation itself now writes as

$$\mathcal{V}^m_{iq}(\theta, q) = S'_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{S'_i(q^m(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\}. \tag{B.8}$$

Since $\mathcal{V}^m_{iq}(\theta, q)$ so defined is uniformly bounded in θ, $t^m_i(q)$ as defined in (5.3) or (5.4) is absolutely continuous and, in fact, differentiable at all $q \in \mathring{\mathcal{Q}}^m$ since $\vartheta(q)$ is single-valued under the conditions of the Theorem. Its derivative is

$$t^{m\prime}_i(q) = \mathcal{V}^m_{iq}(\vartheta^m(q), q). \tag{B.9}$$

Suppose now that $\mathring{\Omega}^m_i \neq \emptyset$. The definition (5.3) is independent of the choice $\hat{q}_i \in q^m_{-i}(\Omega^m_i)$. This comes from applying Proposition 1 for the allocation $\overline{q}_{-i} = q^m$ itself, together with the fact that $\mathcal{V}^m_{iq}(\theta, q^m(\theta)) = 0$ for all $\theta \in \Omega^m_i$. This condition can also be written as $\mathcal{V}^m_{iq}(\vartheta^m(q), q) = 0$ for all $q \in q^m(\Omega^m_i)$. Hence, for $(\hat{q}_i, \hat{q}'_i) \in q^m(\Omega_i) \times q^m(\Omega_i)$, we have $\int_{\hat{q}_i}^{\hat{q}'_i} \mathcal{V}^m_{iq}(\vartheta^m(x), x)dx = 0$; so the result.

Suppose instead that $\mathring{\Omega}^m_i = \emptyset$. The definition (5.4) again follows from Proposition 1 and, more specially, (3.9).

Non-Negative Tariffs. Observe that $\mathcal{V}^m_{iq}(\theta, q^m(\theta)) \geq 0$ (resp. $\leq$) for $\theta \in \Omega^{mc}_i$ and $i \in \mathcal{A}$ (resp. $i \in \mathcal{B}$).From this, it follows that $\mathcal{V}^m_{iq}(\vartheta^m(q), q) \geq 0$ (resp. $\leq$) for $q \geq \hat{q}_i$ (resp. $q \leq \hat{q}_i$) and thus, using (5.3) (resp. (5.4)) and the fact that $t^m_i(\hat{q}_i) = 0$ (resp. either $t^m_i(q^m(\overline{\theta})) \geq 0$ or $t^m_i(q^m(\overline{\theta})) \geq 0$) immediately gives us that $t^m_i(q)$ is non-negative.

Denote now the aggregate by $T^m = \sum_{i\in\mathcal{N}} t^m_i$ where $t^m_i(q)$ satisfies (5.3) or (5.4). What remains to be shown is (i) T^m induces the agent with type θ to choose $q^m(\theta)$, and (ii) each principal i, facing the rivals' aggregate T^m_{-i}, finds it optimal to implement $q^m(\theta)$ as well.

Incentive Compatibility. Consider the agent's problem when facing the aggregate payment T^m so constructed. Note that T^m is differentiable and its derivative is

$$T^{m\prime}(q) = \mathcal{V}^m_q(\vartheta^m(q), q). \tag{B.10}$$

Incentive compatibility requires

$$q^m(\theta) \in \arg\max_{q\in\mathcal{Q}} \; S(\theta, q) + T^m(q) \quad \forall\theta \in \Theta. \tag{B.11}$$

The necessary and conditions for an interior maximum is thus

$$S'_0(q^m(\theta)) + \mathcal{V}^m_q(\vartheta^m(q^m(\theta)), q^m(\theta)) = \theta. \tag{B.12}$$

Now, observe that, for $q^m(\theta) \in \mathring{\mathcal{Q}}$, $\mathcal{V}^m_q(\vartheta^m(q^m(\theta)), q^m(\theta)) = \mathcal{V}^m_q(\theta, q^m(\theta))$ and (B.12) writes as (5.2).

