WORKING PAPERS

N° 1606

December 2024

"Dynamic Consumer Search"

Alexei Parakhonyak and Andrew Rhodes

Toulouse School of Economics

Dynamic Consumer Search *

Alexei Parakhonyak† Andrew Rhodes‡

December 20, 2024

Abstract

We consider a model in which consumers wish to buy a product repeatedly over time, but need to engage in costly search to learn prices and find a product that matches them well. The optimal search rule has two reservation values, one for newly-searched products, and another for products that were searched in the past. Depending on the search cost, firms either keep price steady over time, or gradually raise price to take advantage of a growing pool of high-valuation repeat customers. The model generates rich search and purchase dynamics, as consumers may optimally "stagger" search over time, initially trying different products, settling on one and buying it for a while, before choosing to search again for something better. We also show that consumers may be better off when firms can offer personalized prices based on their search history.

Keywords: Consumer search, repeat purchases, price dispersion, turnover.

JEL codes: D43, D83, L13

*We thank Simon Anderson, Heski Bar-Isaac, Joyee Deb, Laura Doval, Bruno Jullien, Jeanine Miklós-Thal, Volker Nocke, Régis Renault, Patrick Rey, Larry Samuelson, Wilfried Sand-Zantman, Sandro Shelegia, Chris Wilson and Jidong Zhou, as well as audiences at Cambridge, DICE, Frankfurt, Helsinki, Karlsruhe, Louvain, Oxford, Paris School of Economics, Toulouse School of Economics, Vrije University, Yale SOM, APIOC (Hong Kong), Bristol Search and Matching workshop, CEPR Virtual IO Seminar, Consumer Search Workshop (Barcelona), Durham Economic Theory Conference, EARIE (Athens), ESWC (Bocconi), IIOC (Boston), Market Studies and Spatial Economics Workshop (Moscow), and Oligo Workshop (Nicosia). Rhodes acknowledges funding from the French National Research Agency (ANR) under the Investments for the Future (Investissements d'Avenir) program (grant ANR-17-EURE-0010) and funding from the European Union (ERC, DMPDE, grant 101088307). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

†Department of Economics, University of Oxford. Email: aleksei.parakhonyak@economics.ox.ac.uk.

‡Toulouse School of Economics. Email: andrew.rhodes@tse-fr.eu.

1 Introduction

Consumers are often poorly informed about the different products that are available in a market, but can discover new ones through costly search.[1] At the same time, in many product markets consumers need to make repeated purchases—consider, for example, apparel and clothing products, health and beauty items, or certain financial and insurance services. Each time consumers wish to make a purchase, they face a choice between either buying a product they already know, or searching for something better. The theoretical search literature has mainly focused on settings where consumers buy only once. However, once we account for repeat purchases, many new and interesting questions arise. For example, is it better for consumers to search thoroughly early on, find a good product and then buy it repeatedly, or should they "stagger" search over time, only gradually learning which product fits them best? How should firms update their prices over time, taking into account consumers' dynamic search strategies? And if firms can identify their repeat customers and offer them personalized prices, how does this affect market performance?

To address these questions, in Section 2 we provide a tractable discrete-time infinite-horizon search model. Each period there is a continuum of horizontally differentiated products, whose match values are drawn IID across consumers. There is turnover on both sides of the market: at the end of each period, consumers and products exit (and are replaced) with some exogenous probability. Consumers wish to buy one product each period, and their match with any given product is constant over time. However, when a consumer first enters the market she is uninformed about these matches, and must search in order to learn them. Each period a consumer can search in two different ways. Firstly, at a relatively high cost $s > 0$, a consumer can sequentially search products she has not looked at before, whereupon she learns her match and the price. Secondly, at a potentially lower cost $r \in (0, s]$, a consumer can return to a product she searched in a previous period (and therefore whose match value she already knows), whereupon she learns the price. Consumers can search as many times as they like each period. We focus throughout on steady state outcomes.

To build some intuition for our main result, we first solve a special case of the model in Section 3 with no product turnover. We look for a stationary symmetric equilibrium where firms charge the same price each period. First, we show that a consumer's optimal search rule has two thresholds: a consumer who searches a product for the first time buys it if her match exceeds a threshold a, while a consumer returns to a product she searched and bought in the past if her match exceeds a higher threshold $b > a$. Intuitively, consumers who have just searched a product have sunk their search cost s, whereas consumers who are contemplating returning need to incur the cost r and hence employ a higher threshold. Because $b > a$ the

[1] Recent empirical work documents the importance of search frictions across a wide range of markets. See, e.g., Honka et al. (2017), Jolivet and Turon (2019), Moraga-González et al. (2023) and Ershov (2024).

model generates endogenous customer turnover—some consumers buy a product only once and then next period search for something better, i.e., they optimally stagger search over time. Second, turning to firms' pricing, we show that relative to the case where consumers buy once and then exit the market, repeat purchases have two countervailing effects on the equilibrium price. On the one hand, because consumers anticipate buying a product multiple times, when they first enter the market they search more intensively. This induces firms to compete more aggressively. On the other hand, once consumers have found a well-matched product, they return and buy it repeatedly. This gives firms market power, so they compete less aggressively. We show that in general either force can dominate.

In Section 4 we introduce product turnover. We derive a stationary equilibrium in which, conditional on surviving, firms (weakly) raise their price as they become older. Intuitively, older products have been searched by more consumers in the past, so they have a larger pool of (high-valuation) repeat customers, which makes their demand less price elastic. Since prices go up over time, this deters some consumers from going back and buying a product they bought previously—thus creating an additional source of endogenous customer turnover. Our model generates rich patterns of consumer search and purchase behavior. For example, early in their lifetime, some consumers end up buying a different product each period, because during the course of their search they find a product with a good but not very good match. Eventually these consumers find a product with a very good match, so they stop searching and buy it repeatedly. However, if its price gradually increases over time, at some point they may stop returning and instead search for a new product. At the end of Section 4 we also discuss some empirical research that is consistent with the findings of our model.

Finally, Section 5 provides extensions and robustness checks. For instance, in our main analysis we assume that firms cannot price discriminate, but in this section we allow firms to offer consumers personalized prices based on their search history (e.g., via cookies). We show, for example, that even though firms can use very fine-tuned price discrimination, in equilibrium they charge at most two different prices.

Related literature There is a large literature on consumer search, but it overwhelmingly focuses on "one-shot" settings where consumers wish to buy a product only once. One strand of this literature looks at price search for homogeneous products, while the other strand looks at match search for differentiated products. Our paper follows the latter strand, building on work by Wolinsky (1986) and Anderson and Renault (1999).[2] Among papers with one-shot search, the closest to ours is Anderson and Renault (2000). They consider a duopoly model in which some consumers are exogenously informed about which product gives them the highest match, while other consumers must search sequentially to acquire this information. They

[2] It is often assumed that consumers can costlessly return to firms they have already searched. In our dynamic model we follow Janssen and Parakhonyak (2014) and assume a (possibly very small) return cost.

show that the equilibrium price increases in the fraction of informed consumers. The reason is that in symmetric equilibrium informed consumers buy their favorite product without search, which makes them less price sensitive than uninformed consumers. In our model firms also face a mix of price-elastic consumers who are looking for a match, and price-inelastic consumers who come back because they already know they like the product. The difference is that in our model the share of these consumers is determined endogenously, as the outcome of a repeated search and pricing game. Moreover, a firm's (relative) share of each group of consumers varies according to how long it has been in the market.[3]

Only a few papers explicitly model consumer search with repeated purchases. Most of these papers consider a setting with homogeneous products, where consumers can costlessly return to the last firm they bought from. Stigler (1961) argued that if firms' prices are correlated over time, consumers should search intensively at the beginning and then return repeatedly to the same firm. McMillan and Morgan (1988) and Benabou (1993) consider discrete-time models with sequential search. They both derive equilibria in which firms keep price constant over time, and consumers do indeed repeatedly buy from the same firm. Burdett and Coles (1997) meanwhile consider a continuous time model with noisy search. Closer to us, they derive an equilibrium where firms initially price below cost, and then raise price over time. However, in their model consumers also repeatedly buy from the same firm.[4] To the best of our knowledge, the only paper with product differentiation and repeated purchases is the working paper by Chen et al. (2019). Like us, they find that repeat purchases have countervailing effects on equilibrium, giving consumers greater incentives to search, but also making demand less elastic due to returning consumers. However, different from us, firms in their model keep price constant over time, and there is no endogenous customer turnover. The reason is that they assume all firms have the same age, and it is costless for consumers to return to a firm they previously searched.

In our model firms of different ages may charge different prices, even though ex ante they are symmetric. In this sense we relate to the literature on price dispersion for homogeneous goods, where dispersion typically arises in mixed strategies, due to heterogeneity in either consumer information (e.g., Varian, 1980) or search costs (e.g., Stahl, 1989).[5] This contrasts with our model, where dispersion is in pure strategies, and arises because consumers' dynamic search strategies result in firms of different ages facing different demands.

[3]In our model, older firms are more "prominent" in the sense that more consumers know they have a high match with their product and directly search them; older firms also charge (weakly) higher prices. This differs from static search models with product differentiation, where more prominent firms tend to set lower prices (see, e.g., Armstrong et al., 2009). See also, e.g., Choi et al. (2018) and Haan et al. (2018) for models in which consumers have (partial) match information and use it to inform their search decisions.

[4]In these papers firm production costs are steady over time. In a model where firms face cost shocks, and consumers have heterogeneous search costs, Fishman and Rob (1995) show that there can exist equilibria where some consumers react to high prices by not returning to the firm they last purchased from.

[5]For a recent exception where price dispersion arises in pure strategies, see Myatt and Ronayne (2024).

We also contribute to a growing literature on price discrimination in consumer search markets. In one strand of this literature firms sell homogeneous products, and firms observe something that conveys information about a consumer's ability or willingness to search; examples include demand size (e.g., Fabra and Reguant, 2020), search duration (e.g., Mauring and Williams, 2024), or sales channel (e.g., Ronayne and Taylor, 2021). In another strand of the literature products are differentiated, and firms observe a signal of how informed a consumer is (e.g., Preuss, 2023), or directly observe match information (e.g., Groh and von Wangenheim, 2024), or observe whether a consumer has already searched them (e.g., Armstrong and Zhou, 2016). For example, in Armstrong and Zhou (2016) consumers buy a good only once, but firms can discriminate based on whether a consumer buys immediately or buys later after sampling other products in the market. Like us, they show that the latter "return price" is relatively high, but the underlying mechanism is very different—in their model firms can commit to prices, and a high return price is used to deter search.[6]

Our paper also relates to wider research where consumers make repeated purchases and face other (non-search) frictions. One example is the literature on switching costs (see, e.g., Farrell and Klemperer, 2007 for a survey). Here consumers have full information about product matches and prices, but switching products is costly. This creates inertia in consumer purchases, even as tastes or prices change. It also leads to "bargains-then-ripoffs" pricing, as firms initially price low to invest in market share, then raise price to harvest loyal customers. In contrast, in our model consumers can switch freely, but exhibit inertia because it is costly to search and find something better. The reason why firms in our model raise price over time is also very different—as we show later, marginal consumers in our model do not return and buy again, so there is no counterpart to the investment effect from models of switching costs. Another related literature is that on experience goods. Here consumers can only learn a product's match value by consuming it. This also creates inertia, because once a consumer finds a well-matched product she prefers to buy it rather than experiment and risk consuming a bad product. Depending on the taste distributions, firms may either increase or decrease prices over time (see, e.g., Bergemann and Välimäki, 2006). In contrast, in our model prices always (weakly) increase over time. The reason is that consumers know their match before consuming a product, and only buy if that match is relatively high; older products have a larger pool of well-matched past customers, and so are more expensive.[7]

[6] Our analysis of price discrimination is also related to the (non-search) literature on personalized pricing (see, e.g., Rhodes and Zhou, 2024) and behavior-based pricing (see, e.g., Fudenberg and Villas-Boas, 2007).

[7] We note that search costs are usually treated separately from switching costs and experience goods. One exception is Wilson (2012), who allows for both search and switching costs, but does not endogenize the initial choice of which firm a given consumer is locked to. Another exception is Chen et al. (2022), who allow for both search costs and experience goods, but consumers in their model only buy once and the dynamic linkage between different periods comes from consumer reviews.

2 Model

Time is discrete and runs from $-\infty$ to $+\infty$. In each period there is a continuum of differentiated products. Each of these products is supplied by a continuum of undifferentiated single-product firms. The marginal cost of each firm is normalized to zero. In each period there is also a measure one of consumers per firm. Each consumer wishes to purchase one of the products, and has an outside option which is normalized to zero. Following Anderson and Renault (2000), a consumer's utility when purchasing a given product is

$$\varepsilon + u(p) \quad \text{where} \quad u(p) \equiv \int_p^\infty Q(x)dx, \tag{1}$$

and where ε is the *product*-specific match value, p is the price paid and $Q(p)$ is the amount purchased. (Thus if two firms supplying the same product charge the same price, they offer the same utility to a consumer.) The match ε is drawn independently (across *products* and consumers) from a distribution function $F(\varepsilon)$ whose support is $[\underline{\varepsilon}, \overline{\varepsilon}]$, with $\underline{\varepsilon} \geq 0$; the density function $f(\varepsilon)$ is strictly positive and has a strictly increasing hazard rate. The quantity purchased $Q(p)$ is decreasing, continuously differentiable, and logconcave in p. A firm's profit when selling to a measure one of consumers at price p is denoted by $\pi(p)$, where

$$\pi(p) \equiv pQ(p). \tag{2}$$

Logconcavity of $Q(p)$ implies that $\pi(p)$ is quasiconcave and maximized at $p^m = \arg\max_p \pi(p)$.

There is exogenous turnover on both sides of the market. Let $\gamma_f > 0$ be the probability that a product survives (along with the firms that supply it) to the next period. With complementary probability $1 - \gamma_f$ a product exits the market at the end of a period and is replaced by a new product (and a new set of suppliers). Let $0 < \gamma_c < 1$ be the probability that a consumer survives to the next period; when a consumer survives, her valuation for each product in the market remains unchanged. With probability $1 - \gamma_c$ a consumer exits the market at the end of a period and is replaced by a new consumer.

When a consumer first enters the market she has no information about how much she values each product, nor the price charged by each seller. In each period a consumer can search in two different ways. Firstly, at cost $s > 0$, a consumer can search sequentially among products that she has not yet visited during her lifetime. Each time she pays s, she learns (i) her match with a randomly-drawn product, (ii) the price charged by a randomly-drawn seller of that product, and (iii) the product's age and which other firms sell it. Secondly, at cost r, where $0 < r \leq s$, a consumer can return to a product she has searched before, and learn the price charged by a randomly-drawn seller of that product. (We assume that a consumer knows whether or not a previously-searched product is still in the market.) We impose the following tie-break rules: if a consumer is indifferent between returning or searching, she returns; if she is indifferent between buying or searching, she buys.

The timing is as follows. At the start of each period, new consumers and firms enter. Firms simultaneously set prices; we assume that when doing this, firms do not observe other firms' past prices.[8] Consumers then incur search and return costs to visit firms, before either taking their zero outside option or buying from one of the firms visited in that period. At the end of the period, some consumers and firms exit. Consumers and firms discount future payoffs using discount factors $\delta_c < 1$ and $\delta_f < 1$ respectively. Throughout the paper we focus on steady state outcomes.

Discussion We briefly discuss some of our assumptions.

No price discrimination. We assume that each firm charges a single price. In Section 5.1 we allow firms to condition price on a consumer's search history.

Downward-sloping demand. We assume that consumers buy more of a given product when its price is lower. In Section 5.2 we show that our main insights extend to the case where consumers have unit demand, although the equilibrium is less rich.

Continuum of undifferentiated sellers. To simplify the analysis, we assume that each product has a continuum of differentiated sellers. In Section 5.3 we show how our main insights can be extended to the case where each product has a finite $n \geq 1$ undifferentiated sellers.

Learning product age. We assume that when a consumer searches a product she learns its age. Without this assumption, consumers might use a firm's price to infer its product's age—and hence also infer future prices for that product. This would introduce new effects into the analysis which we do not believe are of first-order importance.

Search cost s vs. return cost r. Our analysis holds when $s = r$, but we believe it is useful to allow for distinct search and return costs. Note that we can then interpret $r > 0$ as the (potentially very small) cost of traveling to a store or website to pick up a product, and $s - r \geq 0$ as the extra cost needed to inspect a new product and learn one's valuation for it.

3 A Special Case

To illustrate some of the trade-offs we first solve a special case of our model in which (i) $\gamma_f = 1$, meaning there is no product turnover, and (ii) $s < \mathbb{E}\varepsilon - \underline{\varepsilon}$, which will ensure that each period some consumers search more than one firm. We look for a stationary equilibrium where firms charge p^* every period and consumers use a stationary search rule.

[8] This rules out collusive equilibria. The reason is that price deviations would neither be observed by other firms (by assumption) nor be inferred by other firms (since the deviating firm is atomless, and so would have zero effect on its rivals' profits)—and therefore such deviations could not be punished.

3.1 Optimal Search Rule

When a consumer first enters the market she searches at least once, because the expected payoff from doing so is positive: $\mathbb{E}\varepsilon + u(p^*) - s \geq \mathbb{E}\varepsilon - s > \underline{\varepsilon} \geq 0$. Our first result characterizes her subsequent optimal search behavior. (All omitted proofs are in the Appendix.)

Lemma 1. *An optimal stationary search rule consists of a pair of numbers* (a, b) *satisfying* $\underline{\varepsilon} < a < b < \overline{\varepsilon}$*, such that in each period:*

1. *A consumer returns to and buys a product she encountered in a previous period (if any) with the highest* ε*, provided* $\varepsilon \geq b$*.*
2. *Otherwise, she searches until she finds a product with* $\varepsilon \geq a$ *and buys it.*

When deciding whether or not to buy a product, a consumer compares her match ε with one of two thresholds—either a, if she has just searched the product for the first time, or b, if she searched the product in an earlier period. Since $b > a$, a consumer is choosier about a product she searched in the past: intuitively, a consumer needs to incur $r > 0$ to buy such a product, whereas for a product she has just searched the cost s has already been sunk.

Lemma 1 implies that a consumer does not necessarily buy the same product in each period. Specifically, if a consumer discovers a product with $\varepsilon \in [a, b)$ during her first period, she buys it, but in the second period (conditional on surviving) she searches for a new product. Then, if during her second period she again discovers a product with $\varepsilon \in [a, b)$, she buys it once but does not buy it again in the future. Only when the consumer discovers a product with $\varepsilon \geq b$ does she permanently stop searching and buy it repeatedly thereafter. Consequently, the wedge between a and b generates *endogenous consumer turnover*, whereby some consumers buy a product only once and then look for something better.[9]

We now derive implicit expressions for the thresholds a and b. Let V denote the value to a consumer of searching for a new product. We can then write

$$V = -s + F(a)V + \int_a^b [\varepsilon + u(p^*) + \delta_c\gamma_c V] dF(\varepsilon) + \int_b^{\overline{\varepsilon}} \left[\varepsilon + u(p^*) + \sum_{k=1}^{\infty} (\delta_c\gamma_c)^k [\varepsilon + u(p^*) - r]\right] dF(\varepsilon). \quad (3)$$

This is explained as follows.[10] If a consumer searches and draws a product with match value ε below a, she searches again and obtains V. If instead she draws a product with match value between a and b, she buys the product once but then (conditional on surviving) searches

[9] This contrasts with exogenous turnover caused by $1 - \gamma_c$ consumers leaving the market each period.

[10] To write V in this way we require $V \geq 0$, which is true given our assumption that $s < \mathbb{E}\varepsilon - \underline{\varepsilon}$.

again in the next period and so gets V. Finally, if she draws a product with match value above b, she returns and buys it every period until she exits the market. We can also write

$$V = a + u(p^*) + \delta_c \gamma_c V, \tag{4}$$

because $a < b$ implies that a consumer who searches and draws a product with $\varepsilon = a$ is indifferent between immediately searching again, or buying in the current period and then (conditional on surviving) searching again in the next period. Similarly

$$V = \sum_{k=0}^{\infty} (\delta_c \gamma_c)^k \left[b + u(p^*) - r \right] \quad \Leftrightarrow \quad V = b + u(p^*) - r + \delta_c \gamma_c V, \tag{5}$$

because if the best product a consumer found in a previous period has match value $\varepsilon = b$, she is indifferent between searching for a new match or returning and buying that product repeatedly until she exits the market. Combining these three equations, we find that:

Lemma 2. *There is a unique stationary search rule. In this rule $b = a + r$, and $a > \underline{\varepsilon}$ solves*

$$s = \int_a^{\overline{\varepsilon}} (\varepsilon - a) dF(\varepsilon) + \frac{\delta_c \gamma_c}{1 - \delta_c \gamma_c} \int_{a+r}^{\overline{\varepsilon}} (\varepsilon - a - r) dF(\varepsilon). \tag{6}$$

Moreover a and b increase in $\delta_c \gamma_c$ and decrease in s; a decreases in r, but b increases in r.

The search rule has several natural properties. The gap between a and b equals the return cost, because a consumer who considers going back to purchase a product must pay r whereas a consumer who has just searched a product can buy it at no extra cost. The assumption $s < \mathbb{E}\varepsilon - \underline{\varepsilon}$ ensures that $a > \underline{\varepsilon}$, meaning that consumers prefer to search for a new product whenever they encounter a sufficiently low match. Consumers also become choosier when it costs less to search new products (s is lower), and when they put more weight on the future benefits of finding a good match today ($\delta_c \gamma_c$ is higher). Hence consumers become more choosy when they expect to buy a product more in the future. (Indeed, when $\gamma_c = 0$, the a that solves equation (6) is the same as the threshold in a standard Kohn and Shavell, 1974 one-shot search problem.) Finally, as r increases, consumers become choosier about which products to return to (i.e., b goes up), but the value of search also falls, so they become less choosy about products they have just searched for the first time (i.e., a goes down).

In what follows it will be useful to distinguish between *fresh* consumers, who search for a new product, and *return* consumers, who return to and buy a product they searched in the past. Consumers are fresh either because this is their first period in the market (exogenous turnover), or because in the previous period they were fresh and ended up buying a product with match value between a and b and have survived to the current period (endogenous turnover). Therefore letting m be the measure of fresh consumers per firm, we can write

$$m = 1 - \gamma_c + \gamma_c m \frac{F(b) - F(a)}{1 - F(a)} \implies m = \frac{1 - \gamma_c}{1 - \gamma_c[F(b) - F(a)]/[1 - F(a)]}. \tag{7}$$

Since a fresh consumer searches until finding a match value above a, on average she will search $1/[1-F(a)]$ times. Therefore the measure of fresh consumers that search a given firm is $m/[1 - F(a)]$. The measure of return consumers per firm is simply equal to $1 - m$.

3.2 Equilibrium Pricing

We look for an equilibrium where firms charge p^* in each period. Since there is a measure one of consumers per firm, along the equilibrium path a firm earns $\pi(p^*)$ in each period.

We now derive a firm's discounted profit off the equilibrium path. Suppose that at time t firm j makes a one-shot deviation and charges a price $p_j \neq p^*$. Assume "passive beliefs", meaning that any consumer who observes firm j's deviation still expects all other firms to be charging p^* in period t. We begin with the following simple result.

Lemma 3. *Suppose all firms charge p^* in each period along the equilibrium path. Suppose at time t firm j makes a one-shot deviation and charges $p_j \neq p^*$. Then from period $t+1$ onwards all firms optimally charge p^* and earn $\pi(p^*)$ each period.*

To understand Lemma 3, note that only consumers who visit firm j during period t observe its deviation—and they have zero measure. All other consumers do not observe the deviation, and so automatically expect all firms to charge p^* from period $t+1$ onwards. Moreover, by assumption, in each period consumers visit a random supplier of any given product. Consequently, up to differences of measure zero, starting in period $t+1$ every firm is visited by the same fresh and return consumers—and so faces the same optimization problem—as would have been the case absent firm j's deviation. Therefore, given the supposition that p^* is an equilibrium price, from period $t+1$ onwards each firm (including firm j) optimally charges p^* and earns $\pi(p^*)$.[11]

Lemma 4. *Suppose at time t firm j makes a one-shot deviation and charges a price p_j which satisfies $u(p_j) \geq u(p^*) - r$. The firm's discounted profit is*

$$\pi(p_j)\left[1 - m + m\frac{1 - F(a + u(p^*) - u(p_j))}{1 - F(a)}\right] + \frac{\delta_f \pi(p^*)}{1 - \delta_f}. \tag{8}$$

Lemma 4 looks at the case where firm j either reduces its price, so that $u(p_j) > u(p^*)$, or increases its price by a relatively small amount, so that $u(p_j) < u(p^*)$ but $u(p_j) \geq u(p^*) - r$. The first term in equation (8) is firm j's profit in period t, and is explained as follows.

[11] Note that this argument uses our assumption that firm j's deviation is not observed by other firms.

Consumers only observe firm j's price after visiting it, and so its deviation has no effect on which consumers come to it. Therefore, as explained earlier, firm j is visited by a measure $1-m$ of return consumers and a measure $m/[1-F(a)]$ of fresh consumers. Return consumers all buy. Intuitively, return consumers have a relatively high match, $\varepsilon \geq b$, and so prefer to buy rather than search for another product. Moreover, since firm j has either reduced its price or not increased it too much, return consumers do not find it worthwhile to visit another seller of the same product. Fresh consumers buy if and only if $\varepsilon \geq a+u(p^*)-u(p_j)$. Intuitively, fresh consumers only learn their match after searching firm j, and they compare it with the "on-path" reservation value a, adjusted for the change in utility $u(p_j)-u(p^*)$ offered by firm j as a result of its deviation. Moreover, each consumer who buys generates a profit $\pi(p_j)$. Finally, the second term in equation (8) is firm j's discounted profit from period $t+1$ onwards, and follows directly from Lemma 3.

Remark 1. *At $p_j = p^*$, firm j's fresh demand is more price-sensitive than its return demand.*

As explained above, a small increase in p_j around p^* does not affect how many return consumers buy from firm j, but does reduce the number of fresh consumers that buy. Hence fresh demand is (locally) more elastic than return demand. (Recall that since demand is downward-sloping, conditional on buying, all consumers buy less when p_j is higher.)

