WORKING PAPERS

N° 1625

MARCH 2025

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Working paper

OPTIMAL CONTRACTS UNDER MORAL HAZARD, ADVERSE SELECTION AND LIMITED LIABILITY

March 15, 2025

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Abstract. A buyer (the principal) procures a good or service from a risk-neutral seller (the agent). The seller, protected by limited liability, has private information on his marginal cost of production (adverse selection), and exerts a non-verifiable effort that increases surplus (moral hazard). Even when the effort and production technologies are separable, the optimal contract always mixes features that are found separately under with pure moral hazard or pure screening. Screening distortions are mitigated in comparison with the pure screening scenario with the possibility of bunching for the least efficient types even in contexts where full separation would be obtained with pure screening. Effort distortions are also used as a screening device. In comparison with a pure moral hazard scenario, those distortions may be lessened for the most efficient types, up to the point of possibly allowing implementation of the first-best effort, while they are worsened for the worst types. Although our analysis is cast in a simple procurement setting, we illustrate our findings in other economic environments of general interest including economic and environmental regulation, financial contracting, provision of quality in services, and price discrimination.

Keywords. Adverse selection, moral hazard, contract theory.

JEL codes. D82.

1. INTRODUCTION

Adverse selection (hidden information) and moral hazard (hidden action) have been the two dominant paradigms of the *Theory of Contracts* over the last few decades.[1] As testimonies of their successes, those paradigms have illuminated our understanding of many contracting environments.

Adverse selection models have informed us on the design of nonlinear prices in various environments (Mussa and Rosen (1978), Maskin and Riley (1984), Wilson (1993)), the structure of regulatory and procurement policies (Baron and Myerson (1982), Laffont and Tirole (1986), Armstrong and Sappington (2007)), the nature of financial contracting (Freixas and Laffont (1990), Bolton and Scharfstein (1990)) and insurance provision (Stiglitz (1977), Chade and Schlee (2012)) or the shape of optimal taxation schemes (Mirrlees (1971), Diamond (1998)) among other major advances. Equipped with the analytical tools offered by the Revelation Principle (Myerson (1982)), modelers have studied at length how contracts are constrained by the fundamental trade-off between efficiency and rent extraction that pervades this screening literature. Principals are ready

[*] We thank Javier Gonzales-Morin, Liam Lods, François Maréchal as well as seminar participants at *ESEM* 2023 and *AFSE* 2023 for useful comments on an earlier version. This paper has been funded by the *Agence Nationale de la Recherche* under grant $ANR-17-EURE-0010$(Investissements d'Avenir program). The usual disclaimer applies.

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[1] See Hart and Holmström (1987) for a seminal overview and Laffont and Martimort (2002) and Bolton and Dewatripont (2004) for textbook treatments.

to concede inefficiency when trading with their agents as a mean to extract the information rent that those agents withdraw from having private information.

On their side, moral hazard models have shed light on the design of managerial compensations (Ross (1973), Holmström (1979), Shavell (1979), Mirrlees (1999)), the relationships between firms and their financiers (Jensen and Meckling (2019), Holmström and Tirole (1997), Dewatripont, Legros, and Matthews (2003), Chaigneau, Edmans, and Gottlieb (2018)), the theory of agrarian contracts (Stiglitz (1974)) or the incentive purposes of various legal rules and regulation (Shavell (1984)) to select a few areas of significant interest. On that front, the literature has foreshadowed two main issues. First, principals fail to provide enough insurance to risk-averse agents in order to provide incentives for effort provision: a well-known trade-off between risk and incentives (Holmström (1979), Shavell (1979), Grossman and Hart (1983)). Second, principals may also have to forego efficient effort provision to reduce their risk-neutral agents' liability rent that they view as costly (Innes (1990), Poblete and Spulber (2012), Jewitt, Kadan, and Swinkels (2008), Matthews (2001)).

Although the body of existing literature related to each of those paradigms when taken separately is significant, the lessons of those paradigms that still prevail in richer environments where both adverse selection and moral hazard are simultaneously at play remain to a large extent still unknown. This paper stands on that front.