We now check that the necessary conditions (B.12) that define $q^m(\theta)$ are also sufficient for incentive compatibility. Consider now the rent profile

$$U^m(\theta) = \max_{q \in \mathcal{Q}^m} \ S(\theta, q) + T^m(q).$$

It is routine to prove that U^m so defined is absolutely continuous and admits the following integral representation

$$U^m(\theta) - U^m(\hat{\theta}) = \int_{\theta}^{\hat{\theta}} q^m(\tilde{\theta}) d\tilde{\theta} \tag{B.13}$$

where $q^m(\theta)$ satisfies (5.2). We thus rewrite the incentive compatibility conditions (B.25) in terms of U^m as

$$U^m(\theta) \geq U^m(\hat{\theta}) + S(\theta, q^m(\hat{\theta})) - S(\hat{\theta}, q^m(\hat{\theta})) \quad \forall (\theta, \hat{\theta}) \in \Theta^2. \tag{B.14}$$

From the fact that $q^m(\theta)$ is non-increasing, (B.13) implies

$$U^m(\theta) - U^m(\hat{\theta}) \geq (\hat{\theta} - \theta) q^m(\hat{\theta}) = S_0(\theta, q^m(\hat{\theta})) - S_0(\hat{\theta}, q^m(\hat{\theta}))$$

and the incentive compatibility conditions (B.14) hold.

Principals' Optimality. Consider principal i's program. In light of Theorem 1 and Lemma A.1, we need to check that $q^m(\theta)$ is a best response allocation for principal i, i.e., it satisfies

$$q^m(\theta) \in \arg\max_{q \in \mathcal{Q}} \ S(\theta, q) + \mathcal{V}_i^m(\theta, q) + T_{-i}^m(q), \quad a.e. \tag{B.15}$$

where again T_{-i}^m is the aggregate for all principal except i obtained from individual tariffs satisfying (5.3)/(5.4).

Mimicking what we did above for the agent's incentive compatibility problem, we write the corresponding necessary conditions for optimality as

$$S_0'(q^m(\theta)) + \mathcal{V}_{iq}^m(\theta, q^m(\theta)) + T_{-i}^{m'}(q^m(\theta)) = \theta. \tag{B.16}$$

Proceeding again as for the agent's incentive compatibility problem, it is straightforward to check that (B.16) again boils down to (5.2) as requested.

Turning now to sufficiency for the principals' optimality problem, let us define

$$V_i^m(\theta) = \max_{q \in \mathcal{Q}^m} S(\theta, q) + \mathcal{V}_i^m(\theta, q) + T_{-i}^m(q).$$

It is routine to prove that V_i^m so defined is absolutely continuous with the following integral representation

$$\text{(B.17)}\quad V_i^m(\theta) - V_i^m(\hat{\theta}) = \int_\theta^{\hat{\theta}} (q^m(\tilde{\theta}) - \mathcal{V}_{i\theta}^m(\tilde{\theta}, q^m(\tilde{\theta})))d\tilde{\theta} \quad (\theta, \hat{\theta}) \in \Theta^2.$$

where $q^m(\theta)$ satisfies (5.2). We may rewrite the incentive compatibility conditions (B.15) in terms of V_i^m as

$$\text{(B.18)}\quad V_i^m(\theta) \geq V_i^m(\hat{\theta}) + S(\theta, q^m(\hat{\theta})) + \mathcal{V}_i^m(\theta, q^m(\hat{\theta})) - S(\hat{\theta}, q^m(\hat{\theta})) - \mathcal{V}_i^m(\hat{\theta}, q^m(\hat{\theta})) \quad \forall (\theta, \hat{\theta}) \in \Theta^2.$$

The fact that $q^m(\theta)$ is non-increasing and $\mathcal{V}_i^m$ has decreasing differences implies that

$$V_i^m(\theta) - V_i^m(\hat{\theta}) \geq (\theta - \hat{\theta})q^m(\hat{\theta}) - (\mathcal{V}_i^m(\hat{\theta}, q^m(\hat{\theta})) - \mathcal{V}_i^m(\theta, q^m(\hat{\theta}))).$$

It follows that the principal's optimality conditions (B.15) necessarily hold.

Q.E.D.

Proof of Theorem 4 : That $\overline{q}$ is non-increasing follows from observing that both S_0 and $\mathcal{V}^m$ have decreasing differences. Thus, $\overline{q}$ is a.e. differentiable with a countable number of downward-jump discontinuities. Observing that, for any $q \in \overline{Q}$ there exists $\hat{\theta}$ such that $q = \overline{q}(\hat{\theta})$; the surrogate principal's incentive problem can be written as

$$\theta \in \arg\max_{\hat{\theta}\in\Theta} \; S(\theta, \overline{q}(\hat{\theta})) + \mathcal{V}^m(\theta, \overline{q}(\hat{\theta})) \quad \forall \theta \in \Theta.$$

The corresponding first-order necessary condition for optimality with respect to $\hat{\theta}$, at any point θ where $\overline{q}$ is differentiable, writes as (6.2).