Continuing with the derivation of firm j's deviation profit, we have the following result.

Lemma 5. *Suppose at time t firm j makes a one-shot deviation and charges a price p_j which satisfies $u(p_j) < u(p^*) - r$. The firm's discounted profit is $\delta_f \pi(p^*)/(1-\delta_f)$.*

Lemma 5 looks at the case where firm j raises its price by a relatively large amount. Intuitively, in this case any consumer who visits the firm in period t would rather incur r and buy j's product from one of the other sellers at the lower price of p^*. Hence in period t the firm makes zero sales. However, by Lemma 3, from period $t+1$ onwards firm j returns to the equilibrium path and earns $\pi(p^*)$ each period.

Using the above analysis, we can now derive the equilibrium price. Since fresh and return demand have different elasticities, stronger conditions than usual are required to ensure that a firm's maximization problem is well-behaved. We therefore make the following assumption:

Assumption 1. *For all $\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]$ the following holds:*

$$\gamma_c \leq -\frac{d}{d\varepsilon}\left(\frac{1-F(\varepsilon)}{f(\varepsilon)}\right) \equiv 1 + \frac{f'(\varepsilon)[1-F(\varepsilon)]}{f(\varepsilon)^2}. \tag{9}$$

Note that since we assume $f(\varepsilon)$ has a strictly increasing hazard rate, there exist values of γ_c sufficiently low that Assumption 1 holds. Indeed, in the case where $f'(\varepsilon) \geq 0$, the assumption is satisfied for all values of γ_c. It is convenient to define

$$\Psi(p) = -\frac{\pi(p)u'(p)}{\pi'(p)}, \tag{10}$$

which (as we show in Lemma A1 in the Appendix) is strictly increasing for all $p \in [0, p^m)$.

Proposition 1. *The unique stationary symmetric equilibrium price is*

$$p^* = \Psi^{-1}\left[\frac{1-F(a)}{mf(a)}\right]. \tag{11}$$

Compared to the canonical setting in which consumers are in the market for a single period (which in our model is equivalent to $\gamma_c = 0$), repeat purchases have countervailing effects on the equilibrium price. On the one hand, as explained earlier, fresh consumers are choosier: a is higher, which from equation (11) leads to a lower p^* due to the assumption of an increasing hazard rate. Intuitively, firms compete more aggressively when fresh consumers search more. On the other hand, not all consumers are fresh: $m = 1$ in the canonical model, but $m < 1$ with repeat purchases. From equation (11) this leads to a higher p^*. Intuitively, firms compete less aggressively when part of their demand comes from consumers who visit because they have a high match value. Depending on parameters, either effect can dominate. Figure 1 illustrates this for the case where $Q(p) = 1 - p$, $F(\varepsilon) = \varepsilon(1 + 4\varepsilon)/5$, $s = 0.5$ and $r = 0.05$. The dotted line depicts the standard one-shot Wolinsky-Anderson-Renault price. The red dashed line depicts outcomes when $\delta_c = 0.2$: consumers are relatively impatient, so the search effect is relatively weak, and hence repeat purchases lead to a higher equilibrium price. The blue solid line, meanwhile, depicts outcomes when $\delta_c = 0.8$: consumers are relatively patient, so the search effect is stronger, and hence equilibrium price is lower for a wide range of γ_c compared to the case without repeat purchases.[12]

4 The General Case

We now allow for (i) any probability $\gamma_f \in (0, 1)$ that a product survives from one period to the next, and (ii) any search cost s and return cost r satisfying $s \geq r > 0$. We assume throughout that Assumption 1 holds. We index products (and the firms that sell them) by their age $i = 0, 1, 2, \ldots$, that is the number of periods they have been in the market. Since we focus on steady state, the distribution of product ages in each period is stationary.

We look for a symmetric stationary equilibrium with increasing prices, which we formally define as follows:

[12] Note that irrespective of δ_c, as γ_c approaches 1 the measure of fresh consumers per firm vanishes, and so the equilibrium price approaches $p^m = 1/2$ (which strictly exceeds the Wolinsky-Anderson-Renault price).

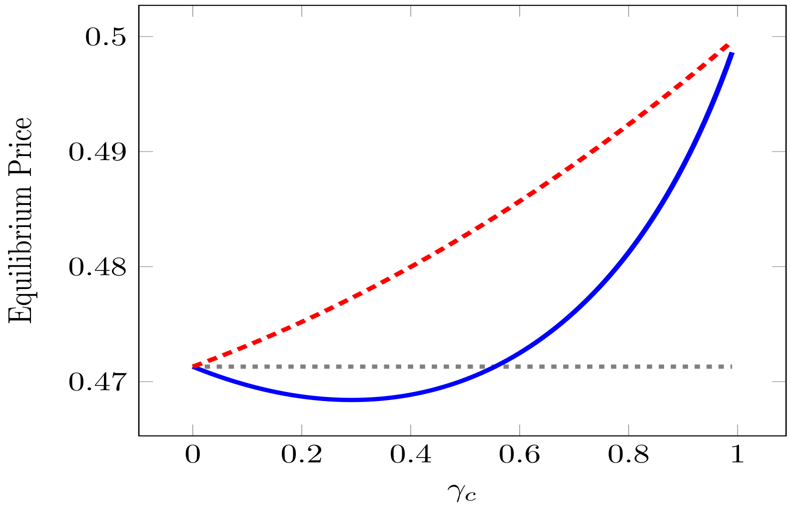


Figure 1: The effect of repeat purchases on equilibrium price for low δ_c (red dashed line) and high δ_c (blue solid line), compared to the equilibrium of a one-shot model (dotted line).

Definition 1. *In a symmetric stationary equilibrium with increasing prices:*

1. *In each period, all firms whose product has age i charge the same price p_i^*, and*

2. *Firms with older products set weakly higher prices, i.e., $p_i^* \leq p_{i+1}^*$ for all $i = 0, 1, 2, \ldots$*

In the Online Appendix we show that with myopic consumers any symmetric stationary equilibrium must take this form. Although we have not been able to prove the same result for an arbitrary δ_c, we have also not been able to construct any equilibrium where price decreases at one or more ages.

We now proceed as follows. Section 4.1 fixes an arbitrary sequence of weakly increasing prices and solves for a consumer's optimal search rule. Section 4.2 fixes an arbitrary consumer search rule and solves for firms' optimal prices. Section 4.3 then combines the two and solves for equilibrium (in Theorem 1) while Section 4.4 does comparative statics. Finally, Section 4.5 discusses empirical evidence consistent with our results.

4.1 Consumer Search Rule

In this subsection we fix an arbitrary sequence of (weakly increasing) putative equilibrium prices $p_0, p_1, p_2, \ldots$ and solve for a consumer's search rule. We assume that r and s are sufficiently small that a consumer prefers to search rather than not participate in the market. (When we later solve for equilibrium prices, we will derive the critical values of r and s required for this.) Call $\varepsilon + u(p)$ the *surplus* offered by a firm with match value ε that charges price p. We then have the following result:

Lemma 6. *Fix a sequence of weakly increasing prices. An optimal stationary search rule consists of a pair of numbers* $(\overline{a}, \overline{b})$ *satisfying* $\overline{a} < \overline{b}$*, such that in each period:*

1. *A consumer returns to and buys a product she encountered in a previous period (if any) with the highest surplus, provided its surplus exceeds* $\overline{b}$*.*

2. *Otherwise, she searches until she finds a product with surplus above* $\overline{a}$ *and buys it.*

Closely following the simple case, Lemma 6 shows that an optimal search rule consists of two thresholds $\overline{a}$ and $\overline{b}$, and that consumers are choosier towards products they searched in previous periods. A simple corollary of the above lemma is the following:

Corollary 1. *If a consumer returns to a previously-searched product, it must be the product she bought in the previous period.*

Corollary 1 is explained as follows. Because sellers (weakly) increase their prices over time, a consumer's surplus from buying each product (weakly) decreases over time. Consequently, if it was not optimal for a consumer to buy a particular product in the past, it will not be optimal for her to return and buy it in the current period either.

The model suggests four reasons why consumers may *not* return and buy the product that they bought last period. Either (i) the consumer and/or (ii) the product she last bought has exited the market. Otherwise, both the consumer and product still exist, but returning to buy the product again is not optimal either (iii) due to the return cost r or (iv) because the product's price has increased (or both). Analogous to the simple case, the first two are sources of *exogenous* turnover, and the last two are sources of *endogenous* customer turnover.

Consumer search is therefore much richer than in the simple case studied earlier. In particular, in the simple case, once the consumer finds a product with a sufficiently high match she returns to it in each subsequent period. However, in this general case, the consumer may buy a product for a few periods but then decide to search for something else. This happens when the product's surplus is initially above $\overline{b}$, but then its price increases over time such that eventually the surplus it offers to the consumer drops below $\overline{b}$.

We now provide implicit expressions for $\overline{a}$ and $\overline{b}$. Conceptually, these thresholds are derived in the same way as in the simple case. However, since consumer search behavior is richer, the derivation is more involved, and so we relegate the details to the Appendix.

Lemma 7. *Fix a sequence of weakly increasing prices. There is a unique stationary search rule. In this rule* $\overline{b} = \overline{a} + r$*, and* $\overline{a}$ *solves*

$$s = \sum_{i=0}^{\infty}(1-\gamma_f)\gamma_f^i \Bigg[\int_{\overline{a}-u(p_i)}^{\overline{\varepsilon}} (\varepsilon - \overline{a} + u(p_i))dF(\varepsilon) + \sum_{j=1}^{\infty}(\delta_c\gamma_c\gamma_f)^j \int_{\overline{a}-u(p_{i+j})+r}^{\overline{\varepsilon}} (\varepsilon - \overline{a} + u(p_{i+j}) - r)dF(\varepsilon) \Bigg]. \tag{12}$$

As in the simple case, the wedge between $\overline{a}$ and $\overline{b}$ is exactly equal to the return cost r.[13] To understand equation (12), notice that when a consumer searches she draws a product of age i with probability $(1-\gamma_f)\gamma_f^i$.[14] The first term in square brackets represents the current-period gain from finding a product whose surplus exceeds $\overline{a}$, while the second term in square-brackets represents the future gains if the product searched today gives surplus that exceeds $\overline{b}$ in each of the following j periods, for $j = 1, 2, \dots$

For the subsequent analysis of firm pricing, it is convenient to introduce the notation

$$\hat{a}_i = \overline{a} - u(p_i) \quad \text{and} \quad \hat{b}_i = \overline{b} - u(p_i), \tag{13}$$

where the assumption of price (weakly) increasing with age implies that $\hat{a}_0 \leq \hat{a}_1 \leq \dots$ and $\hat{b}_1 \leq \hat{b}_2 \leq \dots$[15] Combining the earlier results from this subsection, we have that:

Corollary 2. *Fix a sequence of weakly increasing prices. The optimal stationary search rule consists of a sequence of thresholds* $\{\hat{a}_i\}_{i=0}^{\infty}$*, and associated thresholds* $\hat{b}_i = \hat{a}_i + r$*, such that:*

1. *In each period, a consumer returns to and buys the product she bought last period (if any) provided its match exceeds* $\hat{b}_i$*, where* i *is that product's current age.*

2. *Otherwise, she searches until she finds a product with match above* $\hat{a}_i$*, where* i *is the product's age, and buys it.*

4.2 Firm Pricing

In this subsection we fix an arbitrary sequence of weakly increasing search thresholds $\{a_i\}_{i=0}^{\infty}$ (and the associated thresholds $b_i = a_i + r$), and solve for the corresponding optimal prices $\{\hat{p}_i\}_{i=0}^{\infty}$. Precisely, $\hat{p}_i$ is the optimal price charged by a firm of age i, given the arbitrary sequence of search thresholds, and given that all other firms charge prices $\{\hat{p}_i\}_{i=0}^{\infty}$.

As with the simple case, it is useful to distinguish between *fresh* consumers, who are looking for a new product, and *return* consumers, who go back to the product they bought in the previous period. As before we let m denote the measure of fresh consumers per-firm, and we derive an expression for it in Lemma A2 in the Appendix.

We now derive a firm's profit along the optimal path. When a firm sells a product of age

[13] Fixing a sequence of prices, the thresholds also have natural comparative statics (as in the simple case). Specifically, $\overline{a}$ and $\overline{b}$ both decrease in s and increase in $\delta_c\gamma_c$, while $\overline{a}$ decreases but $\overline{b}$ increases in r. However γ_f has an ambiguous effect, affecting both the distribution of prices and the future benefits of search.

[14] A consumer draws a group of products of age i with probability $(1-\gamma_f)\gamma_f^i$, and then draws uniformly from a continuum of firms with that age. These draw probabilities represent a well-defined probability distribution with countable support over firm ages, and guarantee an equal measure of consumers per firm.

[15] Notice that for an arbitrary sequence of prices there is no guarantee that $\hat{a}_i$ and $\hat{b}_i$ lie on $[\underline{\varepsilon}, \overline{\varepsilon}]$. If, for example, there is an i such that $\hat{a}_i < \underline{\varepsilon}$ then all consumers who search a product of age i buy it.

l, along the optimal path its flow profit *during that period* is

$$\hat{\Pi}_l = \pi(\hat{p}_l) \frac{m}{\sum_{j=0}^{\infty}(1-\gamma_f)\gamma_f^j[1-F(a_j)]} \left[1 - F(a_l) + [1 - F(b_l)]\,\gamma_c \frac{1-\gamma_c^l}{1-\gamma_c}\right]. \tag{14}$$

This can be understood as follows. Using standard computations, each period a firm is searched by a measure $m/\sum_{j=0}^{\infty}(1-\gamma_f)\gamma_f^j[1-F(a_j)]$ of fresh consumers.[16] When a firm of age l charges the optimal price $\hat{p}_l$, fresh consumers buy with probability $1-F(a_l)$. This explains the first term in square brackets. The second term is return demand. In particular, note that $j = 1, \dots, l$ periods earlier, a fresh consumer who searched this product bought if $\varepsilon \geq a_{l-j}$. Conditional on surviving, she then returned to a seller of the same product in the next period if $\varepsilon \geq b_{l-j+1}$, then in the following period if $\varepsilon \geq b_{l-j+2}$, and so on and so forth, up until the current period, where she returns if and only if $\varepsilon \geq b_l$. Hence a consumer who searched the product j periods ago when she was fresh, survives and returns in the current period to one of its sellers with probability $\gamma_c^j\,[1-F(b_l)]$. When a firm of age l charges $\hat{p}_l$ all its return consumers buy. Summing over the previous l periods' fresh consumers gives the second term in square brackets. Finally, each consumer who buys generates a profit of $\pi(\hat{p}_l)$.

We now derive a firm's profit if it deviates off the optimal path. Suppose a firm of age i makes a one-shot deviation and charges a price $p_i \neq \hat{p}_i$. Assume "passive beliefs", meaning that consumers do not change their beliefs about other firms' prices. We find that:

Lemma 8. *Fix a sequence of weakly increasing search thresholds. Suppose a firm of age i makes a one-shot deviation and charges a price p_i.*

1. *If p_i satisfies $u(p_i) \geq u(\hat{p}_i) - r$ the firm's discounted profit is*

$$\pi(p_i) \frac{m\left[1 - F(a_i + u(\hat{p}_i) - u(p_i)) + [1 - F(b_i)]\,\gamma_c \frac{1-\gamma_c^i}{1-\gamma_c}\right]}{\sum_{j=0}^{\infty}(1-\gamma_f)\gamma_f^j[1-F(a_j)]} + \sum_{l=i+1}^{\infty} (\delta_f\gamma_f)^{l-i}\hat{\Pi}_l. \tag{15}$$

2. *If p_i satisfies $u(p_i) < u(\hat{p}_i) - r$ the firm's discounted profit is $\sum_{l=i+1}^{\infty}(\delta_f\gamma_f)^{l-i}\hat{\Pi}_l$.*

The first part of Lemma 8 considers the case where the firm has either reduced its price, or raised it by a sufficiently small amount that the surplus it offers is no less than r what it would have offered had it charged $\hat{p}_i$. All consumers who visit the deviating firm either buy or search—it is not worthwhile to pay r to acquire the same product from another seller. The intuition is then similar to that from the simple case. In particular, since consumers do not observe the deviation before visiting the firm, the deviating firm is visited by the same measures of fresh and return consumers as along the optimal price path. Fresh consumers

[16] In particular, if a fresh consumer draws a product of age j, she searches again with probability $F(a_j)$. Hence on average a fresh consumer searches $1/[1-\sum_{j=0}^{\infty}(1-\gamma_f)\gamma_f^j F(a_j)]$ times.

buy if their match exceeds a_i adjusted for the change in surplus $u(\hat{p}_i) - u(p_i)$ offered by the firm. Return consumers all buy because they have a high match with the product. Moreover, since the deviating firm is atomless and search is random, starting from the next period onwards it is searched by the same consumers as along the optimal path. Hence, starting from the next period, the firm charges optimal prices and earns the same profit as it would have earned had it not deviated from the optimal path.

The second part of Lemma 8 considers the case where the firm has raised price by a large amount. The firm makes zero sales in the current period, because every consumer would rather incur r and buy the product from another seller at price $\hat{p}_i$. However, because the firm is infinitesimal and search is random, as in the simple case, starting in the following period it faces the same demand as on the optimal path and hence charges optimal prices.

Using the above results, we can now solve for optimal prices. In the case where $a_i \in (\underline{\varepsilon}, \overline{\varepsilon})$, if we take the first derivative of (15) with respect to p_i, impose $p_i = \hat{p}_i$, set it to zero, and then solve it, we obtain $\hat{p}_i = \tilde{p}_i(a_i)$ where

$$\tilde{p}_i(a_i) = \Psi^{-1}\left(\frac{1 - F(a_i) + [1 - F(a_i + r)]\gamma_c \frac{1-\gamma_c^i}{1-\gamma_c}}{f(a_i)}\right). \tag{16}$$

Given the properties of $\Psi(p)$ and $f(\varepsilon)$, notice that $\tilde{p}_i(a_i) < p^m$.

Lemma 9. *Fix a sequence of weakly increasing search thresholds. For a firm of age i:*

1. *If $a_i < \underline{\varepsilon}$ then the optimal price is $\hat{p}_i = p^m$.*

2. *If $a_i = \underline{\varepsilon}$ then any $\hat{p}_i \in [\tilde{p}_i(\underline{\varepsilon}), p^m]$ is an optimal price.*

3. *If $a_i \in (\underline{\varepsilon}, \overline{\varepsilon})$ then the optimal price is $\hat{p}_i = \tilde{p}_i(a_i)$.*

A firm's optimal price depends on the elasticities of its fresh and return demands. As explained above, for p_i sufficiently close to $\hat{p}_i$, all return consumers buy, i.e., the measure of return consumers that buy from a firm is completely price-inelastic. However the price-sensitivity of fresh consumers depends on the level of a_i. In particular, if $a_i < \underline{\varepsilon}$ all fresh consumers strictly prefer to buy rather than search when $p_i = \hat{p}_i$, which gives a firm monopoly power over every consumer who visits, allowing it to charge p^m. If instead $a_i > \underline{\varepsilon}$ some fresh consumers with low matches search on when $p_i = \hat{p}_i$, which makes their purchase decision elastic, and induces a firm to charge a price strictly below p^m which satisfies the (interior) first-order condition (16). Finally, if $a_i = \underline{\varepsilon}$ a firm is at a kink: when the firm charges $p_i = \hat{p}_i$, consumers with the lowest match $\underline{\varepsilon}$ are just indifferent about buying. Hence, if the firm slightly reduces price it does not sell to more fresh consumers, but if it slightly raises price it loses fresh consumers. This generates an outward-kinked demand curve, leading to existence of a continuum of possible optimal prices.

4.3 Equilibrium Characterization

We have now separately solved for the optimal search rule given a sequence of prices, and for optimal prices given a sequence of search thresholds. This subsection combines these results and solves for equilibrium. We remind the reader that we restrict our attention to search and return costs that satisfy $s \geq r > 0$. We start with the following preliminary result:

Lemma 10. *There exists $\bar{r} > 0$ and a strictly decreasing threshold $\bar{s}(r)$, such that there is an equilibrium in which the market is active only if $r \in (0, \bar{r}]$ and $s \in [r, \bar{s}(r)]$.*

In order for the market to be active consumers must be willing to search. We note that, as usual, for sufficiently high s there always exist equilibria where consumers expect very high prices and so do not search. Henceforth we focus on equilibria where the market is active. Lemma 10 shows that such equilibria exist only if (r, s) lies below a frontier. This frontier is decreasing in (r, s) space, such that when r is higher the market can only be active for lower values of s. Our main result is the following:

Theorem 1. *For any $r \in (0, \bar{r}]$, there exists a sequence of thresholds $\{\bar{s}_i(r)\}_{i=0}^{\infty}$ which strictly increase in i and satisfy $\lim_{i\to\infty} \bar{s}_i(r) \leq \bar{s}(r)$, such that:*

1. *If $s < \bar{s}_0(r)$ then a_i^* and p_i^* strictly increase in i, with $a_i^* > \underline{\varepsilon}$ and $p_i^* = \tilde{p}_i^*(a_i^*) < p^m$.*
2. *If $\bar{s}_i(r) \leq s < \bar{s}_{i+1}(r)$ for $i \geq 0$ then:*
   - *(a) For $j \leq i$, a_j^* and $p_j^* \in (\tilde{p}_i(\underline{\varepsilon}), \tilde{p}_{i+1}(\underline{\varepsilon}))$ are constant in j, with $a_j^* = \underline{\varepsilon}$.*
   - *(b) For $j > i$, a_j^* and p_j^* strictly increase in j, with $a_j^* > \underline{\varepsilon}$ and $p_j^* = \tilde{p}_j^*(a_j^*) \in (p_i^*, p^m)$.*
3. *If $\lim_{i\to\infty} \bar{s}_i(r) < s \leq \bar{s}(r)$, then a_i^* and p_i^* are constant in i, with $a_i^* < \underline{\varepsilon}$ and $p_i^* = p^m$.*

In the above three cases equilibrium is unique. Finally:

4. *If $s = \lim_{i\to\infty} \bar{s}_i(r)$ there is a continuum of equilibria. In each equilibrium a_i^* and p_i^* are constant in i, with $a_i^* = \underline{\varepsilon}$ and $p_i^* \leq p^m$.*

Theorem 1 shows that equilibrium search and pricing behavior differ qualitatively depending on whether search costs are low (part 1), intermediate (part 2), or high (part 3). Equilibrium is unique except at one particular (i.e., non-generic) search cost (part 4). Precisely, for a given return cost r, the equilibrium characterization depends on how s compares to a sequence of increasing thresholds $\{\bar{s}_i(r)\}_{i=0}^{\infty}$.

Part 1 of the theorem deals with low search costs, that is $s < \bar{s}_0(r)$. Because search is cheap, fresh consumers are "choosy" and do not necessarily buy from the first firm they encounter (i.e., $a_i^* > \underline{\varepsilon}$ for each firm age i). Firms then face a trade-off between pricing low to attract more of these choosy fresh consumers, or pricing high to exploit return consumers.

Older firms, who have been in the market longer, have accumulated a larger pool of return consumers, and so charge strictly higher prices. Equivalently, older firms face a larger and better-matched demand, which is less price-elastic. Hence, conditional on surviving, each period a firm optimally raises its price. Even though products are ex ante symmetric, ex post they have different prices according to how long they have been in the market.

Part 2 of the theorem deals with intermediate search costs, that is $\bar{s}_0(r) < s < \lim_{i\to\infty} \bar{s}_i(r)$. Because search is relatively costly, fresh consumers are less choosy than in part 1. Young firms take advantage of this by pricing in such a way that *all* fresh consumers who search them make a purchase. Precisely, for any intermediate value of s, there exists a critical age i such that firms with age weakly less than i sell to all their fresh consumers (i.e., $a_j^* = \underline{\varepsilon}$ for all $j \leq i$). Intuitively, young firms have relatively little return demand, so they optimally focus on selling to all their fresh consumers. Firms that are older than age i, meanwhile, have relatively high return demand: as in part 1, they exploit this by charging a relatively high price, which increases with their age, and which induces some fresh consumers to not buy from them (i.e., $a_j^* > \underline{\varepsilon}$ for all $j > i$). Summing up, conditional on surviving, firms keep their price constant up to age i while they accumulate return demand, and then strictly increase their price thereafter. Moreover, this critical age i is increasing in s.

Part 3 of the theorem deals with high search costs, that is $s > \lim_{i\to\infty} \bar{s}_i(r)$. Because search is expensive, in equilibrium fresh consumers buy the first product they encounter even if it has a very bad match (i.e., $a_i^* < \underline{\varepsilon}$ for each firm age i). Firms of all ages therefore have a monopoly over all consumers (both fresh and return) that visit them, and so they all charge p^m. Unlike in the previous two cases, then, firms keep price constant over time and so prices are not dispersed. Finally, part 4 of the theorem deals with the non-generic case where $s = \lim_{i\to\infty} \bar{s}_i(r)$, and shows that there is a continuum of equilibria. Mathematically, as $s \uparrow \lim_{i\to\infty} \bar{s}_i(r)$ all firms charge a price strictly below p^m, while as $s \downarrow \lim_{i\to\infty} \bar{s}_i(r)$ all firms charge p^m; the continuum of equilibria at $s = \lim_{i\to\infty} \bar{s}_i(r)$ then "fills in" this gap. Intuitively, in each of these equilibria fresh consumers with the lowest match $\underline{\varepsilon}$ are just indifferent between buying or searching on (i.e., $a_i^* = \underline{\varepsilon}$ for each firm age i). As explained earlier in Lemma 9, this creates a kink in each firm's demand curve, because small price cuts attract no new buyers, but small price rises lead to a drop in demand. This kink then supports the multiplicity of equilibrium prices.

Figure 2 illustrates these pricing patterns for an example where $Q(p) = 1 - p$, $F(\varepsilon) = \varepsilon$, $\gamma_c = 0.9$, $\gamma_f = 0.95$, $\delta_c = 0.8$, and $r = 0.02$. Each curve depicts, for a particular value of the search cost s, the equilibrium price charged by a seller of age $i = 0, \ldots, 20$.[17] The blue circles depict part 1 of the theorem, where the search cost is low ($s = 0.3$). Each period that a firm survives, it strictly raises its price. Moreover the "age premium" is significant, with a

[17]In this example around two-thirds of sellers have age less than or equal to 20.