The Model. We consider a simple archetypical model of procurement in the spirit of Baron and Myerson (1982). A buyer (the principal, *she*) buys a good or service from a risk-neutral seller (the agent, *he*). The seller has private information on his marginal cost (the adverse selection side of the model). The seller can also undertake a non-verifiable and costly effort which reduces his fixed cost of production (the moral hazard side). Contracting takes place once the seller has already learned his cost parameter. The seller is protected by limited liability; his profit from providing the service must always remain non-negative.

Importantly, the two sources of agency problems, adverse selection and moral hazard, are *a priori* unrelated. On the one hand, the effort does not affect the marginal cost of production in sharp contrast with Laffont and Tirole (1986)'s seminal model of incentive regulation for instance. On the other hand, the marginal cost of production does not affect neither the seller's disutility of effort nor the probability of a fixed cost reduction in contrast with Faynzilberg and Kumar (2000) and Castro-Pires, Chade, and Swinkels (2024) among many others.

In such a context with strong separability, intuition might suggest that the two sources of agency costs should be treated separately. Extracting the adverse selection information rent could thus be handled by means of downward distortions in output; with the standard screening result that all types, except the most efficient one, produce less than the first-best level. Those distortions are certainly attractive in our context as well because they allow to reduce the seller's information rent. Remember indeed that, in a screening environment, a low-cost seller may be attracted by the allocation of a high-cost one; producing the same quantity at a lower marginal cost and pocketing extra payments when doing so. Downward output distortions make those strategies less attractive.

Pursuing on this intuition that the two agency problems could be potentially fixed separately, we might also think that the solution to the moral hazard problem is independent of the adverse selection parameter. Because of limited liability, the seller cannot be punished too harshly once his effort fails and no reduction of the fixed cost is observed. Incentives for fixed cost reduction can only be provided with bonuses in case such an event is observed. Therefore, the seller also withdraws some liability rent. This liability rent is also viewed as being costly by the buyer. Reducing this rent calls for lower bonuses and underprovision of effort. Yet, since the disutility of effort does not depend on the seller's type, his liability rent does not either. A rough intuition could thus suggest that the corresponding downward reduction in effort should be made independent of the marginal cost.

Surprisingly, it turns out that none of those intuitions is actually correct.

Findings. The key reason for the *de facto* non-separability between the adverse selection and moral hazard sides of the incentive problem in our context comes from the seller's limited liability. The condition that profits cannot be negative implies that the seller's information rent associated to his private information on cost has necessarily to be greater than his liability rent coming from moral hazard. In other words, extracting the seller's information rent might thus also require to decrease bonuses to reduce his liability rent. To understand why it is so, remember that, under adverse selection, a seller would like to exaggerate his cost to pocket higher prices for the service. To make this strategy less attractive, the seller's payment and output should be reduced if he were to claim a high cost. Those distortions harden the seller's liability constraint; which in turn exacerbates the moral hazard problem. Effort should thus be reduced and lower bonuses should be offered to higher types.

The flip side of this argument is that, when the seller is sufficiently cost-efficient in providing the service, he may also receive a large payment as a reward for truthtelling. The seller has thus enough cash to post a bound equal to the moral hazard liability rent. This rent is recouped in case effort succeeds and a fixed cost reduction arises. The seller's effort, when sufficiently efficient, thus exerts first-best effort. At the same time, output distortions for those sufficiently efficient types are the usual screening distortions, captured by a virtual cost whose expression is familiar from Baron and Myerson (1982). For the most productive types of the seller, moral hazard and adverse selection can be treated separately.

For a seller with a less efficient type, output distortions take a much less familiar shape. First, those output distortions are mitigated in comparison with a pure screening scenario. The intuition is simple. Increasing the seller's output indeed increases his information rent from adverse selection and thus relaxes his limited liability constraint. In turn, those milder distortions allow to raise the seller's effort. For the least productive types of the seller, moral hazard and adverse selection cannot thus be disentangled.

Second, remember that, when binding, the liability constraint links together the seller's information rent and his liability rent; which itself depends on his effort. Extracting the former means reducing the later. The familiar result of the screening literature that the seller with the highest cost parameter would obtain zero rent cannot hold unless this seller is also asked to exert no effort; a significant efficiency loss if the marginal return on even a tiny effort is sufficiently valuable. Leaving a positive information rent to the seller even with the highest cost means that there should not be any output distortion for the highest type either; a result in sharp contrast with the received lessons from pure screening models.