Consider now the value function $\overline{V}$ as defined in (B.2). We already know that $\overline{V}$ is absolutely continuous. At any point of discontinuity θ_0 for $\overline{q}$, continuity of $\overline{V}$ still implies:

$$\text{(B.19)}\quad \lim_{\theta\to\theta_0^-} \overline{V}(\theta) = \lim_{\theta\to\theta_0^+} \overline{V}(\theta).$$

Consider a discontinuity at θ_0 which is isolated. On the right- and the left-neighborhoods of θ_0, (6.2) thus applies and either $\dot{\overline{q}}(\theta) = 0$ or $\overline{q}(\theta) = \overline{q}^m(\theta)$ defined as (5.2). Moreover, at a point at which $\overline{q}$ is continuous but not differentiable, it must be that either the right- or the left-derivative is zero. Taking stock of those remarks, we are now proving that bunching arises both on a right- and a left-neighborhood of θ_0. We proceed by contradiction. To this end, suppose first that bunching arises on the left-neighborhood only and call thus $\overline{q}(\theta_0^-) = \lim_{\theta\to\theta_0^-} \overline{q}(\theta)$ with $\overline{q}(\theta_0^-) > \overline{q}^m(\theta_0)$ because θ_0 must be a downward-jump discontinuity. Observe that $\overline{q}(\theta_0^-) = \overline{q}^m(\theta_1)$ for some type $\theta_1 < \theta_0$ such that $\theta_1 = \max\{\theta \text{ s.t. } q^m(\theta) \geq \overline{q}(\theta_0^-)\}$ and that $\overline{q}(\theta) = \overline{q}^m(\theta_1)$ for all $\theta \in [\theta_1, \theta_0)$.

Because the agent's information rent $\overline{U}$ is also absolutely continuous at θ_0, we also have:

$$\overline{U}(\theta_0) = \lim_{\theta\to\theta_0^-} S(\theta,\overline{q}(\theta)) + \overline{T}(\overline{q}(\theta)) = S(\theta_0,\overline{q}(\theta_0^-)) + \overline{T}(\overline{q}(\theta_0^-))$$

and

$$\overline{U}(\theta_0) = \lim_{\theta\to\theta_0^+} S(\theta,\overline{q}(\theta) + \overline{T}(\overline{q}(\theta)) = S(\theta_0,\overline{q}^m(\theta_0)) + \overline{T}(q^m(\theta_0)).$$

Therefore, we get:

$$S(\theta_0,\overline{q}(\theta_0^-)) + \overline{T}(\overline{q}(\theta_0^-)) = S(\theta_0,\overline{q}^m(\theta_0)) + \overline{T}(\overline{q}^m(\theta_0)).$$

Inserting this equality into (B.19), taking into account the definition (B.3) and simplifying yields:

$$\lim_{\theta\to\theta_0^-} S(\theta_0,\overline{q}(\theta)) + \mathcal{V}(\theta_0,\overline{q}(\theta)) = \lim_{\theta\to\theta_0^+} S(\theta_0,q^m(\theta)) + \mathcal{V}(\theta_0,q^m(\theta)).$$

Expressing those right- and left-hand side limits gives us:

$$S(\theta_0,\overline{q}(\theta_0^-)) + \mathcal{V}^m(\theta_0,\overline{q}(\theta_0^-)) = S(\theta_0,q^m(\theta_0)) + \mathcal{V}^m(\theta_0,q^m(\theta_0)). \tag{B.20}$$

Because $S(\theta_0,q)+\mathcal{V}^m(\theta_0,q)$ is strictly concave in q, it has a unique maximizer $\overline{q}^m(\theta_0)$ that is supposed to be interior. Therefore, (B.20) necessarily implies that $\overline{q}(\theta_0^-) = \overline{q}^m(\theta_0)$. A contradiction with our starting premise that $\overline{q}(\theta_0^-) > \overline{q}^m(\theta_0)$ at the discontinuity θ_0.