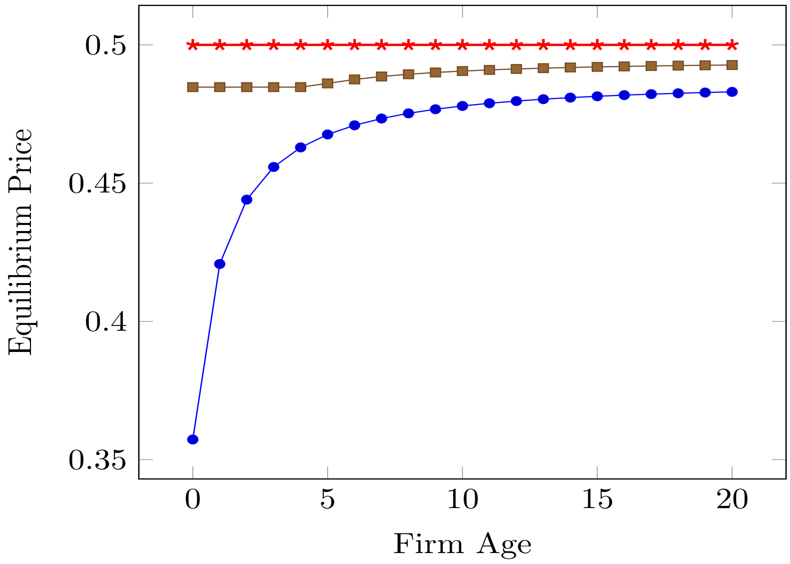


Figure 2: Equilibrium prices charged by firms of different ages, for low s (blue circles), medium s (brown squares), and high s (red stars).

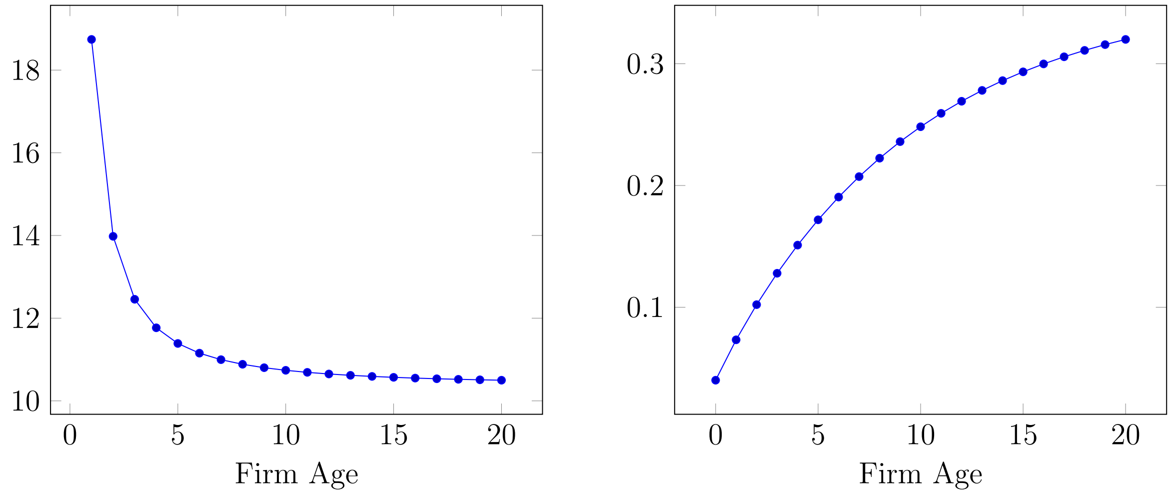
firm of age 20 charging around 35% more than a brand new firm. The brown squares depict part 2 of the theorem, where the search cost is intermediate ($s = 1.53$). After entering the market, firms keep their price constant until they reach age 4, after which they raise price each period until exiting the market. However, because even the price charged by a brand new firm is relatively high, here the age premium is small—a firm of age 20 only charges around 2% more than a brand new firm. The red stars depict part 3 of the theorem, where the search cost is high ($s = 1.6$). Here, firms of all ages charge the monopoly price $p^m = 1/2$, so equilibrium prices exhibit zero age premium.

Equilibrium search and pricing decisions generate rich patterns of customer turnover at the product level. As explained earlier, some turnover is *exogenous*, because with probability $1-\gamma_c$ a consumer leaves the market and so cannot return. Other turnover is *endogenous* and takes two forms. Firstly, if a consumer was fresh last period, she may not return and buy the same product again due to the return cost r. Secondly, both fresh and return consumers may not return to the product they bought last period because its price has increased. When s is high all firms charge the same price, so only the first source of endogenous turnover is present, but otherwise both sources are present.[18] Thus, with a low or intermediate search cost, a consumer may buy a product multiple times but, faced with continual price increases, eventually stop returning and instead re-enter the search pool.

Equilibrium search and pricing decisions also lead older firms to earn more profit:

Proposition 2. *A firm's per-period profit strictly increases with its age i.*

[18] Note that if s is sufficiently high *and* r is sufficiently small, it is possible that $b_i^* < \underline{\varepsilon}$ for each age i, so consumers always return to the product they last bought, and hence there is no endogenous turnover at all.



(a) Product-Level Customer Turnover (%)

(b) Firm Profit

Figure 3: Other equilibrium outcomes for the example depicted in Figure 2 (case of low s).

Intuitively, because older firms sell products which been encountered by more consumers, they face a larger and better-matched demand, which allows them to charge (weakly) higher prices and earn (strictly) more profits. (We can also prove that older firms make strictly higher sales as well, provided either $f(\varepsilon)$ is increasing, or γ_c is sufficiently small.)

Figure 3 depicts customer turnover and firm profit in the numerical example used in the preceding figure. For simplicity, we focus on a single value of the search cost, namely $s = 0.3$. The left panel of Figure 3 displays customer turnover at the product level. Precisely, for each $i = 0, \ldots, 20$, it depicts the percentage of consumers who buy a product at age $i - 1$ and who do *not* return and buy when the product has age i. Recall that in this example $1 - \gamma_c = 0.1$, so each product has exogenous turnover of 10%; any turnover above this level arises endogenously due to the return friction r and the fact that price increases over time. We find that a product of age $i = 1$ has a relatively high turnover of around 18.7%, because from Figure 2 its price is much higher than it was in the previous period. In contrast, we find that a product of age $i = 20$ has a very low turnover of only around 10.5%, because r is small in this example, and because from Figure 2 its price is only slightly higher compared to the previous period. The right panel of Figure 3 depicts per-firm profit. We observe that age has a significant effect on firm performance: compared to a firm with a brand new product, a firm with a product of age $i = 20$ earns around 8 times as much profit.

4.4 Comparative Statics of Equilibrium Prices

We now examine how the equilibrium prices from Theorem 1 vary with parameters of the model. To start with, notice from the example in Figure 2 that each seller charges a higher price when the search cost s is higher. It turns out that this holds in general:

Proposition 3. *Consider $r \in (0, \bar{r}]$ and $s \in (0, \lim_{i\to\infty} \bar{s}_i(r))$. Suppose s increases or δ_c decreases. Then equilibrium prices $\{p_i^*\}_{i=0}^{\infty}$ increase.*

Firms of all ages charge more when the search cost s increases, and when the discount factor δ_c decreases. Intuitively, consumers become less choosy when s is higher (since finding better matches is costlier) and when δ_c is lower (since the future benefit of finding a good match today is valued less). This raises equilibrium prices via two channels. The first channel is standard: as fresh consumers are less choosy, firms compete less aggressively for them. The second channel is new: consumers are more likely to return to the product they bought last period, which makes firms' demands less elastic.

Comparative statics for the parameters γ_c, γ_f and r are more subtle.[19] We begin with the case $\delta_c = 0$. It is straightforward to show that the threshold $\lim_{i\to\infty} \bar{s}_i(r)$ from Theorem 1 is then constant in γ_c, γ_f and r. Moreover, since for $s > \lim_{i\to\infty} \bar{s}_i(r)$ firms of all ages charge p^m, we focus on the more interesting case of $s < \lim_{i\to\infty} \bar{s}_i(r)$.

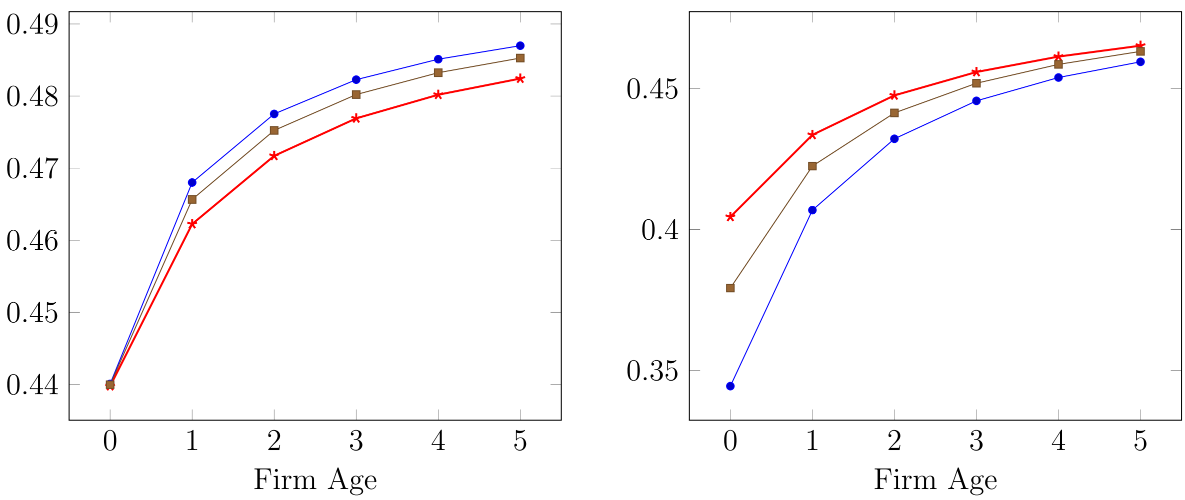
Proposition 4. *Consider $r \in (0, \bar{r}]$ and $s \in (0, \lim_{i\to\infty} \bar{s}_i(r))$. If $\delta_c = 0$ then equilibrium prices $\{p_i^*\}_{i=0}^{\infty}$ increase in γ_c and γ_f, and decrease in r.*

This is explained as follows. First, consider an increase in γ_c or a decrease in r. On the one hand, fresh consumers become choosier—because if they find a good match today they are more likely to return to and benefit from it in the future. Fresh demand is therefore more elastic, which is a force towards lower prices. On the other hand, the ratio of return-to-fresh consumers increases. This makes firms' demands less elastic, which is a force towards higher prices. When $\delta_c = 0$ only the latter effect is relevant, so firms of all ages raise their price. Second, consider an increase in γ_f. On the one hand, fresh consumers become choosier—because if they find a well-matched product today, they are more likely to be able to return and buy it in the future. On the other hand, fresh consumers become less choosy—because the firm-age distribution increases in the FOSD sense, and older firms charge higher prices, so when consumers search they draw from a worse distribution. When $\delta_c = 0$ only the latter effect is relevant, so consumers are less choosy and firms of all ages raise their price.[20]

To illustrate the above discussion, Figure 4 shows that equilibrium prices can either increase or decrease in r, depending on parameters. In this example $Q(p) = 1 - p$, $F(\varepsilon) = \varepsilon(1 + 4\varepsilon)/5$, $\gamma_c = 0.9$, $\gamma_f = 0.95$, and $s = 0.3$. The blue circles depict equilibrium prices for $r = 0.02$, the brown squares for $r = 0.15$, and the red stars for $r = 0.28$. In the left panel, $\delta_c = 0$, and consistent with Proposition 4 price at each firm age is decreasing in r. However, in the right panel, $\delta_c = 0.9$, and the opposite comparative static is observed.[21]

[19]Note that comparative statics for δ_f are simple. As explained earlier, firms choose price to maximize current-period profit only, and so δ_f has *no* effect on equilibrium prices.

[20]We can also study comparative statics for the search thresholds. It is easy to show that all the $\{a_i^*\}_{i=0}^{\infty}$ decrease in s and increase in δ_c. If $\delta_c = 0$, it is easy to show that all the $\{a_i^*\}_{i=0}^{\infty}$ decrease in γ_f. However,



(a) Equilibrium prices for low δ_c (b) Equilibrium prices for high δ_c

Figure 4: Blue circles denote low r, brown squares medium r, and red stars high r.

4.5 Discussion

We now briefly discuss some empirical evidence that is consistent with our findings. First, consider consumer behavior. Our model predicts that consumers should be choosier when $\delta_c\gamma_c$ is larger, i.e., when repeat purchases are more important, because the benefits of finding a good product are higher. Consistent with this, Sorensen (2000) shows that price dispersion is lower for frequently-searched prescriptions drugs; he attributes this to consumers searching more intensively for drugs they will purchase repeatedly, because the benefits of finding a low price are higher. Similarly Sorensen (2001) estimates a model of price search for prescription drugs, and finds that consumers search more actively for drugs used to treat chronic conditions (i.e., drugs which are likely to be used repeatedly). Furthermore, our model suggests that older consumers should exhibit more inertia.[22] Consistent with this, Bornstein (2021) finds that compared to younger households, older households are substantially less likely to switch brands of consumer packaged goods towards new entrants.

Second, consider firm behavior and performance. The model predicts that older firms should charge (weakly) higher prices, sell more, and hence be more profitable, because their products have been searched by more consumers and so they have a larger customer base. Consistent with this, Peters (2020) finds that as a firm ages its mark-up tends to increase. Policymakers have also raised concerns about "price walking" whereby firms gradually raise prices to the detriment of loyal consumers (see, e.g., the report for the UK Competition and Markets Authority report by Heidhues et al., 2020).[23] Turning to sales, Bronnenberg

even if $\delta_c = 0$, an increase in γ_c and r can increase a_i^* for some i and decrease a_j^* for other $j \neq i$.

[21] For clarity we only show this for ages $i = 0, \ldots, 5$. However, the same pattern holds for larger i as well.

[22] For example, in the simple case, once a consumer find a product with a match above b (as defined in Lemma 2) she buys it again and again, and older consumers are more likely to have found such a product.

[23] Argente and Yeh (2022) show that older firms are less likely to change price and, conditional on doing

et al. (2009) show that brands which entered a given geographic market earlier have higher market shares, and that order of entry is a good predictor of which firm is the market leader. Similarly, Afrouzi et al. (2023) find that as a firm ages, its customer base becomes larger and total sales increase. More generally, Einav et al. (2022) show the importance of a firm's customer base for explaining its sales, and document high levels of turnover across a variety of product categories.

5 Extensions and Robustness

We first examine equilibrium outcomes when firms can charge different consumers different prices depending on their search history. We then show how our main results from earlier hold when consumers have unit demand, and each product has a finite number of suppliers.

5.1 Price Discrimination

So far we have assumed that a firm must charge each consumer that visits it the same price. In this section we allow firms to price discriminate. Specifically, we now assume that when a consumer visits a firm, the firm observes her search history h, where h denotes all the product-firm pairs previously visited by the consumer. The firm can then use this data to personalize the price it charges the consumer. We look for a symmetric stationary equilibrium in which all firms of age i facing a consumer with given search history h charge the same price. To rule out uninteresting equilibria we make the following assumption:[24]

Assumption 2. *Consider an equilibrium in which a consumer with history h does not visit a firm of age i on the equilibrium path. If, off the equilibrium path, such a consumer visits the firm, the firm charges her p^m.*

We first establish that firms also charge p^m to any consumer who returns along the equilibrium path:

Lemma 11. *Firms charge the monopoly price $p_r^* = p^m$ to each consumer who returns along the equilibrium path.*

Intuitively, because return consumers have already incurred $r > 0$, they are willing to pay slightly more than whatever price they expected. This gives firms (local) monopoly power over return consumers, enabling them to charge the monopoly price p^m as in Diamond (1971).

We now characterize equilibrium prices and search behavior. To ease the exposition we focus on the main qualitative features of equilibrium. (Detailed expressions for equilibrium

so, change it by less; this is consistent with the case of low search costs depicted in Figure 2.

[24]This rules out equilibria in which consumers do not search, or do not return, because they expect very high prices. It also pins down return prices in equilibria where no return occurs on the equilibrium path.

prices and search thresholds are available in the proof.) Recalling the thresholds $\bar{r}$ and $\overline{s}(r)$ from Lemma 10, and $\lim_{i\to\infty}\overline{s}_i(r)$ from Theorem 1, we find that:

Proposition 5. *Consider $r \in (0,\bar{r}]$ and $s \in [r, \overline{s}(r)]$. There is a symmetric stationary equilibrium, in which firms charge p_f^* to fresh consumers and $p_r^* = p^m$ to return consumers, and consumers buy when fresh if and and only if $\varepsilon \geq a^*$, and return if and only if $\varepsilon \geq b^*$. Moreover, there exist $\overline{s}_b(r) \leq \overline{s}_a(r) \leq \lim_{i\to\infty}\overline{s}_i(r)$ such that:*

1. *If $s < \overline{s}_a(r)$ then $a^* > \underline{\varepsilon}$ and $p_f^* < p^m$.*

2. *If $\overline{s}_a(r) < s \leq \lim_{i\to\infty}\overline{s}_i(r)$ then $a^* = \underline{\varepsilon}$ and $p_f^* \leq p^m$.*

3. *If $\lim_{i\to\infty}\overline{s}_i(r) < s \leq \overline{s}(r)$ then $a^* < \underline{\varepsilon}$ and $p_f^* = p^m$.*

In the above cases the equilibrium is unique. Otherwise:

4. *If $s = \overline{s}_a(r)$ there is at least one equilibrium, and depending on parameters there could be multiple equilibria. All equilibria have $a^* = \underline{\varepsilon}$ and $p_f^* \leq p^m$.*

Finally, if $s < \bar{s}_b(r)$ then $b^* > \overline{\varepsilon}$ and if $s > \bar{s}_b(r)$ then $b^* < \overline{\varepsilon}$.

Even though firms can engage in very fine-tuned price discrimination, in equilibrium they charge at most two prices—p_f^* to fresh consumers who have never searched their product before, and $p_r^* = p^m$ otherwise. Fresh consumers use one reservation threshold a^*, and return consumers use another reservation threshold b^*. In the following paragraphs we explain how equilibrium search behavior and pricing varies with s.

Part 1 of the proposition deals with low search costs, that is $s < \bar{s}_a(r)$. Because search is cheap, fresh consumers do not necessarily buy the first product they encounter (i.e., $a^* > \underline{\varepsilon}$), and so firms offer them a lower price compared to what they charge return consumers (i.e., $p_f^* < p_r^* = p^m$). Interestingly, when search is very cheap, that is $s < \bar{s}_b(r)$, equilibrium outcomes are *exactly the same* as in a standard one-shot Wolinsky-Anderson-Renault model. In particular, even consumers who previously found a product with match $\overline{\varepsilon}$ prefer to search again rather than return and be charged p^m (i.e., $b^* > \overline{\varepsilon}$). Expecting to buy a product only once, fresh consumers' search threshold a^* is therefore exactly the same as in a standard one-shot problem. Hence firms' equilibrium fresh price p_f^* is also the same as in a standard one-shot search model. Notice that firms' ability to charge high prices to return consumers leads to complete customer turnover each period—and means that firms earn the same profit irrespective of their age.[25]

[25] This is different from the main model, where some consumers always return in equilibrium. The reason is that earlier we assumed that firms charge fresh and return consumers the same price; if no consumers returned, firms would charge a low price to cater to fresh consumers, but then past customers with high matches would prefer to return.

Part 3 of the proposition deals with the opposite case of high search costs, that is $\lim_{i\to\infty} \overline{s}_i(r) < s \leq \overline{s}(r)$. Here, equilibrium is exactly the same as in Theorem 1. Intuitively, because search is expensive, in equilibrium fresh consumers buy the first product they encounter (i.e., $a^* < \underline{\varepsilon}$). Firms therefore have (local) monopoly power over both fresh and return consumers, so they charge everybody p^m. Hence, even though firms can price discriminate, in equilibrium they choose not to. As in our main model, there is partial customer turnover: consumers with sufficiently high matches return to the same product repeatedly, while consumers with lower matches search again. Moreover, older firms earn strictly higher profit since they attract larger return demand.

Part 2 of the proposition deals with intermediate search costs, that is $\overline{s}_a(r) < s \leq \lim_{i\to\infty} \overline{s}_i(r)$. For all s *strictly* in this interval there is real price discrimination: fresh consumers pay less than return consumers (i.e., $p_f^* < p_r^*$), and some consumers come back and pay the higher return price (i.e., $b^* < \overline{\varepsilon}$).[26] Intuitively, s is small enough that firms compete for fresh consumers by charging less than p^m, but s is also large enough that consumers who previously found a well-matched product prefer to return and buy it again. As detailed in the proof, there is relatively high customer turnover in the first period after purchasing a product (i.e., $b^* > a^* + r$), because return consumers incur r *and* face a price increase. However, conditional on returning once to a product, the consumer keeps returning until either she or the product exits the market. Older firms again attract larger return demand and hence earn higher profits. Finally, part 4 of the proposition deals with the non-generic case where $s = \overline{s}_a(r)$ and shows that there may be a continuum of equilibria.[27] Like in our main analysis, this arises due to a kink in firms' profit functions and "fills in" any discontinuity in equilibrium prices as we transit from low to intermediate s.

We close this section by considering the impact of price discrimination on consumers. Notice that when $s > \lim_{i\to\infty} \overline{s}_i(r)$ consumer surplus is the same with or without price discrimination, because in either case all consumers are charged p^m. We therefore focus on lower values of s, starting with the following result:

Proposition 6. *If $s < \lim_{i\to\infty} \overline{s}_i(r)$ and also $\delta_c = 0$ then price discrimination increases lifetime consumer surplus.*

Intuitively, in general price discrimination introduces a trade-off. On the one hand, return consumers are charged a higher price. This directly harms those consumers who pay the higher price, and indirectly harms those who choose to search again rather than pay it. On the other hand, with price discrimination firms can target price-elastic fresh consumers with lower prices. This latter effect dominates when $\delta_c = 0$ or, by continuity,

[26] The case $\lim_{i\to\infty} \overline{s}_i(r)$ is a boundary case where $p_f^* = p_r^* = p^m$ such that there is no price discrimination.

[27] Precisely, the proof shows that when $\delta_c \gamma_c \gamma_f = 0$ there are always multiple equilibria at this value of s, while when $\delta_c \gamma_c \gamma_f > 0$ there are multiple equilibria if and only if nobody returns.

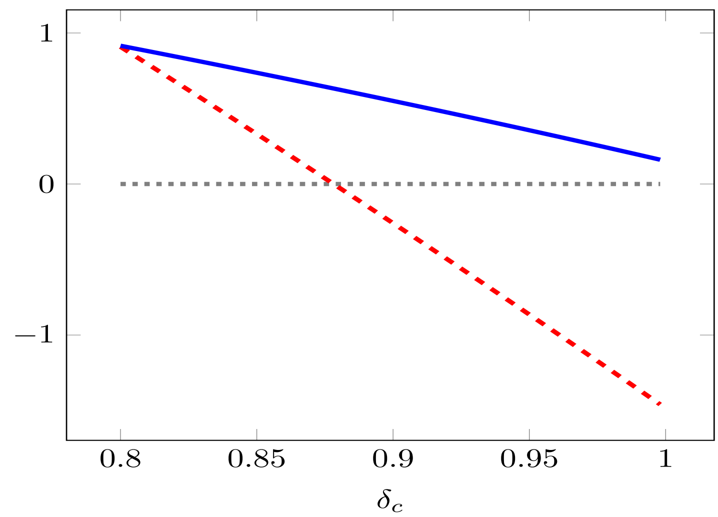


Figure 5: The percentage change in consumer surplus due to price discrimination at high values of δ_c, for respectively low s (red dashed curve) and high s (blue solid curve).

when consumers are relatively impatient. However, at larger values of δ_c, either effect can dominate. Figure 5 illustrates this for the case where $Q(p) = 1 - p$, $F(\varepsilon) = \varepsilon(1 + 4\varepsilon)/5$, $r = 0.005$, $\gamma_c = 0.5$ and $\gamma_f = 1$. The red dashed curve depicts the percentage change in consumer surplus, when moving from no price discrimination to price discrimination, for $s = 0.04$; the blue solid curve depicts the same thing for $s = 0.4$. Depending on parameters, price discrimination can either benefit or harm consumers. For example, at $\delta_c = 0.99$, price discrimination reduces consumer surplus by around 1.4% for $s = 0.04$, but increases it by around 0.2% for $s = 0.4$.

5.2 Unit demand

So far we have assumed that consumers have downward-sloping demand for each product. We now show that our main result holds if instead consumers have unit demand. In particular, suppose now that in each period consumers wish to buy one unit of one product, and obtain surplus $\varepsilon - p$ if they buy a product with match value ε at a price p. To simplify the analysis, assume that the lowest possible match value satisfies $\underline{\varepsilon} = 0$. Noting that the monopoly price is now defined as $p^m \equiv \arg\max p[1 - F(p)]$, we find that:

Proposition 7. *There exist $\bar{r} > 0$ and $\bar{s} \geq \bar{r}$ such that for all $r \leq \bar{r}$ and $s \in [r, \bar{s}]$, there is a unique equilibrium in which*

1. *Consumers buy a product of age i when fresh if $\varepsilon \geq a_i^*$, and return to it if $\varepsilon \geq b_i^* \equiv a_i^* + r$. Consumers are choosier towards older firms: a_i^* strictly increases in $i = 0, 1, \ldots$*

2. *Older firms charge higher prices: $p_i^* < p^m$ strictly increases in $i = 0, 1, \ldots$*

When search and return costs are sufficiently small, equilibrium is qualitatively the same as the one with elastic demand in part 1 of Theorem 1. Specifically, a firm's optimal price strictly increases with its age, because as it gets older it has a larger and better-matched demand; this in turn makes consumers strictly choosier towards older products.