There is thus a tension between ensuring adverse selection incentive compatibility and implementing a positive effort for the least productive types. The former calls for reducing output for all but the most efficient type. The latter requires limiting output inefficiency for inefficient types. From this tension, it follows that the output monotonicity condition imposed by incentive compatibility in screening models may be binding for those worst types. This is so even in regular screening environments where the monotone hazard rate property (Bagnoli and Bergstrom (2005)) would suffice to ensure that the solution to the relaxed screening problem, omitting *a priori* this monotonicity condition, ends up satisfying it. Bunching is pervasive in our contracting environment; leading to simple contractual forms.

Literature Review. Models that mix adverse selection and moral hazard already abound in the literature. They mostly differ in terms of the timing of information learning and actions, and the posited assumptions on how those two agency problems enter into the agent's preferences.[2] Given this richness, it would be somehow illusive to make even a brief overview of this literature and we will just content ourselves with stressing a few earlier findings that are relevant for our specific results.

The closest paper to ours is certainly Hiriart and Martimort (2006). As we do thereafter, those authors consider a procurement model *à la* Baron and Myerson (1982) and append to it a moral hazard problem. The seller needs to exert an effort to avoid some environmental damage. Our model below generalizes their work along two directions. First, we consider a richer adverse selection environment with a continuum of types instead of a two-types information structure as they do. With two types, there is only one binding (upward) truthtelling constraint and thus only the high-cost seller's output needs to be distorted. The possibility of bunching that prevails in our context also disappears. A richer pattern of output distortions, eventually exhibiting bunching arises with a continuum of types. Second, Hiriart and Martimort (2006) make the simplifying assumption that the environmental damage is so severe that the seller can never pay for the damage. In other words, his limited liability constraint is always binding. Our analysis demonstrates that, here also, a richer set of configurations is possible with the most efficient types of the seller being able to use their significant adverse selection rent to relax those liability requirements and implements first-best incentives.

Taking a broader perspective, a burgeoning line of research has analyzed whether lessons of pure moral hazard models carry over when private information is added up. The concomitant presence of both agency issues introduces the possibility that the agent could potentially deviate both in terms of his report but also in terms of the recommended action. On this front, Faynzilberg and Kumar (2000), Faynzilberg and Kumar (1997), Ollier (2007), and Castro-Pires, Chade, and Swinkels (2024) have proposed decoupling techniques that delineate circumstances under which those deviations can be treated separately. In contrast with us, those authors allow for risk aversion on the agent's side[3] and consider that his private information may enter into either the

[2]See Laffont and Martimort (2002), Chapter 7, for an earlier overview.
[3]An exception is Ollier (2007).

production technology or the disutility function. Maybe more surprisingly, even when assuming a strong separability between the moral hazard and adverse selection technological sides of the model as we do thereafter in the context of risk neutrality, the solutions to both agency problems are necessarily linked.

An extreme form of interaction between moral hazard and adverse selection actually arises when the optimal contract cannot depend on the adverse selection parameter. Robust contracts exhibit such bunching in contexts with arbitrarily rich private information about the distribution of outputs and the cost of effort as in Gottlieb and Moreira (2022). Along the same vein, Lewis and Sappington (2001), Ollier and Thomas (2013), Escobar and Pulgar (2017) and Castro-Pires and Moreira (2021a) have derived inflexible contracts in environments that entail more structured preferences with non-separabilities between the adverse selection and moral hazard sides of the agency problem; and sometimes ex post participation rather than limited liability constraints. Bunching is also found in a variety of applications to financial contracting (At and Thomas (2019)), sharecropping (At and Thomas (2017)), R&D or managerial compensations (Rietzke and Chen (2020)) or health economics (Maréchal and Thomas (2021)). Its sources vary across models and depend in fine details of the contracting instruments available and on how stringent limited liability constraints are.[4] One possibility is that, because it constrains payments, limited liability makes it difficult to screen the agent's type as in Lewis and Sappington (2001), At and Thomas (2019) and Ollier and Thomas (2013). Risk aversion may have a similar effect as shown by Maréchal and Thomas (2018). Another possibility is that moral hazard requires bonuses that should ideally vary with the adverse selection parameters in opposite directions to the monotonicity required for incentive compatibility; an instance of non-responsiveness à la Guesnerie and Laffont (1984) that is found in Escobar and Pulgar (2017) and Castro-Pires and Moreira (2021b). In our paper, the source of bunching is radically different. It comes from the existing conflict between the principal's desire to extract the agent's information rent and her willingness to implement a positive effort even for the worst type.