Similarly, we could also rule out the case where bunching only arises on the right-neighborhood of θ_0 at a value $\overline{q}(\theta_0^+) = \lim_{\theta\to\theta_0^+}\overline{q}(\theta)$.

Taking stock of these findings, we necessarily have $\overline{q}(\theta_0^-) > \overline{q}(\theta_0^+)$ at a discontinuity point θ_0. Moreover, bunching arises on both sides of θ_0 which means $\overline{q}(\theta) = \overline{q}(\theta_0^-)$ (resp. $\overline{q}(\theta) = \overline{q}(\theta_0^+)$) for θ on this left- (resp. right-) neighborhood. Because $q^m(\theta)$ is strictly decreasing, there thus exist $\theta_1 < \theta_0 < \theta_2$ such that $\overline{q}(\theta_0^-) = \overline{q}^m(\theta_1)$ and $\overline{q}(\theta_0^+) = \overline{q}^m(\theta_2)$. In fact $\overline{q}(\theta) = \overline{q}^m(\theta_1)$ for all $\theta \in [\theta_1,\theta_0)$. Suppose not. Then, $\overline{q}$ would have a downward discontinuity at some $\theta_0' \in (\theta_1,\theta_0)$. The same argument as above shows that at any such putative discontinuity, we should have $\overline{q}(\theta_0^{'-}) > q^m(\theta_0^{'}) > \overline{q}(\theta_0^{'+})$ and $\overline{q}(\theta_0^{'+}) \geq \overline{q}^m(\theta_0)$. Since $q^m(\theta)$ is decreasing, this is a contradiction with the definition of $\theta_0^{'}$.

Because the agent's rent $\overline{U}$ is continuous at θ_0, we also have:

$$\overline{U}(\theta_0) = \lim_{\theta\to\theta_0^-} S(\theta,\overline{q}(\theta)) + \overline{T}(\overline{q}(\theta)) = S(\theta_0,\overline{q}^m(\theta_1)) + \overline{T}(\overline{q}^m(\theta_1))$$

and

$$\overline{U}(\theta_0) = \lim_{\theta\to\theta_0^+} S(\theta,\overline{q}(\theta)) + \overline{T}(\overline{q}(\theta)) = S(\theta_0,\overline{q}^m(\theta_2)) + \overline{T}(\overline{q}^m(\theta_2)).$$

It follows that:

$$S(\theta_0, \overline{q}^m(\theta_1)) + \overline{T}(\overline{q}^m(\theta_1)) = S(\theta_0, \overline{q}^m(\theta_2)) + \overline{T}(\overline{q}^m(\theta_2)).$$

Inserting this equality into (B.19), taking into account the definition (B.3) and simplifying now yields:

$$\lim_{\theta \to \theta_0^-} S(\theta, \overline{q}(\theta)) + \mathcal{V}(\theta, \overline{q}(\theta)) = \lim_{\theta \to \theta_0^+} S(\theta, \overline{q}(\theta)) + \mathcal{V}(\theta, \overline{q}(\theta))$$

or, expressing those right- and left-hand side limits,

$$S(\theta_0, \overline{q}^m(\theta_1)) + \mathcal{V}(\theta_0, \overline{q}^m(\theta_1)) = S(\theta_0, \overline{q}^m(\theta_2)) + \mathcal{V}(\theta_0, \overline{q}^m(\theta_2))$$

which is (6.3). *Q.E.D.*

PROOF OF THEOREM 5 : Observe that $\overline{q}$ as defined in (6.5) satisfies the necessary conditions (6.2) and (6.3). To prove that $\overline{q}$ is actually an equilibrium allocation, we construct individual tariffs that implement this allocation at equilibrium.

PRELIMINARIES. Because $\mathcal{A} = \mathcal{N}$, the marginal virtual surplus (3.5) relative to the maximal allocation writes as

$$\text{(B.21)} \quad \mathcal{V}_{iq}^m(\theta, q) = S_i'(q) - \min\left\{\frac{F(\theta)}{f(\theta)}; S_i'(q^m(\theta))\right\} \quad \forall i \in \mathcal{N}$$

When evaluated at the maximal allocation itself, this marginal virtual surplus becomes

$$\text{(B.22)} \quad \mathcal{V}_{iq}^m(\theta, q^m(\theta)) = \max\left\{S_i'(q^m(\theta)) - \frac{F(\theta)}{f(\theta)}; 0\right\} \quad \forall i \in \mathcal{N}.$$

An interior maximal allocation q^m is then defined as

$$\text{(B.23)} \quad S_0'(q^m(\theta)) + \sum_{i \in \mathcal{N}} \max\left\{S_i'(q^m(\theta)) - \frac{F(\theta)}{f(\theta)}; 0\right\} = \theta.$$

From Theorem 3, this maximal allocation is actually an equilibrium sustained with the non-negative tariffs $\overline{t}_i^m(q)$ as defined in (5.3) and/or (5.4).