At the same time, with unit demand, equilibrium is less rich than before. Specifically, it is not possible to have an equilibrium like in parts 2-4 of Theorem 1. To see why, notice that such equilibria rely on the existence of some age i for which $a_i^* \leq \underline{\varepsilon} = 0$. However, because the marginal fresh consumer does not return, it must also be that $a_i^* - p_i^* + \delta_c \gamma_c V = V$. But since prices are positive, the value of search V would then be negative—and so no consumer would search. Hence only equilibria of the form in Proposition 7 can exist.

5.3 Finite Number of Suppliers

Recall that in our model, there is a continuum of differentiated products. So far, we have assumed that each of these products has a continuum of undifferentiated suppliers. We now explain why this second assumption greatly simplifies our analysis. We then show that, under some conditions, we can dispense with this assumption and the equilibrium in Theorem 1 still remains an equilibrium.

Assuming a continuum of undifferentiated suppliers greatly simplifies analysis of off-path price deviations. In particular, note that with atomistic firms, a deviation by one firm has a zero measure effect on consumer search behavior and firm profits. Hence, from the following period, all firms charge the same price they would have charged absent the deviation (as in Lemma 3). However, if instead each product has but a single supplier, or more generally has a finite number n suppliers, at least three difficulties may arise. Firstly, when a firm deviates, it may induce some consumers who otherwise would have returned next period to visit other firms. This changes the firm's future demand, and hence also changes its future optimal price path. We then need to solve for that new price path, and verify that the initial deviation was unprofitable. Secondly, if the deviation induces a change in the firm's optimal future price path, consumers may no longer be able to form rational expectations about future prices (which they need, to decide whether and when to return to this firm). Specifically, if a consumer searches a firm and sees an unexpected price, in order to rationalize this unexpected price, and correctly predict future prices, she needs to observe the whole pricing history. Thirdly, even if consumers can correctly predict future prices, solving a firm's (present-discounted) deviation profit is complicated because we need to keep track of how many consumers visited it in the past, did not buy due to the deviation, but may consider returning in the future if prices are expected to be low enough.

Nevertheless, we now show how these difficulties can be overcome. Suppose each product has finite $n \geq 1$ undifferentiated sellers, and normalize the measure of consumers per-firm to one. Also suppose that (for $n > 1$) consumers randomize over which supplier of a product to visit when indifferent. Also assume the following:

Assumption 3. *For $n = 1$: (i) when a consumer searches a firm she learns its past prices, (ii) a consumer can only return to a firm if she bought from it last period, and (iii) $\delta_c = 0$.*

Under this assumption we can prove that the equilibrium in Theorem 1 exists[28]:

Proposition 8. *Suppose either $n = 1$ and Assumption 3 holds, or $n > 1$. Then there exists an equilibrium in which, each period, firms charge the same prices, and consumers use the same search rule, as in Theorem 1.*

Start with the case of $n = 1$. Note that part (i) of Assumption 3 enables consumers to correctly predict future prices following a deviation, while part (ii) of the assumption simplifies computation of a firm's deviation profit. The proof constructs a new optimal price path following a deviation, which involves the firm charging (weakly) lower prices in future periods than it would otherwise have charged. We then use part (iii) of the assumption to show that any initial deviation is unprofitable. Specifically, because a deviation at time t causes consumers to expect lower prices from $t+1$ onwards, a deviation can cause consumers to be *less* price sensitive at t, i.e., they are more willing to bear higher prices at t, in order to pay less in the future. In principle this could make the initial deviation profitable for the firm. Assuming consumers are sufficiently impatient (e.g., $\delta_c = 0$) rules this out.

The case of $n > 1$ is simpler. We construct an equilibrium where, following an arbitrary deviation, firms return to the equilibrium price path from the following period. This allows us to avoid the difficulties outlined at the start of this subsection. The idea behind the construction is as follows. Consider a firm of age i. If it deviates to a price $p_i < p_i^*$ it sells to more fresh consumers, but they have low matches and so do not return next period; if instead it deviates to a price $p_i > p_i^*$ satisfying $u(p_i) \geq u(p_i^*) - r$ it sells to fresh fewer consumers, but they have intermediate matches and so wouldn't have returned in the future anyway. Hence, in both cases, the firm's optimization problem from age $i + 1$ onwards is the same as if it had not deviated. Finally, if the firm deviates to p_i satisfying $u(p_i) < u(p_i^*) - r$, all consumers who visit the firm and would have bought in equilibrium prefer to visit another supplier of the same good. Hence, from age $i + 1$ onwards, the pool of return consumers for the firm's product is the same as before the deviation, so if return consumers randomly choose a supplier to visit, all firms price in the same way as they would have absent the deviation.

[28]Unlike in Theorem 1, we do not establish uniqueness of equilibrium, even in the class of price-increasing equilibria, but also have not been able to construct alternative equilibria.

6 Conclusion

We provide a framework to analyze market outcomes when consumers wish to buy a product repeatedly, but need to search in order to find a well-matched product. Older firms charge higher prices and earn higher profit, because they face a larger and better-matched demand, which consists mainly of return consumers with high valuations. This gradual increase in a product's prices over time generates endogenous customer turnover, as consumers who initially return and buy a product several time, react to the price increases by searching again for a new product. We also showed that when firms can personalize prices based on a consumer's search history, depending on parameters firms may discriminate against loyal consumers, but nevertheless total consumer surplus can be higher.

There are some natural ways in which our analysis could be extended. For example, we have assumed throughout that turnover is exogenous. However, given that older firms earn higher profit, one might expect them to be less likely to exit the market compared to young firms. While adding this into the model would make the analysis more complicated, we believe that the key trade-offs and results—such as increasing price paths and endogenous turnover—would be robust. Another example is that we have assumed that fresh consumers only learn product age after search. However, some consumers may observe product age and use it to direct search. These consumers would search among new (low-price) firms, leaving older firms to focus more on returning consumers. As in the extension to price discrimination, we would expect this to exacerbate the difference in prices between young and old firms, and reduce the latter's profit advantage over the former.

A Appendix

Proof of Lemma 1. Start with the return decision. First, a consumer whose best past match is $\underline{\varepsilon}$ does not return. If she were to return, her payoff this period would be $\underline{\varepsilon} + u(p^*) - r$. If instead she searches once, her expected payoff this period is $\mathbb{E}\varepsilon + u(p^*) - s > \underline{\varepsilon} + u(p^*) - r$, where the inequality uses the assumption $s < \mathbb{E}\varepsilon - \underline{\varepsilon}$; her expected future payoffs would also be higher, since next period her best past match would exceed $\underline{\varepsilon}$. Second, a consumer whose best past match is $\overline{\varepsilon}$ does return. If she returns she gets the maximum possible payoff $\overline{\varepsilon} + u(p^*) - r$ this period. If instead she were to search, she would get strictly lower payoff this period, and not higher payoffs in the future. Third, given a stationary search rule, a consumer who is indifferent this period between searching and returning, is also indifferent between searching this period and returning in this and every subsequent period. Moreover, the payoff $(\varepsilon + u(p^*) - r)/(1 - \delta_c\gamma_c)$ from returning in this and every subsequent period to a product is strictly increasing in its match ε. Hence there exists a reservation value $b \in (\underline{\varepsilon}, \overline{\varepsilon})$.

Now suppose the consumer has no past match with $\varepsilon \geq b$. First, the consumer will search at least once because the payoff from doing so is $\mathbb{E}\varepsilon + u(p^*) - s > 0$, where the inequality uses the assumptions $s < \mathbb{E}\varepsilon - \underline{\varepsilon}$ and $\underline{\varepsilon} \geq 0$. Second, a consumer who discovers a match $\underline{\varepsilon}$ searches again, because if she searches one more time she strictly improves her expected payoff this period (since $\mathbb{E}\varepsilon + u(p^*) - s > \underline{\varepsilon} + u(p^*)$) and weakly increases her future payoff since her best match increases. Third, a consumer who discovers a match $\varepsilon = b$ does not search, because by definition the match is good enough that she would even incur $r > 0$ to take it rather than search again. Finally, since the consumer's payoff is strictly increasing in her current match value, we conclude that there exists a reservation value $a \in (\underline{\varepsilon}, b)$. □

Proof of Lemma 2. Equation (6) is derived by substituting (4) and (5) into (3) and simplifying. The righthand side of (6) is strictly decreasing in a, strictly larger than s when $a = \underline{\varepsilon}$ (because by assumption $s < \mathbb{E}\varepsilon - \underline{\varepsilon}$), and strictly smaller than s when $a = \overline{\varepsilon} - r$ because

$$\int_{\overline{\varepsilon}-r}^{\overline{\varepsilon}} (\varepsilon - \overline{\varepsilon} + r) dF(\varepsilon) < \int_{\overline{\varepsilon}-r}^{\overline{\varepsilon}} r dF(\varepsilon) \leq r \leq s.$$

Hence equation (6) has a unique solution, and it satisfies $a > \underline{\varepsilon}$. Equations (4) and (5) also imply that $b = a + r$.

Comparative statics in s and $\delta_c\gamma_c$ follow because the lefthand side of (6) is increasing in s, while the righthand side is decreasing in a and increasing in $\delta_c\gamma_c$. Comparative statics in r follow because the righthand side is decreasing in both a and r, whereas when a is replaced by $b - r$ the righthand side is decreasing in b but increasing in r. □

Proof of Lemma 3. The proof follows from arguments in the text and so is omitted. □

Proof of Lemma 4. We derive the first term in equation (8). (The second term follows from Lemma 3.) Firm j is visited by a measure $1 - m$ of return consumers with $\varepsilon \geq b$, and a measure $m/[1 - F(a)]$ of fresh consumers with ε distributed according to $F(\varepsilon)$.

Any consumer who visits j either i) buys from j at price p_j, ii) incurs r and buys j's product from another supplier at price p^*, or iii) searches for a better product. The second option is dominated by the first, because $u(p_j) \geq u(p^*) - r$. The third option is dominated by the second for any consumer with $\varepsilon \geq b$ (by definition). Hence all return consumers, and all fresh consumers with $\varepsilon \geq b$, buy from j. Next, note that any fresh consumer with $\varepsilon < b$ who buys from firm j in period t will search for a new product in period $t + 1$. Hence if $\tilde{\varepsilon}$ is the match value of the marginal fresh consumer, it must satisfy $\tilde{\varepsilon} + u(p_j) + \delta_c \gamma_c V = V$; using equation (4) we then obtain $\tilde{\varepsilon} = a + u(p^*) - u(p_j)$. (Note that $\tilde{\varepsilon} \leq a + r \equiv b$.) Hence each fresh consumer who searches j buys with probability $1 - F(\tilde{\varepsilon}) = 1 - F(a + u(p^*) - u(p_j))$. Finally, each consumer who buys from j generates a profit $\pi(p_j)$. □

Proof of Lemma 5. No consumer who visits firm j in period t buys from it, because the payoff $\varepsilon + u(p_j)$ from doing so is strictly less than the payoff $\varepsilon + u(p^*) - r$ that can be obtained by visiting another seller of the same product. However, by Lemma 3, from period $t + 1$ onwards firm j earns $\pi(p^*)$ each period. □

Recall the definition of $\Psi(p)$ in equation (10). We use the next lemma in several proofs.

Lemma A1. *$\Psi(p)$ strictly increases in $p \in [0, p^m)$, with $\Psi(0) = 0$ and $\lim_{p \to p^m} \Psi(p) = \infty$.*

Proof of Lemma A1. The claims that $\Psi(0) = 0$ and $\lim_{p \to p^m} \Psi(p) = \infty$ follow from direct computation. In order to prove that $\Psi(p)$ is strictly increasing, rewrite it as

$$\Psi(p) = \frac{\pi(p)}{1 + pQ'(p)/Q(p)}. \tag{17}$$

Log-concavity of $Q(p)$ implies that $Q'(p)/Q(p)$ is decreasing, and that $\pi(p)$ is log-concave and so strictly increasing in $p \in [0, p^m)$. Hence the numerator of (17) is strictly increasing in p whilst the denominator is strictly decreasing. Since the denominator is strictly positive for all $p < p^m$, the result then follows. □

Proof of Proposition 1. Note that for $p_j \leq u^{-1}[u(p^*) - r]$ such that $Q(p_j) > 0$, the derivative of equation (8) with respect to p_j is proportional to

$$-\left\{-\frac{1}{\Psi(p_j)} + \frac{mf(a + u(p^*) - u(p_j))}{(1 - m)[1 - F(a)] + m[1 - F(a + u(p^*) - u(p_j))]}\right\}. \tag{18}$$

Substituting $p_j = p^*$ and setting (18) to zero, there is a unique candidate equilibrium price p^* and it is given by equation (11). It follows from equation (11) that $p^* < p^m$ because $m > 0$, $\underline{\varepsilon} < a < \overline{\varepsilon}$ (from Lemma 1), $f(\varepsilon) > 0$ for all $\varepsilon \in [\underline{\varepsilon}, \overline{\varepsilon}]$, and $\Psi(p) < \infty$ for all $p < p^m$.

We now prove that firm j's profit function is quasiconcave in p_j.

First consider $p_j \leq \min\{p^m, u^{-1}[u(p^*) - r]\}$. Equation (18) is decreasing in p_j over this range. The reason is as follows. The first term in curly brackets is strictly increasing in p_j from Lemma A1. The second term in curly brackets is also increasing in p_j. To see this, let $\tilde{\varepsilon} = a + u(p^*) - u(p_j)$, and note that $\frac{1-m}{m}[1 - F(a)] = \frac{\gamma_c}{1-\gamma_c}[1 - F(b)]$. Since $\tilde{\varepsilon} < a + r = b < \overline{\varepsilon}$, the derivative of the second term in curly brackets with respect to p_j is proportional to

$$\frac{f'(\tilde{\varepsilon})[1 - F(\tilde{\varepsilon})]}{[f(\tilde{\varepsilon})]^2} \left[1 + \frac{\gamma_c}{1 - \gamma_c} \frac{1 - F(b)}{1 - F(\tilde{\varepsilon})}\right] + 1 \geq \gamma_c \left[\frac{F(b) - F(\tilde{\varepsilon})}{1 - F(\tilde{\varepsilon})}\right] \geq 0,$$

where the first inequality uses Assumption 1 and the second inequality uses $\tilde{\varepsilon} \leq b < \overline{\varepsilon}$. Hence profit is single-peaked in p_j for $p_j \leq \min\{p^m, u^{-1}[u(p^*) - r]\}$.[29]

Second, note from Lemmas 4 and 5 that discounted profit is weakly higher at $p_j = u^{-1}[u(p^*) - r]$ compared to any $p_j > u^{-1}[u(p^*) - r]$. Hence if $u^{-1}[u(p^*) - r] \leq p^m$ the proof of quasiconcavity is complete. Third, then, suppose instead that $u^{-1}[u(p^*) - r] > p^m$. Note that (18) is strictly negative for all $p_j \in (p^m, u^{-1}[u(p^*) - r]]$ and hence profit is decreasing over this interval. Profit is therefore also quasiconcave in this case. □

Proof of Lemma 6. Let $\mathcal{P}(t)$ denote the set of products that a consumer has previously searched and that are still in the market in period t. Suppose the consumer is indifferent in period t between searching a new product and returning to the product in $\mathcal{P}(t)$ with the highest surplus. Since products weakly increase in price and exit the market over time, in all future periods the consumer will weakly prefer to search a new product rather than return to any product in $\mathcal{P}(t)$. Hence, for this consumer, the continuation value of search in period t does not depend on the set $\mathcal{P}(t)$. Denote this continuation value by V. Then, the consumer returns to the product in $\mathcal{P}(t)$ with the highest surplus if that surplus exceeds $\overline{b}$ defined by $\overline{b} - r + \delta_c \gamma_c V = V$ and otherwise searches.

Now suppose that no product in $\mathcal{P}(t)$ gives a surplus above $\overline{b}$. Then, since prices weakly increase over time, the consumer will not come back to any product in $\mathcal{P}(t)$ in the future. Conditional on searching in period t, the consumer will accept any product with a surplus above $\overline{b} - r$. Therefore, there exists a cut-off $\overline{a} \leq \overline{b} - r$, which does not depend on $\mathcal{P}(t)$, such that the consumer buys a product if its surplus exceeds $\overline{a}$ and searches otherwise. □

[29] Note that equation (18) is (downward) discontinuous at $p_j = u^{-1}[a - \underline{\varepsilon} + u(p_j^*)]$ and positive on both sides of this point.

Proof of Corollary 1. On the way to a contradiction, suppose there is a product j which the consumer did not buy in period $t-1$ but to which she returns in period t. Since the consumer returns to product j in period t, its surplus in period t must exceed $\overline{b}$. Since prices weakly increase over time, product j's surplus must therefore have exceeded $\overline{b}$ in all preceding periods. Moreover, in whichever period the consumer searched product j for the first time, by definition no other product gave her surplus exceeding $\overline{b}$. Hence the consumer should have bought product j in that period and returned to it in each period up to and including period t—which is a contradiction. □

Proof of Lemma 7. A consumer who searches and draws surplus $\overline{a}$ is (by definition) indifferent between searching or buying. Moreover, if she buys, she will not return in the future because $\overline{a} < \overline{b}$ (from Lemma 6) and because prices weakly increase with product age. Hence

$$V = \overline{a} + \delta_c \gamma_c V, \tag{19}$$

where V is the value of search. For the same reason, a consumer for whom the highest available surplus from the products she searched previously is exactly $\overline{b}$, must be indifferent between searching again, or returning and buying for one period and then searching. Hence

$$V = \overline{b} - r + \delta_c \gamma_c V. \tag{20}$$

It follows immediately from equations (19) and (20) that $\overline{b} = \overline{a} + r$. We can also write that

$$\begin{aligned} V = -s + \sum_{i=0}^{\infty} (1-\gamma_f)\gamma_f^i \Bigg[F(\overline{a} - u(p_i))V + \int_{\overline{a}-u(p_i)}^{\overline{\varepsilon}} (\varepsilon + u(p_i)) dF(\varepsilon) + \delta_c \gamma_c \Bigg\{ \int_{\overline{a}-u(p_i)}^{\overline{b}-u(p_{i+1})} V dF(\varepsilon) \\ + \sum_{l=i+1}^{\infty} \int_{\overline{b}-u(p_l)}^{\overline{b}-u(p_{l+1})} \left(\sum_{j=i+1}^{l} (\delta_c\gamma_c\gamma_f)^{j-i-1} \left[\gamma_f(\varepsilon + u(p_j) - r) + (1-\gamma_f)V \right] + (\delta_c\gamma_c\gamma_f)^{l-i} V \right) dF(\varepsilon) \\ + \int_{\overline{b}-\lim_{l\to\infty} u(p_{l+1})}^{\overline{\varepsilon}} \left(\sum_{j=i+1}^{\infty} (\delta_c\gamma_c\gamma_f)^{j-i-1} \left[\gamma_f(\varepsilon + u(p_j) - r) + (1-\gamma_f)V \right] \right) dF(\varepsilon) \Bigg\} \Bigg]. \end{aligned} \tag{21}$$

To understand this, note that when a consumer searches she draws a product of age i with probability $(1-\gamma_f)\gamma_f^i$. a) If her match satisfies $\varepsilon + u(p_i) \leq \overline{a}$ she searches again and gets V. b) If her match satisfies $\varepsilon + u(p_i) > \overline{a}$ she buys the product this period. i) If in addition $\varepsilon + u(p_{i+1}) \leq \overline{b}$, then in the next period, conditional on surviving, the consumer searches and gets V. ii) If instead $\varepsilon \in [\overline{b} - u(p_l), \overline{b} - u(p_{l+1})]$ for $l \geq i+1$, then conditional on surviving, the consumer keeps returning to the product until either the product exits or $l-i$ periods have elapsed, and then she searches again. iii) Finally if $\varepsilon + \lim_{i\to\infty} u(p_i) > \overline{b}$ the consumer returns to the product in each period until either she or the product exits the market. (Note that $\lim_{i\to\infty} u(p_i)$ exists because $u(.)$ is a continuous function and prices are weakly increasing.)

Substituting in equation (19) and simplifying, equation (21) reduces to

$$\begin{aligned} s = \sum_{i=0}^{\infty}(1-\gamma_f)\gamma_f^i \Bigg[\int_{\overline{a}-u(p_i)}^{\overline{\varepsilon}} (\varepsilon - \overline{a} + u(p_i)) dF(\varepsilon) \\ + \sum_{l=i+1}^{\infty} \int_{\overline{b}-u(p_l)}^{\overline{b}-u(p_{l+1})} \left(\sum_{j=i+1}^{l} (\delta_c\gamma_c\gamma_f)^{j-i} \left[\varepsilon - \overline{a} + u(p_j) - r\right] \right) dF(\varepsilon) \\ + \int_{\overline{b}-\lim_{l\to\infty} u(p_{l+1})}^{\overline{\varepsilon}} \left(\sum_{j=i+1}^{\infty} (\delta_c\gamma_c\gamma_f)^{j-i} \left[\varepsilon - \overline{a} + u(p_j) - r\right] \right) dF(\varepsilon) \Bigg]. \end{aligned} \tag{22}$$

After rearranging terms and using equation (20), equation (22) simplifies to equation (12). Finally, since the righthand side of (12) is strictly decreasing in $\overline{a}$ it has a unique solution. □

We will use the following lemma in subsequent proofs:

Lemma A2. *Fix a sequence of weakly increasing search thresholds. The measure of fresh consumers per firm in each period is*

$$m = \frac{\sum_{j=0}^{\infty} \gamma_f^j [1 - F(a_j)]}{\sum_{j=0}^{\infty} \gamma_f^j \left[1 - F(a_j) + [1 - F(b_j)]\, \gamma_c \frac{1-\gamma_c^j}{1-\gamma_c}\right]}. \tag{23}$$

Proof of Lemma A2. Note that a firm is searched by a measure $m/[\sum_{j=0}^{\infty}(1-\gamma_f)\gamma_f^j\,[1 - F(a_j)]]$ of fresh consumers each period. We can then write

$$\begin{aligned} m \;=\; & 1 - \gamma_c + \gamma_c (1 - \gamma_f) \\ & + \gamma_c\gamma_f m \sum_{i=0}^{\infty}(1-\gamma_f)\gamma_f^i \frac{[F(b_{i+1}) - F(a_i)]}{\sum_{j=0}^{\infty}(1-\gamma_f)\gamma_f^j\,[1 - F(a_j)]} \\ & + \gamma_c\gamma_f m \sum_{i=1}^{\infty}(1-\gamma_f)\gamma_f^i \frac{[F(b_{i+1}) - F(b_i)]}{\sum_{j=0}^{\infty}(1-\gamma_f)\gamma_f^i\,[1 - F(a_j)]} \gamma_c \frac{1-\gamma_c^i}{1-\gamma_c}. \end{aligned}$$

Consumers are fresh (first line) if they are new to the market, or they survived from the previous period but the product they purchased that period did not survive. Consumers are also fresh (second line) if in the previous period they were fresh and they discovered a product with age $i = 0, 1, \ldots$ that was good enough to buy ($\varepsilon \geq a_i$) but is not good enough to return to this period ($\varepsilon < b_{i+1}$), even though both they and the product have survived to the current period. Consumer are also fresh (third line) if last period they returned to a firm of age $i = 1, 2, \ldots$ (and so have $\varepsilon \geq b_i$), having discovered that product when they were fresh in one of the $j = 1, ..., i$ previous periods and having survived to the last period (which happens with probability $\sum_{j=1}^{i} \gamma_c^j = \gamma_c \frac{1-\gamma_c^i}{1-\gamma_c}$), but do not return this period (and so have $\varepsilon < b_{i+1}$) even though both they and the product have survived to the current period.

Solving this linear equation for m gives the expression in equation (23). $\square$

Proof of Lemma 8. The proof of the first part is similar to that of Lemma 4, and the proof of the second part is similar of that of Lemma 5, so both are omitted. $\square$

Proof of Lemma 9. Note that for $p_i \leq u^{-1}[u(\hat{p}_i) - r]$ such that $Q(p_i) > 0$ and $a_i + u(\hat{p}_i) - u(p_i) < \overline{\varepsilon}$, the derivative of equation (15) with respect to p_i is proportional to

$$-\left\{ \frac{1}{-\Psi(p_i)} + \frac{f(a_i + u(\hat{p}_i) - u(p_i))}{1 - F\left(a_i + u(\hat{p}_i) - u(p_i)\right) + \left[1 - F\left(a_i + r\right)\right] \gamma_c \frac{1-\gamma_c^i}{1-\gamma_c}} \right\}. \tag{24}$$

We now solve for candidate optimal price(s). Note that $\hat{p}_i \leq p^m$: if on the contrary $\hat{p}_i > p^m$, firm i could deviate to $p_i = p^m$ and strictly increase its profit $\pi(p_i)$ per consumer and also weakly increase its demand in the current period. Therefore henceforth suppose $\hat{p}_i \leq p^m$.

1. Suppose $a_i < \underline{\varepsilon}$. Note that (24) is continuous in p_i around $\hat{p}_i$, and when evaluated at $p_i = \hat{p}_i$ it equals $1/\Psi(\hat{p}_i)$, which is strictly positive for $\hat{p}_i < p^m$. Thus p^m is the optimal price.

2. Suppose $a_i = \underline{\varepsilon}$. Note that (24) equals $1/\Psi(p_i) > 0$ for all $p_i < \hat{p}_i$. Also note that (24) is right-continuous in p_i around $\hat{p}_i$. Moreover, when evaluated at $p_i = \hat{p}_i$, (24) is strictly positive for $\hat{p}_i < \tilde{p}_i(\underline{\varepsilon})$, zero at $\hat{p}_i = \tilde{p}_i(\underline{\varepsilon})$, and strictly negative for $\hat{p}_i \in (\tilde{p}_i(\underline{\varepsilon}), p^m]$. Hence any $\hat{p}_i \in [\tilde{p}_i(\underline{\varepsilon}), p^m]$ is an optimal price.

3. Suppose $a_i \in (\underline{\varepsilon}, \overline{\varepsilon})$. Note that (24) is continuous in p_i around $\hat{p}_i$, and equals zero at the $\tilde{p}_i(a_i)$ from equation (16). Hence $\tilde{p}_i(a_i)$ is the optimal price.

Finally, quasiconcavity of the profit of a firm of age i follows exactly the same steps as in the proof of Proposition 1. Specifically, the first term in the curly brackets in (24) is increasing in p_i. Meanwhile, defining $\tilde{\varepsilon} = a_i + u(\hat{p}_i) - u(p_i)$, the second term is zero for $\tilde{\varepsilon} < \underline{\varepsilon}$, and for $\tilde{\varepsilon} > \underline{\varepsilon}$ its derivative with respect to p_i is proportional to

$$\frac{f'(\tilde{\varepsilon})[1 - F(\tilde{\varepsilon})]}{[f(\tilde{\varepsilon})]^2} \left[1 + \frac{\gamma_c(1-\gamma_c^i)}{1-\gamma_c} \frac{1 - F(a_i + r)}{1 - F(\tilde{\varepsilon})}\right] + 1 \geq \gamma_c \left[1 - (1 - \gamma_c^i) \frac{1 - F(a_i + r)}{1 - F(\tilde{\varepsilon})}\right] \geq 0,$$

where the first inequality uses Assumption 1 and the second uses $\tilde{\varepsilon} < a_i + r$ and $\tilde{\varepsilon} < \overline{\varepsilon}$. $\square$

We require Lemmas A3-A8 in the proof of Theorem 1. We will assume that s is sufficiently low that a consumer prefers to search rather than not participate in the market. We let $\{a_i^*\}_{i=0}^{\infty}$ and $\{p_i^*\}_{i=0}^{\infty}$ denote the sequences of equilibrium search thresholds and prices.

Lemma A3. *For all $i = 0, 1, \ldots$ equilibrium search thresholds satisfy $a_i^* < \overline{\varepsilon} - r$.*

Proof of Lemma A3. We first prove that $a_i^* < \overline{\varepsilon}$ for all $i = 0, 1, \ldots$. If to the contrary there exists some i such that $a_i^* \geq \overline{\varepsilon}$, then a firm with a product of age i makes zero sales and earns zero profit. However, the firm could deviate and charge a small but positive price and earn strictly positive profits from consumers with match values close enough to $\overline{\varepsilon}$.

We now prove that $a_i^* < \overline{\varepsilon} - r$ for all $i \geq 1$. Suppose to the contrary there exists some $i \geq 1$ where $a_i^* \in [\overline{\varepsilon} - r, \overline{\varepsilon})$, i.e., no consumer returns to a product of age i. It follows that there exists at least one other product with a price strictly below p_i^*. Since we look for an equilibrium with increasing prices we must therefore have $p_0^* < p_i^*$. Equation (13) then implies that $a_0^* < a_i^*$. There are then two cases to consider. One case is that $a_i^* \leq \underline{\varepsilon}$. This implies that $a_0^* < \underline{\varepsilon}$. Using Lemma 9 we then have $p_0^* = p^m \geq p_i^*$, which yields a contradiction. The other case is that $a_i^* > \underline{\varepsilon}$, in which case Lemma 9 implies that

$$p_i^* = \Psi^{-1}\left(\frac{1 - F(a_i^*)}{f(a_i^*)}\right) < p^m. \tag{25}$$

However, Lemma 9 also implies that if $a_0^* < \underline{\varepsilon}$ then $p_0^* = p^m > p_i^*$, whilst if $a_0^* \geq \underline{\varepsilon}$ then

$$p_0^* \geq \Psi^{-1}\left(\frac{1 - F(a_0^*)}{f(a_0^*)}\right), \tag{26}$$

which is also incompatible with $p_0^* < p_i^*$ because Ψ and $f(\varepsilon)/[1 - F(\varepsilon)]$ are both increasing. We therefore reach a contradiction.

We have proved that $a_i^* < \overline{\varepsilon} - r$ for all $i \geq 1$. Since a_i^* is weakly increasing in i we therefore also have $a_0^* < \overline{\varepsilon} - r$. □

Lemma A4. *Suppose that $r \geq \overline{\varepsilon} - \underline{\varepsilon}$. Then there is a unique equilibrium where $p_i^* = p^m$ and $a_i^* = a$ for all firm ages $i = 0, 1, \ldots$, where $a < \underline{\varepsilon}$ solves*

$$s = \int_a^{\overline{\varepsilon}} (\varepsilon - a) dF(\varepsilon) + \sum_{j=1}^{\infty} (\delta_c \gamma_c \gamma_f)^j \int_{a+r}^{\overline{\varepsilon}} (\varepsilon - a - r) dF(\varepsilon). \tag{27}$$

Proof of Lemma A4. First, note that given $r \geq \overline{\varepsilon} - \underline{\varepsilon}$, Lemma A3 implies that $a_i^* < \underline{\varepsilon}$ for all $i = 0, 1, \ldots$. Lemma 9 then implies that $p_i^* = p^m$ and for all $i = 0, 1, \ldots$, and equation (13) then implies $a_i^* = a$ for all $i = 0, 1, \ldots$. Substituting this into equation (12) and simplifying then gives (27); one can check that the unique solution to equation (27) satisfies $a < \underline{\varepsilon}$. □

Lemma A5. *Suppose that $r < \overline{\varepsilon} - \underline{\varepsilon}$. Then for any $a \in (\underline{\varepsilon}, \overline{\varepsilon} - r)$, $\tilde{p}_i(a)$ is continuous and strictly decreasing in a, and is strictly increasing in i.*

Proof of Lemma A5. Rewrite equation (16) as

$$\tilde{p}_i(a) = \Psi^{-1}\left(\frac{1-F(a)}{f(a)}\left(1+\frac{[1-F(a+r)]\gamma_c\frac{1-\gamma_c^i}{1-\gamma_c}}{1-F(a)}\right)\right). \tag{28}$$

The argument of $\Psi^{-1}(.)$ is continuous in a, and also strictly decreasing in a because by assumption $[1-F(a)]/f(a)$ is strictly decreasing, which in turn also implies that $[1-F(a+r)]/[1-F(a)]$ is strictly decreasing. The argument of $\Psi^{-1}(.)$ is also strictly increasing in i. Since Ψ^{-1} is continuous and strictly increasing the results then follow. □

Lemma A6. *Suppose $r < \overline{\varepsilon} - \underline{\varepsilon}$. Define the thresholds*

$$\overline{V}_i = \lim_{a_i \uparrow \overline{\varepsilon}-r} \frac{a_i + u(\tilde{p}_i(a_i))}{1-\delta_c\gamma_c}, \qquad \underline{V}_i = \lim_{a_i \downarrow \underline{\varepsilon}} \frac{a_i + u(\tilde{p}_i(a_i))}{1-\delta_c\gamma_c}, \qquad V' = \frac{\underline{\varepsilon} + u(p^m)}{1-\delta_c\gamma_c}. \tag{29}$$

Firstly, $\overline{V}_0 = \overline{V}_1 = \cdots \equiv \overline{V}$. Secondly, $\overline{V} > \underline{V}_i > \underline{V}_{i+1} > V'$ for each $i = 0, 1, \ldots$

Proof of Lemma A6. For the first part, note that for all i

$$\lim_{a_i \uparrow \overline{\varepsilon}-r} \tilde{p}_i(a_i) = \Psi^{-1}\left(\frac{1-F(\overline{\varepsilon}-r)}{f(\overline{\varepsilon}-r)}\right).$$

Hence $\overline{V}_i$ is the same for each $i = 0, 1, \ldots$. For the second part, Lemma A5 implies that $\lim_{a \uparrow \overline{\varepsilon}-r} u(\tilde{p}_i) \geq \lim_{a \downarrow \underline{\varepsilon}} u(\tilde{p}_i)$ and so $\overline{V} > \underline{V}_i$ for each i. Lemma A5 (as well as the assumption that $f(\underline{\varepsilon}) > 0$) also implies that $\lim_{a \downarrow \underline{\varepsilon}} \tilde{p}_i(a) < \lim_{a \downarrow \underline{\varepsilon}} \tilde{p}_{i+1}(a)$, and so $\underline{V}_i > \underline{V}_{i+1}$ for each i. Finally, equation (16) implies that $\lim_{a \downarrow \underline{\varepsilon}} \tilde{p}_i(a) < p^m$ and so $\underline{V}_i > V'$ for each i. □

Lemma A7. *Suppose that $r < \overline{\varepsilon} - \underline{\varepsilon}$. If an equilibrium exists, and V is the associated value of search, then:*

1. *If $V < V'$ then $p_i^* = p^m$ and $a_i^* = V(1-\delta_c\gamma_c) - u(p^m) < \underline{\varepsilon}$.*
2. *If $V \in [V', \underline{V}_i]$ then $p_i^* = u^{-1}[(1-\delta_c\gamma_c)V - \underline{\varepsilon}]$ and $a_i^* = \underline{\varepsilon}$.*
3. *If $V \in (\underline{V}_i, \overline{V})$ then $p_i^* = \tilde{p}_i(a_i^*)$ where $a_i^* \in (\underline{\varepsilon}, \overline{\varepsilon} - r)$ uniquely solves $a_i^* + u(p_i^*(a_i^*)) = V(1-\delta_c\gamma_c)$.*

Proof of Lemma A7. As a first step, note that equations (13) and (19) imply

$$V = \frac{a_i^* + u(p_i^*)}{1-\delta_c\gamma_c}. \tag{30}$$

Also recall from Lemma A3 that $a_i^* < \overline{\varepsilon} - r$.

Suppose that $V < V'$. First, it is impossible that $a_i^* \geq \underline{\varepsilon}$: Lemma 9 would imply that $p_i^* \leq p^m$, and then equation (30) would imply that $V \geq V'$, but this yields a contradiction. Second, then, if an equilibrium exists it has $a_i^* < \underline{\varepsilon}$. Lemma 9 implies that $p_i^* = p^m$. Equation (30) can then be solved to obtain $a_i^* = V(1-\delta_c\gamma_c) - u(p^m)$. (Using $V < V'$ and the definition of V' it is easy to verify that indeed $a_i^* < \underline{\varepsilon}$.)

Suppose that $V \in [V', \underline{V}_i]$. First, it is impossible that $a_i^* < \underline{\varepsilon}$: Lemma 9 would imply that $p_i^* = p^m$, and then equation (30) would imply that $V < V'$, which yields a contradiction. Second, it is also impossible that $a_i^* > \underline{\varepsilon}$: Lemma 9 would imply that $p_i^* = \tilde{p}_i(a_i^*)$, then Lemma A5 would imply that $p_i^* < \lim_{a_i \downarrow \underline{\varepsilon}} \tilde{p}_i(a_i)$, and then equation (30) would imply that $V > \underline{V}_i$, which again yields a contradiction. Finally, then, if an equilibrium exists it has $a_i^* = \underline{\varepsilon}$. Directly solving (30) gives $p_i^* = u^{-1}[(1-\delta_c\gamma_c)V - \underline{\varepsilon}]$, which is strictly decreasing in V, equals p^m as $V \to V'$, and equals $\lim_{a_i \downarrow \underline{\varepsilon}} \tilde{p}_i(a_i)$ as $V \to \underline{V}_i$. Note that these correspond to the highest and lowest equilibrium prices in Lemma 9 when $a_i^* = \underline{\varepsilon}$.

Suppose that $V \in (\underline{V}_i, \overline{V})$. First, it is impossible that $a_i^* \leq \underline{\varepsilon}$: Lemma 9 would imply that $p_i^* \geq \lim_{a_i \downarrow \underline{\varepsilon}} \tilde{p}_i(a_i)$, and then equation (30) would imply that $V \leq \underline{V}_i$, which yields a contradiction. Second, then, by Lemma A3 if an equilibrium exists it has $a_i^* \in (\underline{\varepsilon}, \overline{\varepsilon} - r)$. Lemma 9 then implies that $p_i^* = \tilde{p}_i(a_i^*)$. Note that $a_i^* + u(p_i^*(a_i^*))$ is continuous and strictly increasing in $a_i^* \in (\underline{\varepsilon}, \overline{\varepsilon} - r)$ (from Lemma A5), equals $\underline{V}_i$ as $a_i^* \downarrow \underline{\varepsilon}$, and equals $\overline{V}$ as $a_i^* \uparrow \overline{\varepsilon} - r$. Hence for each $V \in (\underline{V}_i, \overline{V})$ there exists a unique a_i^* such that equation (30) holds. □

Lemma A8. *Suppose that $r < \overline{\varepsilon} - \underline{\varepsilon}$. If an equilibrium exists and V is the associated value of search, then let $a_i^*(V)$ and $p_i^*(V)$ be the associated search threshold and price for a firm of age i, for each $i = 0, 1, \ldots$*

1. *$a_i^*(V)$ is continuous in V, strictly increasing in V if $V < V'$ or $V > \underline{V}_i$, and constant in V otherwise.*

2. *$p_i^*(V)$ is continuous in V, strictly decreasing in V if $V > V'$, and constant in V otherwise.*

3. *If $V \in (\underline{V}_i, \overline{V})$ then $a_j^*(V) < a_{j+1}^*(V)$ and $p_j^*(V) < p_{j+1}^*(V)$ for each $j \geq i$.*

Proof of Lemma A8. We start by proving parts 1 and 2. First, from Lemma A7, for $V < V'$, $a_i^*(V)$ is continuous and strictly increasing in V, while $p_i^*(V)$ is constant in V. Second, Lemma A7 also shows that $a_i^*(V)$ and $p_i^*(V)$ are continuous around V' because $\lim_{V \uparrow V'} a_i^*(V) = \underline{\varepsilon}$ and $\lim_{V \downarrow V'} p_i^*(V) = p^m$. Third, Lemma A7 also shows that for $V \in [V', \underline{V}_i]$, $a_i^*(V)$ is constant in V while $p_i^*(V)$ is strictly decreasing in V. Fourth, Lemma A5 implies that $a + u(\tilde{p}_i(a))$ is continuous and strictly increasing in $a \in (\underline{\varepsilon}, \overline{\varepsilon} - r)$. Lemma A7 then implies that for $V \in (\underline{V}_i, \overline{V})$, $a_i^*(V)$ is continuous and strictly increasing in V, while $p_i^*(V)$ is continuous and strictly decreasing in V. Moreover, Lemma A7 also implies that $a_i^*(V)$ and $p_i^*(V)$ are continuous around $\underline{V}_i$ because $\lim_{V \downarrow \underline{V}_i} a_i^*(V) = \underline{\varepsilon}$ and $\lim_{V \downarrow \underline{V}_i} p_i^*(V) = \tilde{p}_i(\underline{\varepsilon})$.

Finally, we prove part 3. Recall from Lemma A6 that $\underline{V}_j$ strictly decreases in j. Therefore since $V \in (\underline{V}_i, \overline{V})$, it is also true that for each $j \geq i$ we have $V \in (\underline{V}_j, \overline{V})$. Lemma A7 then further implies that, for each $j \geq i$, $p_j^*(V) = \tilde{p}_j(a_j^*)$ and $a_j^*(V)$ is the unique solution to $a_j^* + u(\tilde{p}_j(a_j^*)) = V(1 - \delta_c\gamma_c)$. On the way to a contradiction, suppose that $a_j^*(V) \geq a_{j+1}^*(V)$. This would imply that $p_j^*(V) < p_{j+1}^*(V)$, because Lemma A5 shows that $\tilde{p}_i(a)$ is strictly decreasing in a and strictly increasing in i. This would in turn imply that $a_j^* + u(p_j^*(a_j^*)) > a_{j+1}^* + u(p_{j+1}^*(a_{j+1}^*))$. But this is impossible because it violates equation (30). Hence $a_j^*(V) < a_{j+1}^*(V)$. Equation (30) then implies that $p_j^*(V) < p_{j+1}^*(V)$. $\square$

Proof of Lemma 10. We first derive the set of (r, s) such that $V = 0$. Using Lemmas A4 and A7 (depending on whether r is above or below $\overline{\varepsilon} - \underline{\varepsilon}$), in this set $a_i^* = -u(p^m)$ and $p_i^* = p^m$ for all i. Combining this with equations (12) and (13) we obtain

$$s = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} (\varepsilon + u(p^m))dF(\varepsilon) + \frac{\delta_c\gamma_c\gamma_f}{1 - \delta_c\gamma_c\gamma_f} \int_{r-u(p^m)}^{\overline{\varepsilon}} (\varepsilon + u(p^m) - r)dF(\varepsilon). \tag{31}$$

Denote by $\overline{s}(r)$ the righthand side of this equation, which for any r gives the unique s such that $V = 0$. It is easy to see there is a unique $\overline{r}$ such that $\overline{s}(\overline{r}) = \overline{r}$, where $\overline{r} < \overline{\varepsilon} + u(p^m)$. Note that $\overline{s}(r)$ is strictly decreasing on $(0, \overline{r}]$, and that for $r > \overline{r}$ we have $\overline{s}(r) < r$.

We now prove that for $s > \max\{\overline{s}(r), r\}$ the market is inactive. For the case $r \geq \overline{\varepsilon} - \underline{\varepsilon}$ this follows directly from Lemma A4. Now consider the case $r < \overline{\varepsilon} - \underline{\varepsilon}$. Towards a contradiction, suppose the market is active and so $V > 0$. From Lemma A8, $a_i^*(V) \geq a_i^*(0)$ for each i. But equations (12) and (13) then imply that $s \leq \max\{\overline{s}(r), r\}$, which is a contradiction. $\square$

Proof of Theorem 1. We begin with the case $r < \overline{\varepsilon} - \underline{\varepsilon}$. Consider $V \in [0, \overline{V}]$ and define a function $S(r, V)$ by substituting $a_i^*(V) = \overline{a} - u(p_i^*)$ into equation (12):

$$S(r, V) = \sum_{i=0}^{\infty} (1 - \gamma_f)\gamma_f^i \Bigg[\int_{a_i^*(V)}^{\overline{\varepsilon}} (\varepsilon - a_i^*(V))dF(\varepsilon) \\ + \sum_{j=1}^{\infty} (\delta_c\gamma_c\gamma_f)^j \int_{a_{i+j}^*(V)+r}^{\overline{\varepsilon}} (\varepsilon - a_{i+j}^*(V) - r)dF(\varepsilon) \Bigg]. \tag{32}$$

First, note that $S(r, V)$ is strictly decreasing in V on $[0, V') \cup (\underline{V}_\infty, \overline{V}]$, where $\underline{V}_\infty = \lim_{i\to\infty} \underline{V}_i$. This is because i) Lemma A3 implies that $a_i^*(V) < \overline{\varepsilon}$, and hence the right-hand side of (32) is strictly decreasing in each a_i^*, and ii) from Lemma A8 all $a_i^*(V)$ are weakly increasing and there exists at least one i such that $a_i^*(V)$ is strictly increasing for V on $[0, V') \cup (\underline{V}_\infty, \overline{V}]$. Second, also note that by Lemma A8 we have that for $V \in [V', \underline{V}_\infty]$ $a_i^*(V) = \underline{\varepsilon}$ and therefore $S(r, V)$ is constant in V on that interval.

Now for each $i = 0, 1, \ldots$ we define

$$\bar{s}_i(r) = S(r, \underline{V}_i). \tag{33}$$

Note that $\bar{s}_i(r)$ strictly increases in i, because from Lemma A6 $\underline{V}_\infty < \underline{V}_{i+1} < \underline{V}_i$ for all i.

Consider $s < \bar{s}_0(r)$. It is immediate that $V > \underline{V}_0$. From Lemma A7 we have $p_i^* = \tilde{p}_i(a_i^*) < p^m$ and $a_i^* > \underline{\varepsilon}$. From Lemma A8, a_i^* and p_i^* both strictly increase in i.

Now consider $s \in [\bar{s}_i(r), \bar{s}_{i+1}(r))$ for any $i \geq 0$. It is immediate that $V \in (\underline{V}_{i+1}, \underline{V}_i]$. Lemma A7 implies that $p_0^* = \cdots = p_i^* = u^{-1}[(1 - \delta_c\gamma_c)V - \underline{\varepsilon}] \in [\tilde{p}_i(\underline{\varepsilon}), \tilde{p}_{i+1}(\underline{\varepsilon}))$ and $a_0^* = \cdots = a_i^* = \underline{\varepsilon}$. Lemma A7 also implies that $a_j^* > \underline{\varepsilon}$ for each $j > i$, which from equation (30) further implies that $p_j^* = \tilde{p}(a_j^*) \in (p_i^*, p^m)$ for each $j > i$. Finally, for each $j > i$ both a_j^* and p_j^* are strictly increasing in j by Lemma A8.

Now suppose that $s = \lim_{i\to\infty} \bar{s}_i(r)$. Hence $V \in [V', \underline{V}_\infty]$, so by Lemma A7 there is a continuum of equilibria, in which (a_i, p_i^*) is the same for all $i = 0, 1, \ldots,$ $a_i = \underline{\varepsilon}$ and $p_i^* = u^{-1}[(1 - \delta_c\gamma_c)V - \underline{\varepsilon}]$ where $V \in [V', \underline{V}_\infty]$ is the equilibrium value of search.

Next, consider $\lim_{i\to\infty} \bar{s}_i(r) < s \leq \bar{s}(r)$. It is immediate that $V \in [0, V')$. Lemma A7 then implies that $a_i^* < \underline{\varepsilon}$ and $p_i^* = p^m$ for each $i = 0, 1, \ldots.$ Moreover, as $V' > 0$ we have that $a > -u(p^m)$ and so $\lim_{i\to\infty} \bar{s}_i(r) < \bar{s}(r)$ for $r < \bar{r}$.

We now turn to the case of $r > \bar{\varepsilon} - \underline{\varepsilon}$. Lemma A4 proved that (a_i^*, p_i^*) is the same for all i, with $a_i^* < \underline{\varepsilon}$ and $p_i^* = p^m$. Hence it suffices to choose $\{\bar{s}_i(r)\}_{i=0}^\infty$ to be any strictly increasing sequence with $\lim_{i\to\infty} \bar{s}_i(r) < r$, thus ensuring we are in case 3 of Theorem 1. □

Proof of Proposition 2. Let $R_i^* = [1 - F(a_i^* + r)]\gamma_c \frac{1-\gamma_c^i}{1-\gamma_c}$, which is proportional to the equilibrium return demand of a seller of age i.

We begin by establishing that $R_{i+1}^* > R_i^*$ for each i. There are two cases to consider. First, suppose $p_i^* = p_{i+1}^*$. Equation (30) implies that $a_i^* = a_{i+1}^*$, so it is immediate from the definition of R_i^* that $R_{i+1}^* > R_i^*$. Second, suppose $p_i^* < p_{i+1}^*$. Using Theorem 1 we know that $s < \bar{s}_{i+1}(r)$ and hence $p_{i+1}^* = \tilde{p}_{i+1}(a_{i+1}^*)$. Using equation (16) we can therefore write

$$\frac{1}{-\Psi(p_{i+1}^*)} + \frac{f(a_{i+1}^*)}{1 - F(a_{i+1}^*) + R_{i+1}^*} = 0. \tag{34}$$

Using the same steps as in the proof of Lemma 9, $f(a)/[1 - F(a) + R_{i+1}^*]$ is increasing in $a \in [\underline{\varepsilon}, a_{i+1}^*]$. Also, Theorem 1 implies $a_i^* \geq \underline{\varepsilon}$, while equation (30) and $p_i^* < p_{i+1}^*$ imply $a_i^* < a_{i+1}^*$. Since Ψ is strictly increasing, equation (30), equation (34) then implies that

$$\frac{1}{-\Psi(p_i^*)} + \frac{f(a_i^*)}{1 - F(a_i^*) + R_{i+1}^*} < 0. \tag{35}$$

Meanwhile, Theorem 1 implies that $p_i^* \geq \tilde{p}_i(a_i^*)$, and so from equation (16) we have

$$\frac{1}{-\Psi(p_i^*)} + \frac{f(a_i^*)}{1 - F(a_i^*) + R_i^*} \geq 0. \tag{36}$$

Equations (35) and (36) are only compatible if $R_{i+1}^* > R_i^*$.

We have established that $R_{i+1}^* > R_i^*$. A seller of age $i+1$ therefore earns strictly more profit than a seller of age i, because it could always charge p_i^* and earn the same profit on fresh consumers as the seller of age $i+1$ does, but earn strictly more than it from return consumers (since strictly more consumers return to it, and given $p_i^* \leq p_{i+1}^*$ they all buy). □

Proof of Proposition 3. Suppose either s increases or δ_c decreases. Recall equation (32) and note that $s = S(r, V)$ must hold before and after the parameter change. This implies that some a_i^* must change. Lemma A7 implies that if one a_i^* strictly increases, no a_j^* strictly decreases, which implies that $S(r, V)$ decreases, leading to a contradiction. As $s < \lim_{i\to\infty} \bar{s}_i(r)$, for at least one i we have $a_i^* > \underline{\varepsilon}$. Then, part 3 of Lemma A7 implies that $(1-\delta_c\gamma_c)V$ decreases following the parameter change. Hence, all the prices weakly increase. □

Proof of Proposition 4. First, note that we must have $r < \bar{\varepsilon} - \underline{\varepsilon}$. This is because from Lemma A4, if $r \geq \bar{\varepsilon} - \underline{\varepsilon}$ then all firms charge p^m, which from Theorem 1 is incompatible with $s < \lim_{i\to\infty} \bar{s}_i(r)$. Second, note that at $V = \underline{V}_\infty$ we must have $a_i^* = \underline{\varepsilon}$ for all $i = 0, 1, \ldots$, otherwise from Lemma A7 there would exist i with $a_i^* < \underline{\varepsilon}$ and $p_i^* = p^m$ which would violate (30). It follows that $\lim_{i\to\infty} \bar{s}_i(r) = \mathbb{E}(\varepsilon) - \underline{\varepsilon}$ which is independent of γ_f, γ_c, and r.

Let $x \in \{\gamma_f, \gamma_c, r\}$ denote the parameter of interest. Rewrite the lefthand side of equation (16) as $\tilde{p}_i(a_i, x)$, and denote the thresholds in equation (29) by $\overline{V}(x), \underline{V}_i(x), V'(x)$. Using Lemma A7 there is a system of equations:

$$a_i^* + u(p_i^*) = V, \qquad p_i^* = \begin{cases} p^m, & V < V'(x) \\ u^{-1}(V - \underline{\varepsilon}), & V \in [V'(x), \underline{V}_i(x)] \\ \tilde{p}_i(a_i^*, x), & V \in (\underline{V}_i(x), \overline{V}(x)) \end{cases}$$

whose solution $a_i^*(V, x)$ and $p_i^*(V, x)$ always exists and is unique. Moreover, $a_i^*(V, x)$ is weakly increasing in V, and $p_i^*(V, x)$ is weakly decreasing in V. Rewrite equation (32) as

$$s = \sum_{i=0}^{\infty} (1-\gamma_f)\gamma_f^i \left[\int_{a_i^*(V,x)}^{\bar{\varepsilon}} (\varepsilon - a_i^*(V, x)) dF(\varepsilon) \right], \tag{37}$$

and note that the righthand side is decreasing in V. Also recall from the proof of Theorem 1 that because $s < \lim_{i\to\infty} \bar{s}_i(r)$ this equation has a unique solution, which we denote by $V^*(x)$.