TARIFFS. Observe that the tariff $\overline{t}_i(q)$ as defined in (6.8) is non-negative, so is the aggregate payment $\overline{T}(q) = \sum_{i \in \mathcal{N}} \overline{t}_i(q)$.

By construction, we have

$$\left[\overline{t}_i(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \left[t_i^m(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \int_{q^m(\theta_2)}^{q^m(\theta_1)} \mathcal{V}_{iq}(\vartheta^m(q), q) dq.$$

Because (6.5) holds, (B.22) implies that

$$[t_i^m(q)]_{q^m(\theta_2)}^{q^m(\theta_1)} = \int_{q^m(\theta_2)}^{q^m(\theta_1)} \left(S_i'(q) - \frac{F(\vartheta^m(q))}{f(\vartheta^m(q))} \right) dq = [S_i(q)]_{q^m(\theta_2)}^{q^m(\theta_1)} - \int_{q^m(\theta_2)}^{q^m(\theta_1)} \frac{F(\vartheta^m(q))}{f(\vartheta^m(q))} dq$$

$$= [S_i(q)]_{q^m(\theta_2)}^{q^m(\theta_1)} - \frac{F(\theta_0)}{f(\theta_0)}(q^m(\theta_1) - q^m(\theta_2))$$

where the last equality follows from (6.6).

Summing over $i \in \mathcal{N}$, we get

$$\text{(B.24)} \quad \left[\overline{T}(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \left[\sum_{i\in\mathcal{N}} S_i(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} - \frac{F(\theta_0)}{f(\theta_0)}(q^m(\theta_1) - q^m(\theta_2)).$$

Incentive Compatibility. Incentive compatibility can be expressed as

$$\text{(B.25)} \quad \overline{q}(\theta) \in \arg\max_{q\in\mathcal{Q}} S(\theta, q) + \overline{T}(q) = \arg \max_{q\in\mathcal{Q}^m/[q^m(\theta_2),q^m(\theta_1)]} S(\theta, q) + T^m(q) \quad \forall \theta \in \Theta.$$

For $\theta \in [\underline{\theta}, \theta_1] \bigcup \left[\theta_2, \overline{\theta}\right]$, the arg max above is of course achieved for $q^m(\theta)$ since, for such θ, we have $q^m(\theta) \in \mathcal{Q}^m / \left[q^m(\theta_2), q^m(\theta_1)\right]$.

Consider now θ_0. Using (B.24), we observe that

$$\text{(B.26)} \quad \left[S(\theta_0, q) + \overline{T}(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = [S_0(\theta_0, q) + \mathcal{V}^m(\theta_0, q)]_{q^m(\theta_2)}^{q^m(\theta_1)} = 0$$

where the last equality follows from (6.4); which proves that an agent with type θ_0 is indifferent between choosing $q^m(\theta_2)$ or $q^m(\theta_1)$.

Consider now $\theta \in [\theta_1, \theta_0)$ (resp. $\theta \in (\theta_0, \theta_2]$). Because of increasing differences, we thus have

$$\text{(B.27)} \quad \left[S(\theta, q) + \overline{T}(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} \geq 0.$$

Because $q^m(\theta)$ satisfies (B.23), S_i and S_0 are concave, we necessarily have

$$\text{(B.28)} \quad [S(\theta, q) + \mathcal{V}^m(\theta, q)]_{q^m(\theta_1)}^{q} \leq 0 \quad \forall q \geq q^m(\theta_1)$$

and

$$\text{(B.29)} \quad [S(\theta, q) + \mathcal{V}^m(\theta, q)]_{q}^{q^m(\theta_2)} \leq 0 \quad \forall q \leq q^m(\theta_2).$$

Gathering (B.27), (B.28) and (B.29) yields that $\overline{q}(\theta) = q^m(\theta_1)$ for $\theta \in [\theta_1, \theta_0)$.