Start with the case $x = \gamma_f$. Consider $\gamma_f \in \{\gamma_f^l, \gamma_f^h\}$ where $0 \leq \gamma_f^l < \gamma_f^h < 1$. Notice that $a_i^*(V, \gamma_f^l) = a_i^*(V, \gamma_f^h)$ and $p_i^*(V, \gamma_f^l) = p_i^*(V, \gamma_f^h)$. Hence it follows from equation (37) that

$V^*(\gamma_f^l) \geq V^*(\gamma_f^h)$. This in turn implies that

$$p_i^*(\gamma_f^h) \equiv p_i^*(V^*(\gamma_f^h), \gamma_f^h) \geq p_i^*(V^*(\gamma_f^l), \gamma_f^h) = p_i^*(V^*(\gamma_f^l), \gamma_f^l) \equiv p_i^*(\gamma_f^l),$$

where the inequality uses $V^*(\gamma_f^l) \geq V^*(\gamma_f^h)$ and $p_i^*(V, \gamma_f)$ weakly decreasing V, while the following equality uses $p_i^*(V, \gamma_f^l) = p_i^*(V, \gamma_f^h)$.

Now consider the case $x = \gamma_c$. Consider $\gamma_c \in \{\gamma_c^l, \gamma_c^h\}$ where $0 \leq \gamma_c^l < \gamma_c^h < 1$. Then, $\tilde{p}_i(a_i^*, \gamma_c)$ is increasing in γ_c for $i \geq 1$, and constant in γ_c for $i = 0$. Hence, $\overline{V}(\gamma_c), \underline{V}_0(\gamma_c), V'(\gamma_c)$ are constant in γ_c, while $\underline{V}_i(\gamma_c)$ is decreasing in γ_c for $i \geq 1$. Therefore $p^*(V, \gamma_c^l) \leq p^*(V, \gamma_c^h)$, which then implies that $a^*(V, \gamma_c^l) \leq a^*(V, \gamma_c^h)$. Hence it follows from equation (37) that $V^*(\gamma_c^l) \geq V^*(\gamma_c^h)$. This in turn implies that

$$p_i^*(\gamma_c^h) \equiv p_i^*(V^*(\gamma_c^h), \gamma_c^h) \geq p_i^*(V^*(\gamma_c^l), \gamma_c^h) \geq p_i^*(V^*(\gamma_c^l), \gamma_c^l) \equiv p_i^*(\gamma_c^l),$$

where the first inequality uses $V^*(\gamma_c^l) \geq V^*(\gamma_c^h)$ and $p_i^*(V, \gamma_c)$ weakly decreasing in V, and the second inequality uses $p^*(V, \gamma_c^l) \leq p^*(V, \gamma_c^h)$.

Finally, consider the case $x = r$. Consider $r \in \{r^l, r^h\}$, where $0 < r^l < r^h < \overline{\varepsilon} - \underline{\varepsilon}$. The proof follows the same steps as the $x = \gamma_c$ case and so for brevity is omitted. □

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Online Appendix: Not For Publication

Uniqueness of Equilibria with Increasing Prices

Here we provide a proof of the claim on page 13 that for $\delta_c = 0$ all stationary symmetric equilibria must have increasing prices, as in Definition 1.

Proposition 9. *Suppose $\delta_c = 0$. There is no stationary symmetric equilibrium (with active search) where equilibrium prices satisfy $p_j^* > p_{j+1}^*$ for some $j = 0, 1, \ldots$*

Proof. First, note that conditional on a consumer deciding to search for a new product, her value of search is independent of her past search history. The reason is that the discounted value of search is equal to the current-period value of search, which is independent of the history given the consumer has decided not to return to any previously-searched product. Moreover, since we focus on an equilibrium with stationary prices, the value of search must be stationary; denote it by V, and notice that it is a function of the equilibrium price sequence. It then follows that a fresh consumer buys a product with match ε and price p if and only if $\varepsilon + u(p) \geq V$ and $u(p) \geq u(p^*) - r$ (because a consumer can always buy the same product from a different supplier at price p^*). Note also that the measure of fresh consumers searching a firm each period is stationary.

Next, we proceed by induction. We start by proving that $p_0^* \leq p_1^*$. Note that

$$p_0^* = \arg\max \pi(p)[1 - F(V - u(p))]\mathbf{1}_{u(p)\geq u(p_0^*)-r}, \quad \text{and}$$
$$p_1^* = \arg\max \pi(p)[1 - F(V - u(p)) + R_1(p_1^*)]\mathbf{1}_{u(p)\geq u(p_1^*)-r},$$

where $R_1(p_1^*)$ is the probability that a consumer who was fresh and visited this product when it had age 0 returns to this firm at age 1. Note that $R_1(p_1^*)$ is independent of the actual price charged by the firm, and satisfies $R_1(p_1^*) \leq \gamma_c[1 - F(V - u(p_1^*) + r)]$.[30] Following the same steps as in Proposition 1 (and using the upper bound on $R_1(p_1^*)$) it is straightforward to show that profit at both ages is quasiconcave, so both optimization problems have a unique maximizer. We now prove that $p_0^* \leq p_1^*$. On the way to a contradiction, suppose $p_0^* > p_1^*$. Note that by definition $\pi(p)[1 - F(V - u(p))]$ must be strictly increasing in p at $p = p_1^*$, given that $p_0^* \leq p^m$. But then it follows that $\pi(p)[1 - F(V - u(p)) + R_1(p_1^*)]$ is also strictly increasing in p at $p = p_1^*$, which is impossible.

Next, consider an arbitrary age $j \geq 1$, and suppose that $p_0^* \leq \cdots \leq p_j^*$. We now prove

[30] A necessary condition to return is that $\varepsilon + u(p_1^*) - r \geq V$, so $1 - F(V - u(p_1^*) + r)$ is an upper bound on the probability that a surviving consumer returns. As usual, once the consumer has incurred r, she will buy from this supplier if and only if $u(p) \geq u(p_1^*) - r$.

that $p_j^* \le p_{j+1}^*$. Note that

$$p_j^* = \arg\max \pi(p) \left[1 - F(V - u(p)) + \sum_{k=1}^{k=j} R_{j,k}(p_j^*) \right] \mathbf{1}_{u(p) \ge u(p_j^*) - r}, \quad \text{and}$$

$$p_{j+1}^* = \arg\max \pi(p) \left[1 - F(V - u(p)) + \sum_{k=1}^{k=j+1} R_{j+1,k}(p_{j+1}^*) \right] \mathbf{1}_{u(p) \ge u(p_{j+1}^*) - r},$$

where, for example, $R_{j,k}(p_j^*)$ is the probability that a consumer returns to the firm when it has age j, given the consumer was fresh and searched the product k periods earlier, and given that the firm is expected to charge p_j^*. Note that $R_{j,k}(p_j^*) \in \left(0, \gamma_c^k [1 - F(V - u(p_j^*) + r)]\right]$ and is independent from the actual price charged by the firm. We now establish that $R_{j,k}(p_j^*) = R_{j+1,k}(p_j^*)$, i.e., the probability that a consumer who was fresh k periods ago returns is the same for a firm of age j and $j+1$ if both firms charge the same price p_j^*. To this end, let H_k denote the distribution over the set of products that a consumer who was fresh k periods ago had encountered before she searched the current product in question, and which still exist in the market today. Note that H_k is stationary. A consumer returns to a firm at age $j, j+1$ only if $\varepsilon + u(p_j^*) - r \ge V$. Since by assumption the prices were increasing for ages up to j, it follows that a consumer who was fresh k periods ago and returns today, did not search for new products in the meantime. Hence we can write

$$R_{j,k}(p_j^*) = R_{j+1,k}(p_j^*) = \int \Pr[\varepsilon + u(p_j^*) - r \ge \max_{h \in H}\{V, \varepsilon_h + u(p_k^*) - r\}] dH_k(H).$$

We have therefore established that $R_{j,k}(p_j^*) = R_{j+1,k}(p_j^*)$. It then follows that if $p_j^* > p_{j+1}^*$, then $R_{j,k}(p_j^*) \le R_{j+1,k}(p_{j+1}^*)$. Since at age $j+1$ the firm has an extra cohort of return consumers, we must have $\sum_{k=1}^{k=j+1} R_{j+1,k}(p_{j+1}^*) \ge \sum_{k=1}^{k=j} R_{j,k}(p_j^*)$. However, the firm's optimization problems at ages $j, j+1$ would then imply that $p_j^* \le p_{j+1}^*$, which is a contradiction. Hence $p_j^* \le p_{j+1}^*$. Moreover, using the upper bounds on $R_{j,k}(p_j^*)$ and $R_{j+1,k}(p_{j+1}^*)$, and adapting the proof of Proposition 1, one can then prove that the firm's profit at ages $j, j+1$ are quasiconcave.

To conclude, we have proved that $p_0^* \le p_1^*$, and we have also proved that if p_i^* weakly increases for all $i \le j$, then $p_j^* \le p_{j+1}^*$. It then follows that equilibrium price weakly increases in firm age, for all firm ages. □

Proofs for Section 5.1 (Price Discrimination)

Proof of Lemma 11. The proof follows from arguments in the text and so is omitted. □

Proof of Proposition 5. Let $p^*_{i,h}$ denote the equilibrium price charged by firms of age i to consumers with search history h. Let $a^*_{i,h}$ and $b^*_{i,h}$ be defined such that a consumer with search history h buys a product of age i when fresh if and only if $\varepsilon \geq a^*_{i,h}$, and returns to a product of age i if and only if $\varepsilon \geq b^*_{i,h}$. Let V denote the stationary value of search.

Assumption 2 and Lemma 11 say that $p^*_{i,h} = p^*_r = p^m$ for all firm ages i and all search histories h such that the consumer has previously searched a firm's product. A consumer with search history h and match $\varepsilon = b^*_{i,h}$ must be indifferent between returning to a firm of age i for one period and then searching, or just searching this period. Hence

$$b^*_{i,h} + u(p^m) - r + \delta_c\gamma_c V = V \implies b^*_{i,h} = b^* = V(1-\delta_c\gamma_c) + r - u(p^m). \tag{38}$$

Now consider prices charged to fresh consumers. In any equilibrium with search, no firm charges more than p^m. Therefore, given $r > 0$, a fresh consumer with search history h who draws a product of age i with match $\varepsilon = a^*_{i,h}$ will not return in the future. Hence

$$a^*_{i,h} + u(p^*_{i,h}) + \delta_c\gamma_c V = V. \tag{39}$$

Suppose a firm deviates and charges a fresh consumer $p_{i,h} \leq p^m$. Incurring r and visiting another seller of the same product is dominated since that seller will charge the consumer p^m (by Assumption 2). Hence the consumer either buys or searches a new product; let $\tilde{\varepsilon}$ be the match of the fresh consumer who is indifferent between these two options. Note that this consumer will not return next period, since she will pay a weakly higher price and need to incur return cost r. Hence $\tilde{\varepsilon} + u(p_{i,h}) = V(1-\delta_c\gamma_c) = a^*_{i,h} + u(p^*_{i,h})$. This implies that the firm's current-period profit from fresh consumers is proportional to

$$\pi(p_{i,h})[1 - F(V(1-\delta_c\gamma_c) - u(p_{i,h}))] = \pi(p_{i,h})[1 - F(a^*_{i,h} + u(p^*_{i,h}) - u(p_{i,h}))]. \tag{40}$$

Following the same reasoning as for Lemma 3, the firm's future profit is independent of $p_{i,h}$ so it chooses $p_{i,h}$ to maximize (40). Closely following the proof of Proposition 1, (40) is quasiconcave in $p_{i,h}$, and i) $p^*_{i,h} = p^m$ when $a^*_{i,h} < \underline{\varepsilon}$, ii) $p^*_{i,h} \in [\Psi^{-1}[1/f(\underline{\varepsilon})], p^m]$ when $a^*_{i,h} = \underline{\varepsilon}$, and iii) $p^*_{i,h} = \Psi^{-1}[[1-F(a^*_{i,h})]/f(a^*_{i,h})]$ when $a^*_{i,h} > \underline{\varepsilon}$. It then follows that $a^*_{i,h} = a^*$ for all $i = 0, 1, \ldots$ and for all h: if not, there would exist some (j, h') and $(k, h'') \neq (j, h')$ such that $a^*_{j,h'} < a^*_{k,h''}$, but then $p^*_{j,h'} \geq p^*_{k,h''}$ from cases i)-iii) above, and then equation (39) could not hold. Since $a^*_{i,h} = a^*$ for all $i = 0, 1, \ldots,$ it then follows from equation (39) that $p^*_{i,h} = p^*_f$ for all $i = 0, 1, \dots$ as well. Combining this with equations (38) and (39), we obtain

$$b^* = a^* + r + u(p_f^*) - u(p^m). \tag{41}$$

Moreover, using the same steps involved in deriving equation (12), we can write that

$$s = \int_{a^*}^{\overline{\varepsilon}} (\varepsilon - a^*) dF(\varepsilon) + \frac{\delta_c \gamma_c \gamma_f}{1 - \delta_c \gamma_c \gamma_f} \int_{b^*}^{\overline{\varepsilon}} (\varepsilon - b^*) dF(\varepsilon). \tag{42}$$

Note that the righthand side of (42) evaluated at $a^* = \underline{\varepsilon}$ and $p_f^* = p^m$ coincides with $S(r, \underline{V}_\infty)$ defined in the proof of Theorem 1. Note also that the righthand side of (42) evaluated at $a^* = -u(p^m)$ and $p_f^* = p^m$ is equal to $\bar{s}(r)$ as defined in the proof of Lemma 10.

We start with $r < \overline{\varepsilon} - \underline{\varepsilon}$. From the proof of Theorem 1:

$$\lim_{i \to \infty} \bar{s}_i(r) = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} (\varepsilon - \underline{\varepsilon}) dF(\varepsilon) + \frac{\delta_c \gamma_c \gamma_f}{1 - \delta_c \gamma_c \gamma_f} \int_{\underline{\varepsilon}+r}^{\overline{\varepsilon}} (\varepsilon - \underline{\varepsilon} - r) dF(\varepsilon).$$

It will also be convenient to define

$$\overline{s}_a(r) = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} (\varepsilon - \underline{\varepsilon}) dF(\varepsilon) + \frac{\delta_c \gamma_c \gamma_f}{1 - \delta_c \gamma_c \gamma_f} \int_{b^\ddagger}^{\overline{\varepsilon}} (\varepsilon - b^\ddagger) dF(\varepsilon),$$

where

$$b^\ddagger \equiv \underline{\varepsilon} + r + u(\Psi^{-1}[1/f(\underline{\varepsilon})]) - u(p^m).$$

It is easy to see that $0 < \overline{s}_a(r) \leq \lim_{i \to \infty} \overline{s}_i(r)$.

First, consider the case $s > \lim_{i \to \infty} \overline{s}_i(r)$. We claim that $a^* < \underline{\varepsilon}$. On the way to a contradiction, suppose $a^* \geq \underline{\varepsilon}$; from earlier work, we would have $p_f^* \leq p^m$, and so from equation (41) we would have $b^* \geq \underline{\varepsilon} + r$, but then equation (42) would imply that $s \leq \lim_{i \to \infty} \overline{s}_i(r)$, which is a contradiction. Hence $a^* < \underline{\varepsilon}$. Using earlier work, this implies $p_f^* = p^m$, and so from equation (41) we have $b^* = a^* + r$. Substitute this into equation (42), and note that the righthand side is continuous and strictly decreasing in $a^* < \underline{\varepsilon}$, equals $\lim_{i \to \infty} \overline{s}_i(r)$ as $a^* \uparrow \underline{\varepsilon}$, and is strictly larger than s for sufficiently low values of a^*. Hence a unique $a^* < \underline{\varepsilon}$ solves equation (42). Finally, since $b^* = a^* + r$ and $a^* < \underline{\varepsilon}$ and $r < \overline{\varepsilon} - \underline{\varepsilon}$ we have $b^* < \overline{\varepsilon}$.

Second, consider the case $s < \overline{s}_a(r)$. We claim that $a^* > \underline{\varepsilon}$. On the way to a contradiction, suppose $a^* \leq \underline{\varepsilon}$; from earlier work, we would have $p_f^* \geq \Psi^{-1}[1/f(\underline{\varepsilon})]$, and so from equation (41) we would have $b^* \leq b^\ddagger$, but then equation (42) would imply that $s \geq \overline{s}_a(r)$, which is a contradiction. Hence $a^* > \underline{\varepsilon}$. Using earlier work, this implies $p_f^* = \Psi^{-1}[[1 - F(a^*)]/f(a^*)]$. Substitute equation (41) into equation (42), and note that the righthand side is continuous and strictly decreasing in $a^* > \underline{\varepsilon}$, tends to zero as $a^* \uparrow \overline{\varepsilon}$, and equals $\overline{s}_a(r)$ as $a^* \downarrow \underline{\varepsilon}$. Hence a unique $a^* > \underline{\varepsilon}$ solves equation (42). It is straightforward to see that there is a unique b^*, which is continuous and strictly decreasing in s, and moreover as $s \uparrow \bar{s}_a(r)$ we have $a^* \to \underline{\varepsilon}$ and $b^* \to b^\ddagger$. There are then two subcases to consider. 1) If $b^\ddagger \geq \bar{\varepsilon}$ it follows that $b^* > b^\ddagger$ for all $s < \bar{s}_a(r)$. 2) Suppose $b^\ddagger < \bar{\varepsilon}$. If at $s = r$ we have $b^* \geq \bar{\varepsilon}$, then there is a unique $s \in [r, \bar{s}_a(r))$ for which $b^* = \bar{\varepsilon}$. If instead at $s = r$ we have $b^* < \bar{\varepsilon}$ then $b^* < \bar{\varepsilon}$ for all $s \in [r, \bar{s}_a(r))$.

Third, consider the case $s \in [\bar{s}_a(r), \lim_{i\to\infty} \bar{s}_i(r)]$. We claim that $a^* = \underline{\varepsilon}$. On the way to a contradiction, suppose $a^* > \underline{\varepsilon}$; from earlier work, we would have $p_f^* < \Psi^{-1}[1/f(\underline{\varepsilon})]$, and so from equation (41) we would have $b^* > b^\ddagger$, but then equation (42) would imply that $s < \bar{s}_a(r)$, which is a contradiction. On the way to another contradiction, suppose $a^* < \underline{\varepsilon}$; from earlier work, we would have $p_f^* = p^m$, and so from equation (41) we would have $b^* < \underline{\varepsilon} + r$, but then equation (42) would imply that $s > \lim_{i\to\infty} \bar{s}_i(r)$, which is a contradiction. Hence $a^* = \underline{\varepsilon}$. Using earlier work, this implies that in any equilibrium $p_f^* \in [\Psi^{-1}[1/f(\underline{\varepsilon})], p^m]$. We now determine p_f^* and b^* in different subcases. 1) Suppose $\delta_c\gamma_c\gamma_f = 0$. Note that in this subcase $\bar{s}_a(r) = \lim_{i\to\infty} \bar{s}_i(r)$. It is clear that at $s = \bar{s}_a(r) = \lim_{i\to\infty} \bar{s}_i(r)$ any $p_f^* \in [\Psi^{-1}[1/f(\underline{\varepsilon})], p^m]$ satisfies equations (41) and (42) and is hence an equilibrium. For each equilibrium p_f^* there is a unique b^* which solves equation (41). 2) Suppose $\delta_c\gamma_c\gamma_f > 0$. Note that in this subcase $\bar{s}_a(r) < \lim_{i\to\infty} \bar{s}_i(r)$. a) Suppose $b^\ddagger \geq \bar{\varepsilon}$. When $s = \bar{s}_a(r)$ equation (42) implies that $b^* \geq \bar{\varepsilon}$; hence for any $b^* \in [\bar{\varepsilon}, b^\ddagger]$ there exist corresponding $p_f^* \in [\Psi^{-1}[1/f(\underline{\varepsilon})], u^{-1}[u(p^m) + \bar{\varepsilon} - \underline{\varepsilon} - r]]$ such that pair (b^*, p_f^*) satisfies equations (41) and (42) and is hence an equilibrium. When $s > \bar{s}_a(r)$ it follows from equation (41) that $b^* < \bar{\varepsilon}$. Substitute equation (41) into equation (42), and note that the righthand side is continuous and strictly increasing in p_f^*, equals $s = \bar{s}_a(r)$ at $p_f^* = \Psi^{-1}[1/f(\underline{\varepsilon})]$, and equals $s = \lim_{i\to\infty} \bar{s}_i(r)$ at $p_f^* = p^m$. It then follows there exists a unique p_f^*. Substituting it into equation (41) gives a unique b^*. b) Suppose $b^\ddagger < \bar{\varepsilon}$. We have already established that $p_f^* \geq \Psi^{-1}[1/f(\underline{\varepsilon})]$, and hence equation (41) implies that $b^* \leq b^\ddagger < \bar{\varepsilon}$. We can then follow the same steps as in the above case a).

We now define $\bar{s}_a(r)$ and $\bar{s}_b(r)$ for the different cases. (The following applies for both $\delta_c\gamma_c\gamma_f = 0$ and $\delta_c\gamma_c\gamma_f > 0$. The only difference is that for the former $\bar{s}_a(r) = \lim_{i\to\infty} \bar{s}_i(r)$, whereas for the latter $\bar{s}_a(r) < \lim_{i\to\infty} \bar{s}_i(r)$.) 1) If $b^\ddagger \geq \bar{\varepsilon}$ then $\bar{s}_b(r) = \bar{s}_a(r)$. 2) If $b^\ddagger < \bar{\varepsilon}$ and we have $b^* \geq \bar{\varepsilon}$ at $s = r$, we established earlier there is a unique $s \in [r, \bar{s}_a(r))$ such that $b^* = \bar{\varepsilon}$; denote that critical s by $\bar{s}_b(r)$. 3) If $b^\ddagger < \bar{\varepsilon}$ and we have $b^* < \bar{\varepsilon}$ at $s = r$, then without loss we can set $\bar{s}_b(r) \leq \min\{r, \bar{s}_a(r)\}$.

We now turn to the case $r \geq \bar{\varepsilon} - \underline{\varepsilon}$. We claim that $a^* < \underline{\varepsilon}$. On the way to a contradiction, suppose $a^* \geq \underline{\varepsilon}$; from equation (41) we would have $b^* \geq \bar{\varepsilon}$, but then equation (42) would imply that $s = \int_{a^*}^{\bar{\varepsilon}} (\varepsilon - a^*) dF(\varepsilon) < \bar{\varepsilon} - \underline{\varepsilon} \leq r$, which is a contradiction. Hence $a^* < \underline{\varepsilon}$. Using earlier work, this implies that $p_f^* = p^m$. It then follows from equation (41) that $b^* = a^* + r$. Substitute this into equation (42), and note that the righthand side is continuous and strictly decreasing in a^*, which implies that a^* is unique. Hence it suffices to choose any $\bar{s}_a(r)$ and $\lim_{i\to\infty} \bar{s}_i(r)$ satisfying $\bar{s}_a(r) \leq \lim_{i\to\infty} \bar{s}_i(r) < r$, thus ensuring we are in case 3 of Proposition 5. Finally, we prove that $b^* < \bar{\varepsilon}$. On the way to a contradiction, suppose that $b^* \equiv a^* + r \geq \bar{\varepsilon}$. Equation (42) simplifies to $s = \mathbb{E}\varepsilon - a^*$. But then we must have $s \leq \mathbb{E}\varepsilon - \bar{\varepsilon} + r < r$ which is impossible. Hence $b^* < \bar{\varepsilon}$. It then suffices to choose any $\bar{s}_b(r) \leq \bar{s}_a(r)$. □

Proof of Proposition 6. Note from the proof of Proposition 5 that for $\delta_c = 0$ we have $\bar{s}_a(r) = \lim_{i\to\infty} \bar{s}_i(r)$, and hence that with price discrimination fresh consumers pay $p_f^* = \Psi^{-1}[[1 - F(a^*)]/f(a^*)]$ where $a^* > \underline{\varepsilon}$ is the unique solution to

$$s = \int_{a^*}^{\bar{\varepsilon}} (\varepsilon - a^*) dF(\varepsilon). \tag{43}$$

Also note from equation (32) that without price discrimination we have

$$s = \sum_{i=0}^{\infty} (1 - \gamma_f)\gamma_f^i \left[\int_{a_i^*}^{\bar{\varepsilon}} (\varepsilon - a_i^*) dF(\varepsilon) \right]. \tag{44}$$

We now prove that without price discrimination, $a_0^* < a^*$. On the way to a contradiction, suppose that $a_0^* \geq a^*$. Since $a^* > \underline{\varepsilon}$, Theorem 1 implies that $a_i^* > \underline{\varepsilon}$ for all $i = 0, 1, \ldots$. But then equations (43) and (44) cannot hold simultaneously, yielding a contradiction. Hence $a_0^* < a^*$. We can then write that

$$p_i^* \geq p_0^* \geq \Psi^{-1} \left[\frac{1 - F(a_0^*)}{f(a_0^*)} \right] > \Psi^{-1} \left[\frac{1 - F(a^*)}{f(a^*)} \right] = p_f^*$$

where the first two inequalities follow from Theorem 1, and the third inequality uses $a_0^* < a^*$ as well as $\Psi^{-1}(.)$ being strictly increasing and $f(\varepsilon)$ having a strictly increasing hazard rate. Hence price discrimination strictly reduces the price charged by each firm to fresh consumers, and thus it strictly increases lifetime consumer surplus. □

Proofs for Section 5.2 (Unit Demand)

Proof of Proposition 7. First, note that Lemma 6 holds as stated, Lemma 7 holds as stated after substituting $u(p) = -p$, and Lemma 8 also holds as stated after substituting $u(p) = -p$ and $\pi(p) = p$. Also note that $r < \bar{\varepsilon} - \underline{\varepsilon} = \bar{\varepsilon}$; given that a consumer's maximum payoff in any period is $\bar{\varepsilon}$, if instead $s \geq r \geq \bar{\varepsilon} - \underline{\varepsilon} = \bar{\varepsilon}$ then consumers would not search or return in any period.

Second, we prove that $a_i^* > \underline{\varepsilon} = 0$ for all $i = 0, 1, \ldots$. Using equations (13) and (19) we have that $a_i^* - p_i^* = V(1 - \delta_c \gamma_c)$ for each i, and so we must have $a_i^* - p_i^* \geq 0$. Given that $p_i^* \geq 0$ we cannot have $a_i^* < 0$. Adapting arguments in the proof of Lemma A7 we also cannot have $a_i^* = 0$, because if we did we would have $p_i^* > 0$ and hence $a_i^* - p_i^* < 0$.

Next, note that Lemma A5 holds after substituting $\Psi(p) = p$. Adapting the proofs of part 3 of Lemmas A7 and A8, for any value of search $V \geq 0$ we have that $a_i^*(V)$ is strictly increasing in V and $p_i^*(V)$ is strictly decreasing in V, with $a_i^*(V) < a_{i+1}^*(V)$ and $p_i^*(V) < p_{i+1}^*(V)$ for each $i = 0, 1, \ldots$. Following the proof of Theorem 1, for each $V \geq 0$ there exists a unique sequence of thresholds $\{a_i^*(V)\}_{i=0}^{\infty}$ and prices $\{p_i^*(V)\}_{i=0}^{\infty}$. Finally, to obtain $\bar{s}(r)$, substitute $V = 0$ into the righthand side of equation (32). Then, note that $\lim_{r \to 0} \bar{s}(r) > 0$. To see this, note that if instead $\lim_{r \to 0} \bar{s}(r) = 0$ then we must have $\lim_{r \to 0} a_i^* = \bar{\varepsilon}$ for each $i = 0, 1, \ldots$, but then given $V = 0$ we would require that $\lim_{r \to 0} p_i^* = \bar{\varepsilon}$ for each $i = 0, 1, \ldots$, but this is incompatible with equation (28) from the proof of Lemma A5. We have therefore established that $\lim_{r \to 0} \bar{s}(r) > 0$; it follows by continuity that there exists some $\bar{r} > 0$ such that $\bar{s}(r) \geq r$ for all $r \in (0, \bar{r}]$. □

Proofs for Section 5.3 (Finite Number of Suppliers)

We prove Proposition 8 separately for the cases $n = 1$ and $n > 1$, starting with the latter.

Proof of Proposition 8 (case with $n > 1$). We prove that it is a Nash equilibrium for firms to charge prices $\{p_i^*\}_{i=0}^{\infty}$ from Theorem 1. This equilibrium is supported by the following strategy: regardless of firms' pricing histories, in each period, a firm of age $i = 0, 1, \ldots$ charges p_i^*. Notice that for any weakly increasing price path, consumer behaviour is described by Lemmas 6 and 7 (and hence also Corollary 2). Therefore, irrespective of the number of suppliers, for given price sequence $\{p_i^*\}_{i=0}^{\infty}$ the value of search is given by V in Theorem 1.

First, consider a firm of age $i = 0$ selling product $\mathcal{P}$. We will prove that its present-discounted profit is the same as in Lemma 8 (setting $i = 0$, replacing $\hat{p}_0$ by p_0^*, and for each $j = 0, 1, \ldots$ replacing a_j by a_j^* and b_j by b_j^*), and so it is optimal for the firm to charge p_0^*. To this end, suppose the firm makes a one-shot deviation and charges a price $p_0 \neq p_0^*$. We proceed in two steps.

i) Consider present-period profit. a) If $u(p_0) < u(p_0^*) - r$ then the deviating firm earns zero profit this period, because a consumer's utility from visiting another supplier of product $\mathcal{P}$ (i.e., $\varepsilon + u(p_0^*) - r$) is greater than her utility from buying from the deviating firm (i.e., $\varepsilon + u(p_0)$). b) If $u(p_0) \geq u(p_0^*) - r$ then, employing a similar logic, a consumer who searches the deviating firm will not visit another supplier of product $\mathcal{P}$. Instead, following the same steps as in Section 4, she buys from the deviating firm if $\varepsilon \geq \tilde{\varepsilon}_0 \equiv a_0^* + u(p_0^*) - u(p_0)$, and otherwise she searches for a new product. Hence in this case, fresh consumers who search the deviating firm buy with probability $1 - F(\tilde{\varepsilon}_0)$. (Note that since the firm has age 0 there are no returning consumers.)

ii) Consider future profits. Notice that in the period where the firm deviates, all fresh and return consumers with $\varepsilon \geq b_1$ who visit the deviating firm, either buy from the deviating firm or visit another supplier of product $\mathcal{P}$. (Specifically, because $\varepsilon \geq b_1 \geq b_0 = a_0 + r$, searching for a new product is dominated by incurring r and visiting another supplier of product $\mathcal{P}$ and paying p_0^*). Because in the following period all suppliers of product $\mathcal{P}$ (including the deviating firm) are expected to charge p_1^*, all these consumers with $\varepsilon \geq b_1$ randomly return (conditional on surviving) to one of those suppliers. This implies that when the suppliers have age 1, they are visited by exactly the same fresh and return consumers as would have been the case absent the deviation. Hence, by the definition of equilibrium prices in Theorem 1, starting from age 1 the firms optimally charge $\{p_i^*\}_{i=1}^{\infty}$ and earn the same profits as they would have absent the deviation.

Combining the above analysis, the deviating firm's present-discounted profit at age 0 is the same as in Lemma 8 (setting $i = 0$, replacing $\hat{p}_0$ by p_0^*, and for each $j = 0, 1, \ldots$ replacing a_j by a_j^* and b_j by b_j^*). Hence the deviation is unprofitable—by the definition of p_0^*, the firm optimally charges p_0^*.

Second, consider a firm of age 1 which, along with all other suppliers of its product, charged p_0^* in the previous period. Following the same logic as above, one can show that the firm's present-discounted profit from charging price $p_1 \neq p_1^*$ is the same as in Lemma 8 (setting $i = 1$, replacing $\hat{p}_1$ by p_1^*, and for each $j = 0, 1, \ldots$ replacing a_j by a_j^* and b_j by b_j^*). Hence, it is optimal for this firm to charge p_1^* this period. Proceeding by induction, in each period it is an equilibrium for all firms of age i to charge p_i^*. □

We now turn to the $n = 1$ case. The proof will require Lemmas A9 and A10.

Lemma A9. *Suppose $n = 1$ and Assumption 3 holds. Suppose all firms except one exogenously play the prices from Theorem 1 in each time period. Let V denote the associated value of search. Then for the remaining firm, the following strategy (hereafter denoted strategy †) forms a subgame perfect Nash equilibrium: given age $i \geq 1$ and arbitrary past prices $\mathbf{p_{i-1}} = (p_0, p_1, \ldots, p_{i-1})$, charge $p_i^\dagger(\mathbf{p_{i-1}})$ where*

1. *If $V \leq \underline{\varepsilon} + u(p^m)$, then $p_i^\dagger(\mathbf{p_{i-1}}) = p^m$.*

2. *If* $\underline{\varepsilon} + u(p^m) < V \leq V_i^\dagger(\mathbf{p_{i-1}})$, *then* $p_i^\dagger(\mathbf{p_{i-1}}) = u^{-1}(V - \underline{\varepsilon})$.

3. *If* $V > V_i^\dagger(\mathbf{p_{i-1}})$, *then* $p_i^\dagger(\mathbf{p_{i-1}})$ *is the unique solution to*

$$\frac{1}{\Psi(p_i^\dagger(\mathbf{p_{i-1}}))} - \frac{f(V - u(p_i^\dagger(\mathbf{p_{i-1}})))}{1 - F(V - u(p_i^\dagger(\mathbf{p_{i-1}}))) + \sum_{j=1}^{i} \gamma_c^j R_{i,j}^\dagger(\mathbf{p_{i-1}})} = 0. \tag{45}$$

The variables $V_i^\dagger(\mathbf{p_{i-1}})$ *and* $R_{i,j}^\dagger(\mathbf{p_{i-1}})$ *are defined as follows:*

$$R_{i,j}^\dagger(\mathbf{p_{i-1}}) = 1 - F\left(\max\left\{\zeta_{i,j}(\mathbf{p_{i-1}}), V - u(p_i^\dagger(\mathbf{p_{i-1}})) + r\right\}\right)$$
$$V_i^\dagger(\mathbf{p_{i-1}}) = \underline{\varepsilon} + u\left(\Psi^{-1}\left[\frac{1 + \sum_{j=1}^{i} \gamma_c^j [1 - F(\max\{\zeta_{i,j}(\mathbf{p_{i-1}}), \underline{\varepsilon} + r\})]}{f(\underline{\varepsilon})}\right]\right),$$

where

$$\zeta_{i,1}(\mathbf{p_{i-1}}) = V - u(p_{i-1})$$
$$\zeta_{i,j}(\mathbf{p_{i-1}}) = \max\left\{\begin{array}{c} V - u(p_{i-j}), V - u(p_{i-j+1}), \dots, V - u(p_{i-1}), \\ V - u(p_{i-j+1}^\dagger(\mathbf{p_{i-j}})) + r, \dots, V - u(p_{i-1}^\dagger(\mathbf{p_{i-2}})) + r \end{array}\right\}, \text{ for } j = 2, \dots i.$$

Proof of Lemma A9. Note that because there is a continuum of firms with differentiated products, all but one of which exogenously charges the prices from Theorem 1, and because $\delta_c = 0$, the value of search V is (for given parameters) the same as in Theorem 1 (in particular, it is independent of the pricing behavior of the "remaining" firm). The proof now proceeds in several steps:

- Step 1: Show that the prices $\{p_i^\dagger(\mathbf{p_{i-1}})\}_{i=1}^\infty$ in strategy † are well-defined.
- Step 2: Derive optimal consumer search and purchase behavior of consumers who visit the "remaining" firm when it charges arbitrary price p_i at age i.
- Step 3: Derive an expression for the firm's current-period profit.
- Step 4: Prove that current-period profit is maximized at $p_i^\dagger(\mathbf{p_{i-1}})$.
- Step 5: Prove that given an arbitrary pricing history, if from age i onwards the firm follows strategy †, then its prices weakly increase as it ages.
- Step 6: Prove that if the firm will follow strategy † from age $i+1$ onwards, then at age i it maximizes its lifetime discounted profit by charging $p_i^\dagger(\mathbf{p_{i-1}})$.
- Step 7: Argue that strategy † induces a subgame perfect Nash equilibrium.

Step 1. We start by showing that the prices $\{p_i^\dagger(\mathbf{p_{i-1}})\}_{i=1}^\infty$ are well-defined. Notice that, for any given p_0, $p_1^\dagger(p_0)$ is well-defined from strategy †. Notice also that, for any given p_0, the associated $p_1^\dagger(p_0)$, and the actual price p_1 charged at age 1, $p_2^\dagger(\mathbf{p_1})$, where $\mathbf{p_1} = (p_0, p_1)$, is also well-defined from strategy †. Indeed, for any $i > 1$, given $(p_0, \ldots, p_{i-1})$ and $(p_1^\dagger(\mathbf{p_0}), \ldots, p_{i-1}^\dagger(\mathbf{p_{i-2}}))$, $p_i^\dagger(\mathbf{p_{i-1}})$ is well-defined. Hence, by induction, $\{p_i^\dagger(\mathbf{p_{i-1}})\}_{i=1}^\infty$ are well-defined.

Step 2. We now derive optimal consumer search and purchase behavior of consumers who visit the "remaining" firm. Since $\delta_c = 0$, fresh consumers either buy and get $\varepsilon + u(p_i)$, or search and get V; hence they buy if and only if $\varepsilon \geq V - u(p_i)$. Consumers who bought in the previous period know the firm's pricing history (by Assumption 3), and so from strategy † they expect the firm to charge $p_i^\dagger(\mathbf{p_{i-1}})$, and therefore they return if $\varepsilon \geq V - u(p_i^\dagger(\mathbf{p_{i-1}})) + r$; once they have returned they learn the actual price p_i and buy if $\varepsilon \geq V - u(p_i)$. (Note that by Assumption 3 consumers who visited the firm in the past but did not buy last period cannot return and purchase this period.)

Step 3. We now derive the firm's current-period profit. Again, let p_i denote the actual price charged by this firm. Since all other firms charge (by assumption) the prices from Theorem 1, the "remaining" firm is visited in each period by the same measure of fresh consumers $m / \sum_{j=0}^\infty (1-\gamma_f)\gamma_f^j [1 - F(a_j)]$ as in our main analysis (see page 16 for details). Now recall Step 2 above. Fresh consumers who search this period buy if and only if $\varepsilon \geq V - u(p_i)$. In addition, consumers who first searched the firm $j = 1, \ldots, i$ periods ago, bought in that period if and only if $\varepsilon \geq V - u(p_{i-j})$, then returned and bought in all intervening periods including the current one if and only if $\varepsilon \geq V - u(p_k^\dagger(\mathbf{p_{k-1}})) + r$ and $\varepsilon \geq V - p_k$ for $k = i - j + 1, \ldots, i$. Hence we can write the firm's current-period profit as

$$\frac{m}{\sum_{j=0}^\infty (1-\gamma_f)\gamma_f^j [1 - F(a_j)]} \pi(p_i) \left[1 - F(V - u(p_i)) + \sum_{j=1}^{i} \gamma_c^j R_{i,j}(\mathbf{p_{i-1}}, p_i) \right], \tag{46}$$

where

$$R_{i,j}(\mathbf{p_{i-1}}, p_i) = 1 - F\left(\max\left\{ \zeta_{i,j}(\mathbf{p_{i-1}}), V - u(p_i), V - u(p_i^\dagger(\mathbf{p_{i-1}})) + r \right\} \right). \tag{47}$$

Again, note for future reference that the $p_i^\dagger(\mathbf{p_{i-1}})$ term that appears in equation (47) is consumers' expectation about the firm's price (based on strategy †), and it does not depend on the actual price p_i that the firm charges.

Step 4. We now prove that the profit expression (46) is maximized at $p_i = p_i^\dagger(\mathbf{p_{i-1}})$. Clearly the firm never maximizes profit by charging $p_i > p^m$, so henceforth focus on $p_i \leq p^m$. Clearly $p_i^\dagger(\mathbf{p_{i-1}})$ must also satisfy $V - u(p_i^\dagger(\mathbf{p_{i-1}})) < \overline{\varepsilon}$. (If to the contrary $V - u(p_i^\dagger(\mathbf{p_{i-1}})) \geq \overline{\varepsilon}$, the firm would make zero profit. But this is impossible, because the firm could always charge just above cost and make strictly positive profit from fresh consumers.)

Consider part 1 of strategy †. We will prove that for these parameters, $p_i^\dagger(\mathbf{p_{i-1}}) = p^m$. On the way to a contradiction, suppose that $p_i^\dagger(\mathbf{p_{i-1}}) < p^m$. Note that because $V \leq \underline{\varepsilon} + u(p^m)$, then $V - u(p_i^\dagger(\mathbf{p_{i-1}})) < \underline{\varepsilon}$. But this implies that the derivative of (46) with respect to p_i evaluated at $p_i = p_i^\dagger(\mathbf{p_{i-1}})$ is proportional to $\pi'(p_i^\dagger(\mathbf{p_{i-1}})) > 0$, a contradiction.

Now consider parts 2 and 3 of strategy †. First, we prove that $p_i^\dagger(\mathbf{p_{i-1}}) < p^m$. On the way to a contradiction, suppose that $p_i^\dagger(\mathbf{p_{i-1}}) = p^m$. Then the derivative of (46) with respect to p_i evaluated at $p_i = p_i^\dagger(\mathbf{p_{i-1}})$ is proportional to $\pi(p^m)u'(p^m)f(V - u(p^m)) < 0$, where the inequality uses $\underline{\varepsilon} < V - u(p^m) < \overline{\varepsilon}$. (Note that in points 2 and 3, $\underline{\varepsilon} < V - u(p^m)$ holds by assumption. Note also that at the beginning of Step 4, we argued that $V - u(p_i^\dagger(\mathbf{p_{i-1}})) < \overline{\varepsilon}$, so if $p_i^\dagger(\mathbf{p_{i-1}}) = p^m$ then $V - u(p^m) < \overline{\varepsilon}$.) Hence by slightly lowering price the firm could increase (46)—a contradiction. Thus $p_i^\dagger(\mathbf{p_{i-1}}) < p^m$. Second, we prove that $\underline{\varepsilon} + u(p_i^\dagger(\mathbf{p_{i-1}})) \leq V$. On the way to a contradiction, suppose that $\underline{\varepsilon} + u(p_i^\dagger(\mathbf{p_{i-1}})) > V$. Then the derivative of (46) with respect to p_i evaluated at $p_i = p_i^\dagger(\mathbf{p_{i-1}})$ is proportional to $\pi'(p_i^\dagger(\mathbf{p_{i-1}})) > 0$, where the inequality follows because $p_i < p^m$. (Specifically, $\underline{\varepsilon} + u(p_i^\dagger(\mathbf{p_{i-1}})) > V$ implies that $p_i^\dagger(\mathbf{p_{i-1}}) < u^{-1}(V - \underline{\varepsilon})$, while our assumption in parts 2 and 3 of strategy † that $V > \underline{\varepsilon} + u(p^m)$ implies that $u^{-1}(V - \underline{\varepsilon}) < p^m$.) Third, we have therefore established that $\underline{\varepsilon} \leq V - u(p_i^\dagger(\mathbf{p_{i-1}})) < \overline{\varepsilon}$. The right derivative evaluated at $p_i = p_i^\dagger(\mathbf{p_{i-1}})$ is proportional to

$$\frac{1}{\Psi(p_i^\dagger(\mathbf{p_{i-1}}))} - \frac{f(V - u(p_i^\dagger(\mathbf{p_{i-1}})))}{1 - F(V - u(p_i^\dagger(\mathbf{p_{i-1}}))) + \sum_{j=1}^{i} \gamma_c^j R_{i,j}^\dagger(\mathbf{p_{i-1}})}. \tag{48}$$

Adapting steps from earlier proofs, it is straightforward to show that (48) is strictly decreasing in $p_i^\dagger(\mathbf{p_{i-1}})$.[31] Therefore if (48) is weakly negative when evaluated at $p_i^\dagger(\mathbf{p_{i-1}}) = u^{-1}(V - \underline{\varepsilon})$, we must have $p_i^\dagger(\mathbf{p_{i-1}}) = u^{-1}(V - \underline{\varepsilon})$ as the solution; note that (48) evaluated at $p_i^\dagger(\mathbf{p_{i-1}}) = u^{-1}(V - \underline{\varepsilon})$ is weakly negative if and only if $V \leq V_i^\dagger$, i.e., we are in part 2 of strategy †. Now suppose instead that (48) evaluated at $p_i^\dagger(\mathbf{p_{i-1}}) = u^{-1}(V - \underline{\varepsilon})$ is strictly positive, i.e., we are in part 3 of strategy †. It is straightforward to show that (48) is strictly

[31] The first term is strictly decreasing since $\Psi(p_i^\dagger(\mathbf{p_{i-1}})) > 0$ is strictly increasing in $p_i^\dagger(\mathbf{p_{i-1}})$. It remains to show that the second term (after the minus sign) is weakly increasing. Since each $R_{i,j}^\dagger(\mathbf{p_{i-1}})$ term weakly decreases in $p_i^\dagger(\mathbf{p_{i-1}})$, the derivative of this second term with respect to $p_i^\dagger(\mathbf{p_{i-1}})$ is at least

$$\frac{f'(V - u(p_i^\dagger(\mathbf{p_{i-1}})))}{1 - F(V - u(p_i^\dagger(\mathbf{p_{i-1}}))) + \sum_{j=1}^{i} \gamma_c^j R_{i,j}^\dagger(\mathbf{p_{i-1}})} + \left[\frac{f(V - u(p_i^\dagger(\mathbf{p_{i-1}})))}{1 - F(V - u(p_i^\dagger(\mathbf{p_{i-1}}))) + \sum_{j=1}^{i} \gamma_c^j R_{i,j}^\dagger(\mathbf{p_{i-1}})}\right]^2$$

multiplied by $-u'(p_i^\dagger(\mathbf{p_{i-1}}))$. This is in turn proportional to

$$\frac{f'(V - u(p_i^\dagger(\mathbf{p_{i-1}})))[1 - F(V - u(p_i^\dagger(\mathbf{p_{i-1}})))]}{f(V - u(p_i^\dagger(\mathbf{p_{i-1}})))^2}\left[1 + \frac{\sum_{j=1}^{i} \gamma_c^j R_{i,j}^\dagger(\mathbf{p_{i-1}})}{1 - F(V - u(p_i^\dagger(\mathbf{p_{i-1}})))}\right] + 1 \geq \gamma_c^{i+1} \geq 0,$$

where the inequality uses Assumption 1 and the fact that $R_{i,j}^\dagger(\mathbf{p_{i-1}}) \leq 1 - F(V - u(p_i^\dagger(\mathbf{p_{i-1}})))$ for each $j = 1, \ldots, i$.

negative as $p_i^\dagger(\mathbf{p_{i-1}}) \uparrow \min\{p^m, u^{-1}(V-\overline{\varepsilon})\}$, and so there is a unique $p_i^\dagger(\mathbf{p_{i-1}})$ which sets (48) equal to zero.[32]

Continuing with parts 2 and 3 of strategy †, it remains to prove that the profit expression (46) is quasiconcave, i.e., it is globally maximized at $p_i^\dagger(\mathbf{p_{i-1}})$. First, note that for $p_i < u^{-1}(V-\underline{\varepsilon})$ the derivative of (46) with respect to p_i is proportional to $\pi'(p_i) > 0$ because over this interval of prices we have $p_i < p_i^\dagger(\mathbf{p_{i-1}}) < p^m$. Second, note that for $u^{-1}(V-\underline{\varepsilon}) \leq p_i \leq \min\{u^{-1}[u(p_i^\dagger(\mathbf{p_{i-1}})) - r], p^m\}$ each $R_{i,j}(\mathbf{p_{i-1}}, p_i)$ term does not depend on p_i, and therefore the derivative of (46) with respect to p_i is proportional to

$$\frac{1}{\Psi(p_i)} - \frac{f(V-u(p_i))}{1-F(V-u(p_i)) + \sum_{j=1}^{i} \gamma_c^j R_{i,j}(\mathbf{p_{i-1}}, p_i)}.$$

Following the same approach as usual (e.g., the one used in the proof of Lemma 9) this is strictly decreasing in p_i. Finally, if $\min\{u^{-1}[u(p_i^\dagger(\mathbf{p_{i-1}})) - r], p^m\} < p_i < p^m$ then each $R_{i,j}(p_i, \mathbf{p_{i-1}})$ term is weakly decreasing in p_i, hence the derivative of (46) with respect to p_i is weakly less than something which is proportional to the previous displayed equation. Hence, following the same steps, profit is strictly decreasing in p_i over this interval. We conclude that current-period profit is globally quasiconcave.