The case $\theta \in (\theta_0, \theta_2]$ can be treated similarly to obtain that $\overline{q}(\theta) = q^m(\theta_2)$ for such θ. Finally, the output $\overline{q}(\theta)$ so obtained satisfies in (6.7).

By construction, this action profile is non-increasing. Following the same steps as in the Proof of Theorem 3, we can prove sufficiency.

Principals' Optimality. Consider principal i's program. In light of Theorem 1 and Lemma A.1, we need to check that $\overline{q}(\theta)$ is a best response allocation for principal i, i.e., it satisfies

$$\overline{q}(\theta) \in \arg \max_{q \in \mathcal{Q}^m / [q^m(\theta_2), q^m(\theta_1)]} S(\theta, q) + \mathcal{V}_i^m(\theta, q) + \overline{T}_{-i}(q), \quad a.e.$$

where again $\overline{T}_{-i}$ is the aggregate for all principals except i obtained from individual tariffs satisfying (6.8). Hence, we rewrite this optimality condition as

$$\text{(B.30)} \quad \overline{q}(\theta) \in \arg \max_{q \in \mathcal{Q}^m / [q^m(\theta_2), q^m(\theta_1)]} S(\theta, q) + \mathcal{V}_i^m(\theta, q) + T_{-i}^m(q), \quad a.e.$$

For $\theta \in [\underline{\theta}, \theta_1] \bigcup [\theta_2, \overline{\theta}]$, the arg max above is of course achieved for $q^m(\theta)$ since, for such θ, we have $q^m(\theta) \in \mathcal{Q}^m / [q^m(\theta_2), q^m(\theta_1)]$.

Consider now θ_0. Using (B.24), we observe that

$$\text{(B.31)} \quad \left[S(\theta_0, q) + \mathcal{V}_i^m(\theta_0, q) + T_{-i}^m(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \left[S_0(\theta_0, q) + \mathcal{V}^m(\theta_0, q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = 0$$

where the last equality again follows from (6.4); which proves that, at θ_0, principal i is indifferent between choosing $q^m(\theta_2)$ or $q^m(\theta_1)$.

Consider now $\theta \in [\theta_1, \theta_0)$ (resp. $\theta \in (\theta_0, \theta_2]$). Because of increasing differences, we thus have

$$\text{(B.32)} \quad \left[S(\theta, q) + \mathcal{V}_i^m(\theta_0, q) + T_{-i}^m(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} \geq 0.$$

Because $q^m(\theta)$ satisfies (B.23), S_i and S_0 are concave, we necessarily have

$$\text{(B.33)} \quad [S(\theta, q) + \mathcal{V}^m(\theta, q)]_{q^m(\theta_1)}^{q} \leq 0 \quad \forall q \geq q^m(\theta_1)$$

and

$$\text{(B.34)} \quad [S(\theta, q) + \mathcal{V}^m(\theta, q)]_{q}^{q^m(\theta_2)} \leq 0 \quad \forall q \leq q^m(\theta_2).$$

Gathering (B.32), (B.33) and (B.34) yields that $\overline{q}(\theta) = q^m(\theta_1)$ for $\theta \in [\theta_1, \theta_0)$.

The case $\theta \in (\theta_0, \theta_2]$ can be treated similarly to obtain that $\overline{q}(\theta) = q^m(\theta_2)$ for such θ.

Finally, sufficiency for the principals' optimality problem can be proved as Proof of Theorem 3. *Q.E.D.*

ONLINE APPENDIX C: OTHER PROOFS

PROOF OF PROPOSITION 3: PRELIMINARIES. From (8.3) and $MHRC$, there always exists a unique interior solution $\hat{\theta}_i$ to

$$(C.1) \qquad s_i = \begin{cases} \frac{F(\hat{\theta}_i)}{f(\hat{\theta}_i)} & \text{if } i \in \mathcal{A}, \\ \frac{F(\hat{\theta}_i)-1}{f(\hat{\theta}_i)} & \text{if } i \in \mathcal{B}. \end{cases}$$

From there, it follows that principal i's inactivity sets are non-empty and of the form

$$(C.2) \qquad \Omega_i^m = \begin{cases} \left[\hat{\theta}_i, \overline{\theta}\right] & \text{if } i \in \mathcal{A}, \\ \left[\underline{\theta}, \hat{\theta}_i\right] & \text{if } i \in \mathcal{B}. \end{cases}$$