Step 5: Let $p_j^{**}(\mathbf{p_{i-1}})$ denote the price charged by a firm of age $j \geq i$ that follows strategy † from age i onwards. In other words, $p_i^{**}(\mathbf{p_{i-1}}) = p_i^\dagger(\mathbf{p_{i-1}})$, while for all $j > i$ we have

$$p_j^{**}(\mathbf{p_{i-1}}) = p_j^\dagger(\mathbf{p_{i-1}}, p_i^{**}(\mathbf{p_{i-1}}), \ldots, p_{j-1}^{**}(\mathbf{p_{i-1}})).$$

We will now prove that $p_j^{**}(\mathbf{p_{i-1}})$ weakly increases in j. Notice that in order to do this, it is sufficient to prove that

$$p_i^{**}(\mathbf{p_{i-1}}) \leq p_{i+1}^{**}(\mathbf{p_{i-1}}) \tag{49}$$

First, if $V \leq \underline{\varepsilon} + u(p^m)$, then part 1 of strategy † implies that the optimal price for any age and pricing history is p^m, so (49) is trivially satisfied. Second, then, suppose that $V > \underline{\varepsilon} + u(p^m)$ such that we require parts 2 and 3 of strategy †. Recall from Step 4 that i) if (48) is weakly negative when evaluated at $p_i^\dagger(\mathbf{p_{i-1}}) = u^{-1}(V-\underline{\varepsilon})$ then $p_i^{**}(\mathbf{p_{i-1}}) = u^{-1}(V-\underline{\varepsilon})$, and otherwise ii) $p_i^{**}(\mathbf{p_{i-1}}) > u^{-1}(V-\underline{\varepsilon})$ is the unique price which sets (48) to zero. Clearly if $p_i^{**}(\mathbf{p_{i-1}}) = u^{-1}(V-\underline{\varepsilon})$, then because $p_{i+1}^{**}(\mathbf{p_{i-1}}) \geq u^{-1}(V-\underline{\varepsilon})$, (49) is again trivially satisfied. If instead $p_i^{**}(\mathbf{p_{i-1}}) > u^{-1}(V-\underline{\varepsilon})$, then $p_i^{**}(\mathbf{p_{i-1}})$ is the unique $p_i^\dagger(\mathbf{p_{i-1}})$ which sets

[32] If $p^m < u^{-1}(V-\overline{\varepsilon})$ then as $p_i^\dagger(\mathbf{p_{i-1}}) \uparrow p^m$ the first term in (48) equals zero whilst the second term is strictly negative. If instead $p^m \geq u^{-1}(V-\overline{\varepsilon})$ then as $p_i^\dagger(\mathbf{p_{i-1}}) \uparrow u^{-1}(V-\overline{\varepsilon})$ the first term in (48) is finite, while the second one is unboundedly negative because in the limit its denominator equals zero while its numerator is finite and bounded away from zero.

equation (48) to zero, while the equivalent equation for the firm at age $i+1$ is

$$\frac{1}{\Psi(p_{i+1}^{\dagger}(\mathbf{p_i}))} - \frac{f(V-u(p_{i+1}^{\dagger}(\mathbf{p_i})))}{1-F(V-u(p_{i+1}^{\dagger}(\mathbf{p_i}))) + \sum_{j=1}^{i+1}\gamma_c^j R_{i+1,j}^{\dagger}(\mathbf{p_i})}, \tag{50}$$

where $\mathbf{p_i} = (\mathbf{p_{i-1}}, p_i^{**}(\mathbf{p_{i-1}}))$. We now prove that (50) is weakly positive when evaluated at $p_{i+1}^{\dagger}(\mathbf{p_i}) = p_i^{**}(\mathbf{p_{i-1}})$. Note that in this case equation (50) is the same as equation (48) when evaluated at $p_i^{\dagger}(\mathbf{p_{i-1}}) = p_i^{**}(\mathbf{p_{i-1}})$, except for the final terms in the denominators; note also that equation (48) evaluated at $p_i^{\dagger}(\mathbf{p_{i-1}}) = p_i^{**}(\mathbf{p_{i-1}})$ equals zero. Hence we need to prove

$$\begin{aligned}
&\sum_{j=1}^{i+1}\gamma_c^j R_{i+1,j}^{\dagger}(\mathbf{p_i})\Big|_{p_i^{\dagger}(\mathbf{p_{i-1}})=p_{i+1}^{\dagger}(\mathbf{p_i})=p_i^{**}(\mathbf{p_{i-1}})} \geq \sum_{j=1}^{i}\gamma_c^j R_{i,j}^{\dagger}(\mathbf{p_{i-1}})\Big|_{p_i^{\dagger}(\mathbf{p_{i-1}})=p_i^{**}(\mathbf{p_{i-1}})} \\
&\iff \sum_{j=1}^{i+1}\gamma_c^j \left[1-F\left(\max\left\{\zeta_{i+1,j}(\mathbf{p_i}), V-u(p_i^{**}(\mathbf{p_{i-1}}))+r\right\}\right)\right]\Big|_{p_i=p_i^{\dagger}(\mathbf{p_{i-1}})=p_i^{**}(\mathbf{p_{i-1}})} \\
&\qquad \geq \sum_{j=1}^{i}\gamma_c^j \left[1-F\left(\max\left\{\zeta_{i,j}(\mathbf{p_{i-1}}), V-u(p_i^{**}(\mathbf{p_{i-1}}))+r\right\}\right)\right].
\end{aligned} \tag{51}$$

However, notice that

$$1-F\left(\max\left\{\zeta_{i+1,1}(\mathbf{p_i}), V-u(p_i^{**}(\mathbf{p_{i-1}}))+r\right\}\right)\Big|_{p_i=p_i^{**}(\mathbf{p_{i-1}})} = 1-F(V-u(p_i^{**}(\mathbf{p_{i-1}}))+r),$$

and also for any $j=2,\ldots,i+1$, we have

$$\begin{aligned}
&1-F\left(\max\left\{\zeta_{i+1,j}(\mathbf{p_i}), V-u(p_i^{**}(\mathbf{p_{i-1}}))+r\right\}\right)\Big|_{p_i=p_i^{\dagger}(\mathbf{p_{i-1}})=p_i^{**}(\mathbf{p_{i-1}})} \\
&= 1-F\left(\max\left\{\zeta_{i,j-1}(\mathbf{p_{i-1}}), V-u(p_i^{**}(\mathbf{p_{i-1}}))+r\right\}\right).
\end{aligned}$$

Hence we can rewrite the inequality in (51) as

$$\gamma_c[1-F(V-u(p_i^{**}(\mathbf{p_{i-1}}))+r)] \geq \sum_{j=1}^{i}\gamma_c^j(1-\gamma_c)\left[1-F\left(\max\left\{\zeta_{i,j}(\mathbf{p_{i-1}}), V-u(p_i^{**}(\mathbf{p_{i-1}}))+r\right\}\right)\right],$$

which clearly holds. Using the same procedure as in Step 4, one can show that (50) is strictly decreasing in $p_{i+1}^{\dagger}(\mathbf{p_i})$, and hence is strictly positive for all $p_{i+1}^{\dagger}(\mathbf{p_i}) \in [u^{-1}(V-\underline{\varepsilon}), p_i^{**}(\mathbf{p_{i-1}}))$, which implies that (49) holds as required.

Step 6: We now prove that if the firm follows strategy † from age $i+1$ onwards, then at age i it maximizes its lifetime discounted profit by charging $p_i^{\dagger}(\mathbf{p_{i-1}})$. Since we have already shown in Step 4 that $p_i^{\dagger}(\mathbf{p_{i-1}})$ maximizes current-period profit, it is sufficient to show that it (weakly) maximizes discounted future profit as well.

We will use several times the notation $p_j^{**}(\mathbf{p_{i-1}})$ introduced in the previous step. We will also use the notation $V_j^{**}(\mathbf{p_{i-1}}) = V_j^{\dagger}(\mathbf{p_{i-1}}, p_i^{**}(\mathbf{p_{i-1}}), \ldots, p_{j-1}^{**}(\mathbf{p_{i-1}}))$.

First, we prove that for all p_i satisfying $u(p_i) \geq u(p_{i+1}^{**}(\mathbf{p_{i-1}})) - r$, starting from age $i+1$ strategy † induces the firm to charge $\{p_j^{**}(\mathbf{p_{i-1}})\}_{j=i+1}^{\infty}$, and the resulting per-period profit is constant in p_i.

Start by analyzing the firm's pricing problem at age k. Suppose either $k = i+1$, or $k > i+1$ and in this case suppose the firm has charged $p_j = p_j^{**}(\mathbf{p_{i-1}})$ for $j = i+1, \ldots, k-1$. We will prove that at age k the firm charges $p_k^{**}(\mathbf{p_{i-1}})$. It then follows by induction that from age $i+1$ onwards the firm charges $\{p_j^{**}(\mathbf{p_{i-1}})\}_{j=i+1}^{\infty}$. i) If $V \leq \underline{\varepsilon} + u(p^m)$ then, using part 1 of strategy †, it is immediate that $p_k^{\dagger}(\mathbf{p_{k-1}}) = p^m = p_k^{**}(\mathbf{p_{i-1}})$ as claimed. ii) If $\underline{\varepsilon} + u(p^m) < V \leq V_k^{**}(\mathbf{p_{i-1}})$ then $p_k^{**}(\mathbf{p_{i-1}}) = u^{-1}(V - \underline{\varepsilon})$. Since $p_j^{**}(\mathbf{p_{i-1}})$ weakly exceeds $u^{-1}(V - \underline{\varepsilon})$ (from Step 4) and weakly increases in j (from Step 5) we have that $p_{i+1}^{**}(\mathbf{p_{i-1}}) = \cdots = p_k^{**}(\mathbf{p_{i-1}}) = u^{-1}(V - \underline{\varepsilon})$. The facts that $p_{i+1}^{**}(\mathbf{p_{i-1}}) = u^{-1}(V - \underline{\varepsilon})$ and $u(p_i) \geq u(p_{i+1}^{**}(\mathbf{p_{i-1}})) - r$ also imply that $V - u(p_i) \leq \underline{\varepsilon} + r$. Combining everything together, $V_k^{\dagger}(\mathbf{p_{k-1}}) = V_k^{**}(\mathbf{p_{i-1}})$, and so $p_k^{\dagger}(\mathbf{p_{k-1}}) = u^{-1}(V - \underline{\varepsilon}) = p_k^{**}(\mathbf{p_{i-1}})$ as claimed. iii) If $V_k^{**}(\mathbf{p_{i-1}}) < V$ then from Step 4, $p_k^{\dagger}(\mathbf{p_{k-1}}) = p_k^{**}(\mathbf{p_{i-1}})$ and uniquely solves

$$\frac{1}{\Psi(p_k^{\dagger}(\mathbf{p_{k-1}}))} - \frac{f(V - u(p_k^{\dagger}(\mathbf{p_{k-1}})))}{1 - F(V - u(p_k^{\dagger}(\mathbf{p_{k-1}}))) + \sum_{j=1}^{k} \gamma_c^j R_{k,j}^{\dagger}(\mathbf{p_{k-1}})} = 0, \tag{52}$$

given $\mathbf{p_{k-1}} = (\mathbf{p_{i-1}}, p_i^{**}(\mathbf{p_{i-1}}), \ldots, p_{k-1}^{**}(\mathbf{p_{i-1}}))$. Suppose instead that $\mathbf{p_{k-1}} = (\mathbf{p_{i-1}}, p_i)$ if $k = i+1$, and $\mathbf{p_{k-1}} = (\mathbf{p_{i-1}}, p_i, p_{i+1}^{**}(\mathbf{p_{i-1}}), \ldots, p_{k-1}^{**}(\mathbf{p_{i-1}}))$ otherwise. Because $p_j^{**}(\mathbf{p_{i-1}})$ weakly increases in j (again, from Step 5), and because $V - u(p_i) \leq V - u(p_{i+1}^{**}(\mathbf{p_{i-1}})) + r$ (by assumption), one can check that $\sum_{j=1}^{k} \gamma_c^j R_{k,j}^{\dagger}(\mathbf{p_{k-1}})$ evaluated at $p_k^{\dagger}(\mathbf{p_{k-1}}) = p_k^{**}(\mathbf{p_{i-1}})$ is constant in p_i. Hence equation (52) holds when $p_k^{\dagger}(\mathbf{p_{k-1}}) = p_k^{**}(\mathbf{p_{k-1}})$. Therefore even after the deviation at age i, at age k the firm charges $p_k^{**}(\mathbf{p_{k-1}})$ as claimed.

Continuing with the case $u(p_i) \geq u(p_{i+1}^{**}(\mathbf{p_{i-1}})) - r$, now consider per-period profit. We have just proved that after the deviation at age i, from age $i+1$ the firm charges $\{p_j^{**}(\mathbf{p_{i-1}})\}_{j=i+1}^{\infty}$. It is immediate from equation (47) that $R_{k,j}(\mathbf{p_{k-1}}, p_k)$ is constant in p_i, and hence so is the profit expression in equation (46).

Second, consider p_i satisfying $u(p_i) < u(p_{i+1}^{**}(\mathbf{p_{i-1}})) - r$. Let

$$p_j^{***}(\mathbf{p_i}) = p_j^{\dagger}(\mathbf{p_i}, p_{i+1}^{***}(\mathbf{p_i}), \ldots, p_{j-1}^{***}(\mathbf{p_i})),$$

where $\mathbf{p_i} = (\mathbf{p_{i-1}}, p_i)$. From Step 5, $p_k^{***}(\mathbf{p_i})$ weakly increases in k for $k > i+1$. We now prove that $p_k^{***}(\mathbf{p_i}) \leq p_k^{**}(\mathbf{p_{i-1}})$, and that per-period profit is weakly lower than if the firm had charged $p_i = p_i^{**}(\mathbf{p_{i-1}})$. That is, following a large upward deviation in period i, the firm will charge lower prices and receive lower per-period profits in future periods.

Start by analyzing the firm's pricing problem at age k. Suppose either $k = i + 1$, or $k > i + 1$ and in this case suppose the firm has charged $p_j^\dagger(\mathbf{p_{j-1}}) \leq p_j^{**}(\mathbf{p_{i-1}})$ for $j = i+1, \ldots, k-1$. We will prove that at age k the firm will charge $p_k^\dagger(\mathbf{p_{k-1}}) \leq p_k^{**}(\mathbf{p_{i-1}})$. It then follows by induction that from age $i + 1$ onwards the firm charges weakly less than $\{p_j^{**}(\mathbf{p_{i-1}})\}_{j=i+1}^{\infty}$. i) If $V \leq \underline{\varepsilon} + u(p^m)$ then, using part 1 of strategy †, it is immediate that $p_k^\dagger(\mathbf{p_{k-1}}) = p^m = p_k^{**}(\mathbf{p_{i-1}})$. ii) If $\underline{\varepsilon} + u(p^m) < V \leq V_k^{**}(\mathbf{p_{i-1}})$ then strategy † implies $p_k^{**}(\mathbf{p_{i-1}}) = u^{-1}(V - \underline{\varepsilon})$. Consider first $k = i + 1$: notice that $V_{i+1}^\dagger(\mathbf{p_i}) > V_{i+1}^{**}(\mathbf{p_{i-1}})$, and therefore $V < V_{i+1}^\dagger(\mathbf{p_i})$, so strategy † implies $p_{i+1}^\dagger(\mathbf{p_i}) = u^{-1}(V - \underline{\varepsilon}) = p_{i+1}^{**}(\mathbf{p_{i-1}})$. Now consider $k > i + 1$. Since prices under strategy † weakly exceed $u^{-1}(V - \underline{\varepsilon})$ (because $V > \underline{\varepsilon} + u(p^m)$), and since $p_j^{**}(\mathbf{p_{i-1}})$ weakly increases in j (from Step 5), our supposition is that the firm has charged $u^{-1}(V - \underline{\varepsilon})$ at all ages $i + 1, \ldots, k - 1$. Notice that $V_k^\dagger(\mathbf{p_{k-1}}) > V_k^{**}(\mathbf{p_{i-1}})$, and so $V < V_k^\dagger(\mathbf{p_{k-1}})$, and therefore again strategy † implies $p_k^\dagger(\mathbf{p_{k-1}}) = u^{-1}(V - \underline{\varepsilon}) = p_k^{**}(\mathbf{p_{i-1}})$. iii) If $V_k^{**}(\mathbf{p_{i-1}}) < V$ then from Step 4 $p_k^{**}(\mathbf{p_{i-1}})$ is the unique $p_k^\dagger(\mathbf{p_{k-1}})$ which solves equation (52) given $\mathbf{p_{k-1}} = (\mathbf{p_{i-1}}, p_i^{**}(\mathbf{p_{i-1}}), \ldots, p_{k-1}^{**}(\mathbf{p_{i-1}}))$. Now suppose instead that $\mathbf{p_{k-1}} = (\mathbf{p_{i-1}}, p_i)$ if $k = i + 1$, and $\mathbf{p_{k-1}} = (\mathbf{p_{i-1}}, p_i, p_{i+1}^{***}(\mathbf{p_i}), \ldots, p_{k-1}^{***}(\mathbf{p_i}))$ otherwise. Recalling that $p_j^{**}(\mathbf{p_{i-1}})$ weakly increases in j (again, from Step 5), as well as our supposition for $k > i + 1$ that $p_j^\dagger(\mathbf{p_{j-1}}) \leq p_j^{**}(\mathbf{p_{i-1}})$ for $j = i + 1, \ldots, k - 1$, it is straightforward to see that $\sum_{j=1}^{k} \gamma_c^j R_{k,j}^\dagger(\mathbf{p_{k-1}})$ evaluated at $p_k^\dagger(\mathbf{p_{k-1}}) = p_k^{**}(\mathbf{p_{i-1}})$ is weakly lower than it would be under $\mathbf{p_{k-1}} = (\mathbf{p_{i-1}}, p_i^{**}(\mathbf{p_{i-1}}), \ldots, p_{k-1}^{**}(\mathbf{p_{i-1}}))$. Hence the lefthand side of equation (52) is weakly negative when evaluated at $p_k^\dagger(\mathbf{p_{k-1}}) = p_k^{**}(\mathbf{p_{i-1}})$; it follows from Step 4 therefore that $p_k^\dagger(\mathbf{p_{k-1}}) \leq p_k^{**}(\mathbf{p_{i-1}})$.

Continuing with the case $u(p_i) < u(p_{i+1}^{**}(\mathbf{p_{i-1}})) - r$, now consider per-period profit. We have just proved that after the deviation at age i, at each $k > i$ the firm charges $p_k^{***}(\mathbf{p_i}) \leq p_k^{**}(\mathbf{p_{i-1}})$. We will distinguish between $p_k^{***}(\mathbf{p_i}) = p_k^{**}(\mathbf{p_{i-1}})$ and $p_k^{***}(\mathbf{p_i}) < p_k^{**}(\mathbf{p_{i-1}})$.

First, suppose $p_k^{***}(\mathbf{p_i}) = p_k^{**}(\mathbf{p_{i-1}})$. We know from Step 4 that $p_j^{***}(\mathbf{p_i})$ and $p_j^{**}(\mathbf{p_{i-1}})$ both weakly increase in j. It is then immediate from equation (47) that $R_{k,j}(\mathbf{p_{k-1}}, p_k)$ is weakly lower at p_i satisfying $u(p_i) < u(p_{i+1}^{**}(\mathbf{p_{i-1}})) - r$ than at $p_i = p_i^{**}(\mathbf{p_{i-1}})$, and hence the same is true of the profit expression in equation (46).

Second, suppose $p_k^{***}(\mathbf{p_i}) < p_k^{**}(\mathbf{p_{i-1}})$. Denote by $R_{k,j}^{**}(\mathbf{p_{i-1}})$ equation (47) if the firm charges $p_i = p_i^{**}(\mathbf{p_{i-1}})$ at age i, then charges $p_j^{**}(\mathbf{p_{i-1}})$ at age $j = i + 1, \ldots, k$. Similarly, denote by $R_{k,j}^{***}(\mathbf{p_i})$ equation (47) if the firm charges p_i satisfying $u(p_i) < u(p_{i+1}^{**}(\mathbf{p_{i-1}})) - r$ at age i, then charges $p_j^{***}(\mathbf{p_i})$ at ages $j = i + 1, \ldots, k$. We now prove that $\sum_{j=1}^{k} \gamma_c^j R_{k,j}^{***}(\mathbf{p_i}) \leq \sum_{j=1}^{k} \gamma_c^j R_{k,j}^{**}(\mathbf{p_{i-1}})$. Following arguments in Step 4, we must have

$$\frac{1}{\Psi(p_k^{***}(\mathbf{p_i}))} - \frac{f(V - u(p_k^{***}(\mathbf{p_i})))}{1 - F(V - u(p_k^{***}(\mathbf{p_i}))) + \sum_{j=1}^{k} \gamma_c^j R_{k,j}^{***}(\mathbf{p_i})} \leq 0. \tag{53}$$

Moreover, because we have $u^{-1}(V - \underline{\varepsilon}) \leq p_k^{***}(\mathbf{p_i}) < p_k^{**}(\mathbf{p_{i-1}})$, it also follows from arguments in Step 4 that

$$\frac{1}{\Psi(p_k^{**}(\mathbf{p_{i-1}}))} - \frac{f(V - u(p_k^{**}(\mathbf{p_{i-1}})))}{1 - F(V - u(p_k^{**}(\mathbf{p_{i-1}}))) + \sum_{j=1}^{k} \gamma_c^j R_{k,j}^{**}(\mathbf{p_{i-1}})} = 0. \tag{54}$$

However, notice that because $R_{k,j}^{**}(\mathbf{p_{i-1}}) \leq 1 - F(V - u(p_k^{**}(\mathbf{p_{i-1}}))) \leq 1 - F(V - u(p))$ for $p \leq p_k^{**}(\mathbf{p_{i-1}})$, it follows from arguments in Step 4 that the lefthand side of equation (54) is strictly decreasing in $p_k^{**}(\mathbf{p_{i-1}})$, and so would be weakly positive if $p_k^{**}(\mathbf{p_{i-1}})$ were replaced by $p_k^{***}(\mathbf{p_i})$. But then equations (53) and (54) are only consistent if $\sum_{j=1}^{k} \gamma_c^j R_{k,j}^{***}(\mathbf{p_i}) \leq \sum_{j=1}^{k} \gamma_c^j R_{k,j}^{**}(\mathbf{p_{i-1}})$ as claimed. To complete this part of the proof, we now argue that $\sum_{j=1}^{k} \gamma_c^j R_{k,j}^{***}(\mathbf{p_i}) \leq \sum_{j=1}^{k} \gamma_c^j R_{k,j}^{**}(\mathbf{p_{i-1}})$ implies that the firm's profit at age k is weakly higher if it charged $p_i = p_i^{**}(\mathbf{p_{i-1}})$ rather than p_i satisfying $u(p_i) < u(p_{i+1}^{**}(\mathbf{p_{i-1}})) - r$ at age i (and then in both cases followed strategy † thereafter). The reason is that in the former case, at age k the firm could deviate downward and charge $p_k^{***}(\mathbf{p_i})$, and in that period earn the same per-consumer profit and sell to the same measure of fresh consumers but weakly more return consumers than it does in the latter case where it is actually optimal to charge $p_k^{***}(\mathbf{p_i})$; by revealed preference the firm's profit at age k is weakly higher in the former case than the latter case.[33]

Step 7: Finally, notice that to check that strategy † induces a subgame perfect Nash equilibrium, we can invoke the one-shot deviation principle. The previous step has shown that such a deviation at age i from $p_i^\dagger(\mathbf{p_{i-1}})$ strictly reduces current-period profit and weakly reduces profit in all future periods. Hence one-shot deviations are unprofitable. □

We will also require the following lemma:

Lemma A10. *Suppose $n = 1$ and Assumption 3 holds. Suppose all firms except one exogenously play the prices from Theorem 1 in each time period. Let V denote the associated value of search. Suppose that from age 1 onwards the remaining firm follows strategy † defined in Lemma A9. Then at age 0 the firm charges $p_0^\dagger$ where*

1. *If $V \leq \underline{\varepsilon} + u(p^m)$, then $p_0^\dagger = p^m$.*
2. *If $\underline{\varepsilon} + u(p^m) < V \leq \underline{\varepsilon} + u(\Psi^{-1}(1/f(\underline{\varepsilon})))$, then $p_0^\dagger = u^{-1}(V - \underline{\varepsilon})$.*
3. *If $V > \underline{\varepsilon} + u(\Psi^{-1}(1/f(\underline{\varepsilon})))$, then $p_0^\dagger$ is the unique solution to*

$$\frac{1}{\Psi(p_0^\dagger)} - \frac{f(V - u(p_0^\dagger))}{1 - F(V - u(p_0^\dagger))} = 0. \tag{55}$$

Proof of Lemma A10. We have already argued in the proof of Lemma A9 that the value of search V is the same (for given parameters) as in Theorem 1. Moreover, closely following

[33] Note also that, using the first part of this step of the proof, such a downward deviation would have no effect on the firm's profit from period $k + 1$ onwards.

Steps 2 and 3 of that proof, fresh consumers buy from the firm if and only if $\varepsilon \geq V - u(p)$ given price p, and so the firm's profit at age 0 is

$$\frac{m}{\sum_{j=0}^{\infty}(1-\gamma_f)\gamma_f^j[1-F(a_j)]}\pi(p)[1-F(V-u(p))]. \tag{56}$$

Using the same approach as in Step 4 of the proof of Lemma A9, one can show that $p_0^\dagger$ strictly maximizes (56). Using the same approach as in Step 6 of that proof, one can also show that conditional on following strategy † from age 1 onwards, charging $p_0^\dagger$ at age 0 weakly maximizes discounted future profit as well. □

We are now ready to prove Proposition 8 for the $n = 1$ case.

Proof of Proposition 8 (case with $n = 1$). We prove that if all firms but one charge p_i^* from Theorem 1 at age $i = 0, 1, \ldots$, the remaining firm also optimally charges p_i^* at age i.

Start with age 0. Recall Lemma A6 and note that given $\delta_c = 0$, $V' = \underline{\varepsilon} + u(p^m)$ and $\underline{V}_0 = \underline{\varepsilon} + u(\Psi^{-1}(1/f(\underline{\varepsilon})))$. Recall also Lemma A7 and note that, again given $\delta_c = 0$, for all $V > \underline{V}_0$, p_0^* is the unique solution to $p_0^* = \tilde{p}_0(V - u(p_0^*)) = \Psi^{-1}[[1 - F(V - u(p_0^*))]/f(V - u(p_0^*))]$. This expression coincides with equation (55), and has a unique solution, hence $p_0^\dagger = p_0^*$.

Now consider an age $i \geq 1$. Suppose the firm charged p_0^* at age 0, and that now either $i = 1$ or $i > 1$ and the firm has charged $p_j = p_j^\dagger(\mathbf{p_{j-1}}) = p_j^*$ for $j = 1, \ldots, i-1$ and $\mathbf{p_{j-1}} = (p_0^*, \ldots, p_{j-1}^*)$. We will prove that strategy † induces the firm to charge p_i^* in the current period. By induction, we will then have proven that the firm optimally charges prices $\{p_i^*\}_{i=0}^{\infty}$. i) Suppose $V \leq V'$. Lemma A7 says that $p_i^* = p^m$. Since $V' = \underline{\varepsilon} + u(p^m)$, strategy † induces the firm to charge $p_i^\dagger(\mathbf{p_{i-1}}) = p^m = p_i^*$ for all $V \leq V'$. ii) Suppose $V' < V \leq \underline{V}_i$. Lemma A6 says that $\underline{V}_j > \underline{V}_i$ for all $j < i$, so Lemma A7 says that $p_0^* = \cdots = p_i^* = u^{-1}(V - \underline{\varepsilon})$. This implies that $V_i^\dagger(\mathbf{p_{i-1}}) = \underline{V}_i$. Strategy † then induces the firm to charge $p_i^\dagger(\mathbf{p_{i-1}}) = u^{-1}(V - \underline{\varepsilon}) = p_i^*$. iii) Suppose $\underline{V}_i < V$. Lemma A7 says that p_i^* is the unique solution to

$$p_i^* = \tilde{p}_i(V - u(p_i^*)) \iff \frac{1}{\Psi(p_i^*)} - \frac{f(V - u(p_i^*))}{1 - F(V - u(p_i^*)) + [1 - F(V - u(p_i^*) + r)]\gamma_c\frac{1-\gamma_c^i}{1-\gamma_c}} = 0.$$

Notice that this coincides with equation (45) when it is evaluated at $p_i^\dagger(\mathbf{p_{i-1}})$. This follows from our supposition that the firm has charged $p_j = p_j^\dagger(\mathbf{p_{j-1}}) = p_j^*$ for $j = 1, \ldots, i-1$, and the fact that p_j^* weakly increases in j, which in turn implies that $R_{i,j}^\dagger(\mathbf{p_{i-1}})$ evaluated at $p_i^\dagger(\mathbf{p_{i-1}}) = p_i^*$ equals $1 - F(V - u(p_i^*) + r)$. It then follows from arguments in Step 4 of the proof of Lemma A9 that $p_i^\dagger(\mathbf{p_{i-1}}) = p_i^*$. □