CHARACTERIZATION. Inserting the expression of the virtual surplus (8.1) into (5.2), $q^m(\theta)$, when interior, satisfies (8.4). The condition for having an interior solution (i.e., $q_m(\theta) \geq 0$) is that

$$\sum_{i \in \mathcal{A}} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}; 0\right\} + \sum_{i \in \mathcal{B}} \min\left\{s_i + \frac{1-F(\theta)}{f(\theta)}; 0\right\} \geq \theta + C'(0) \quad \forall \theta \in \Theta.$$

Because of $MHRC$, this condition holds when

$$\sum_{i \in \mathcal{A}} \max\left\{s_i - \frac{1}{f(\overline{\theta})}; 0\right\} + \sum_{i \in \mathcal{B}} s_i \geq \overline{\theta} + C'(0)$$

and a sufficient condition is then (8.2).

Following the same steps as in Theorem 3, we thus define a set of transfers $t_i^m(q)$ as in (8.5). *Q.E.D.*

PROOF OF COROLLARY 1: Obvious from the text. *Q.E.D.*

PROOF OF COROLLARY 2: This result follows directly from an application of (8.4). Because $q^m(\theta)$ is weakly increasing in s_i (and strictly increasing in s_i for some positive measure of types), it follows that the maximal equilibrium allocation must weakly increase (and strictly so over the same measure of types). Hence, the aggregate marginal contribution schedule $T^{m'}(\overline{q})$ cannot decrease for any $q \in q^m(\Theta)$ and must strictly increase for at least some range of q that are chosen in equilibrium by the agent. Next consider the marginal payments made by principals $j \neq i$ (whose stakes have remained constant). For any region of types for which $q^m(\theta)$ is decreasing and strictly higher, it follows that $\vartheta^m(q)$ is also decreasing and strictly higher. From the marginal payment equation obtained in (8.5), $t_j^{m'}(q)$ must be lower following the change in principal i's preferences for these q. Of course, we know that $T^{m'}(q)$ is strictly higher for this q, so it follows that $t_i^{m'}(q)$ must be increase more than the reduction of $\sum_{j\neq i} t_j^{m'}(q)$. Hence, crowd out occurs, but it is less than perfect. *Q.E.D.*

Proof of Proposition 4: Recall from (8.4) that

$$q^m(\theta) = \arg\max_{q\in Q} \; S_0(q) - \theta q + \left(\sum_{i=1}^n \max\left\{s_i - \frac{F(\theta)}{f(\theta)}, 0\right\}\right) q$$

Because $\sum_{i=1}^n \max\left\{s_i - \frac{F(\theta)}{f(\theta)}, 0\right\}$ is convex in s_i, it weakly higher under $\tilde{\mathbf{s}}$ compared to $\mathbf{s}$. Define $\hat{\theta}_i$ by $s_i f(\hat{\theta}_i) = F(\hat{\theta}_i)$ and define $\tilde{\theta}_i$ by $\tilde{s}_i f(\tilde{\theta}_i) = F(\tilde{\theta}_i)$. Choose i such that $s_i < \tilde{s}_i$, and thus $\hat{\theta}_i < \tilde{\theta}_i$. Then for any $\theta \in (\hat{\theta}_i, \tilde{\theta}_i)$, the *argmax* above is strictly higher under $\tilde{\mathbf{s}}$ compared to $\mathbf{s}$. It follows that the maximal allocation under $\tilde{\mathbf{s}}$ is weakly higher than that under $\mathbf{s}$ (and it is strictly higher for some types). *Q.E.D.*

Proof of Proposition 5: Consider first the scenario with *ex ante* contracting as depicted in Section 7.3. From (7.5), the efficient allocation is

$$\text{(C.3)} \qquad \overline{q}^{ea}(\theta) \in \arg\max_{q\in Q} \left(\frac{1}{2} - \theta\right) q - \frac{q^2}{2} + b \equiv \frac{1}{2} - \theta \in [0,1], \quad \forall \theta \in \Theta.$$

Define also $\overline{q}_i(\theta)$ as the optimal action when the agent contracts only with principal i. Provided that $b > 1$, we actually have

$$\text{(C.4)} \qquad \overline{q}_1(\theta) \in \arg\max_{q\in Q} \left(\frac{1}{2} - \theta\right) q - \frac{q^2}{2} + bq \equiv 1, \quad \forall \theta \in \Theta.$$

and

$$\text{(C.5)} \qquad \overline{q}_2(\theta) \in \arg\max_{q\in Q} \left(\frac{1}{2} - \theta\right) q - \frac{q^2}{2} + b(1-q) \equiv 0, \quad \forall \theta \in \Theta.$$

From there and using symmetry, it follows that the constant $C_1 = C_2 = C^*$ which is defined in (7.6) and solves the system (7.7), writes here as

$$\text{(C.6)} \qquad \mathbb{E}_\theta \left(\frac{1}{2}\left(\frac{1}{2} - \theta\right)^2 + b\right) - 2C^* = \max\left\{\mathbb{E}_\theta\left(\frac{1}{2}\left(\frac{1}{2} - \theta\right)^2\right); \mathbb{E}_\theta\left(b - \theta\right) - C^*\right\}.$$

Because $\mathbb{E}_\theta\left(\frac{1}{2}\left(\frac{1}{2} - \theta\right)^2\right) = \frac{1}{6}$ and $b > 1$, (C.6) rewrites as

$$\text{(C.7)} \qquad b + \frac{1}{6} - 2C^* = \max\left\{\frac{1}{6}; b - C^*\right\}.$$

The solution to (C.7) is reached when the max on the right-hand side has the agent contracting with a single principal. It is thus $C^* = \frac{1}{6}$ and the agent's expected payoff under *ex ante* contracting is

$$U^{ea} = b + \frac{1}{6} - 2C^* = b - \frac{1}{6}. \tag{C.8}$$

Under *ex post* contracting, the agent's expected payoff in the maximal equilibrium (8.8) is

$$U^m = \mathbb{E}_\theta\left(\left(\frac{1}{2} - \theta\right) q^m(\theta) - \frac{(q^m(\theta))^2}{2} + t_1^m(q^m(\theta)) + t_2^m(q^m(\theta))\right). \tag{C.9}$$

Tedious computations yields

$$U^m = b - \frac{7}{12}. \tag{C.10}$$

Comparing (C.8) and (C.10) yields the result. *Q.E.D.*

Proof of Proposition 6: Our general formula (5.2) applies in this setting and the maximal equilibrium import tariff should satisfy

$$\sum_{i=1}^{2} \min\left\{S_i'(q^m(\theta)) + \frac{1-F(\theta)}{f(\theta)}; 0\right\} = \theta \tag{C.11}$$

where

$$S_i'(q) = \frac{\mathcal{S}_i(p+\gamma(q))}{(1+\lambda)\Gamma(q)\mathcal{M}'(p+\gamma(q)))} < 0 \text{ for } i = 1, 2. \tag{C.12}$$

For θ close enough to zero, we have

$$\min\left\{S_i'(q^m(\theta)) + \frac{1-F(\theta)}{f(\theta)}; 0\right\} < 0.$$

This condition means that both principals are active on such neighborhood. Item 1. follows.

Inserting (C.12) into (C.11) yields

$$\gamma^m(\theta) = \frac{\mathcal{S}(p+\gamma^m(\theta))}{(1+\lambda)\left(\theta - 2\frac{1-F(\theta)}{f(\theta)}\right)(\mathcal{D}'(p+\gamma^m(\theta)) - \mathcal{S}'(p+\gamma^m(\theta)))}.$$

Setting $\gamma^m(\theta) = \Gamma(q^m(\theta))$ and manipulating yields (8.10).

Item 2. follows from observing that $S_1'(q) < S_2'(q) < 0$ and thus the activity sets for principal i are necessarily nested, i.e., $\Omega_2^{cm} \subset \Omega_1^{cm}$. Because the right-hand sides in (8.10) and (8.11) are non zero, group 1 is always active with $\Omega_1^{cm} = \Theta$. Using (C.11) when only group 1 is active yields

$$\gamma^m(\theta) = \frac{\mathcal{S}_1(p+\gamma^m(\theta))}{(1+\lambda)\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)(\mathcal{D}'(p+\gamma^m(\theta)) - \mathcal{S}'(p+\gamma^m(\theta)))}. \tag{C.13}$$

After manipulations, we obtain Condition (8.11). Finally, observe that $\gamma^m(\theta)$ as defined in (C.13) is always positive so that group 1 is actually always active. *Q.E.D.*