From a technical viewpoint, the limited liability constraint is a mixed constraint linking the agent's information rent (a state variable) and his liability rent which itself depends on the bonus for fixed cost reduction (a control variable).[5] Such a constraint imposes no terminal condition on the state variable, in sharp contrast with the case of pure screening. This implies that the associated costate variable is zero at both ends of the type space; which in turn implies no output distortions at those points unless the monotonicity condition is binding. This possibility of bunching induced by a free transversality condition is certainly reminiscent of the optimal taxation literature as exemplified by Lollivier and Rochet (1983), the nonlinear pricing literature with random outside options as in Rochet and Stole (2002) or the regulation of a risk-averse firm as in Salanie (1990) and Laffont and Rochet (1998). To characterize the bunching area without a priori imposing any continuity assumption on the agent's output (only mono-

[4] *A contrario*, a related literature has shown that, when the agent has private information on his performance and has to be induced to report such, the principal can disentangle adverse selection and moral hazard provided that she has enough instruments (audit, payments contingent on some verifiable outcomes and the like...). This point is made in Mookherjee and Png (1989), Gromb and Martimort (2007), Malcomson (2009), Krähmer and Strausz (2011), Roger (2013) among others.

[5] Mixed constraints have received less attention in the contracting literature in comparison with the pure state constraints studied in Lewis and Sappington (1989), Jullien (2000) and Martimort and Stole (2022) among others.

tonicity should matter), we follow and adapt ironing techniques that were developed earlier on by Myerson (1981) and Toikka (2011). In contrast with those papers where the virtual cost is entirely determined by the exogenous distribution of types, ironing the virtual cost now requires consideration of a costate variable which is endogenous to the optimization problem since it depends on where the liability constraint binds. This endogeneity implies that ironing is needed even in contexts where the monotone hazard rate property, familiar from screening models, would suffice to ensure monotonicity in pure screening environments.

Organization of the Paper. Section 2 presents the model and some basic results. Section 3 describes the set of incentive-feasible allocations in our context mixing adverse selection and moral hazard. Section 4 analyzes two benchmarks. In Section 4.1, moral hazard is the sole incentive problem. In Section 4.2, adverse selection is added but the agent has no limited liability. Section 5 considers the more complete scenario with adverse selection, moral hazard, and limited liability. Section 5.1 first focuses on simple scenarios where the monotonicity condition for incentive compatibility holds and characterizes fully separating allocations. Section 5.2 provides sufficient conditions for this scenario. Section 5.3 develops ironing techniques to handle the monotonicity condition when binding. Section 6 briefly analyzes the shape of payments. Finally, Section 7 develops other applications and extensions of our framework, showing its broad applicability. Proofs are relegated to an Appendix.

2. THE PROCUREMENT MODEL

We consider the following procurement model. A buyer (the principal, thereafter sometimes referred to as *she*) contracts with a seller (the agent, thereafter *he*) for the provision of a good or service on her behalf. Both parties are risk neutral and the cashless seller is protected by limited liability. Our analysis below thus heavily relies on the textbook models for adverse selection and moral hazard as it can be found in Laffont and Martimort (2002) (Chapters 2 and 5) but merges those two models in a very simple way.

By purchasing q units of the good, the buyer enjoys a benefit $S(q)$ where S is increasing and strictly concave ($S' > 0$, $S'' < 0$) with $S(0) = 0$. To ensure positive production under all circumstances below, we assume that the following Inada conditions hold $\lim_{q\to 0} S'(q) = +\infty$ and $\lim_{q\to+\infty} S'(q) = 0$. We will sometimes refer to the demand function $D = S'^{-1}$ which is thus decreasing from the fact that S is strictly concave. For technical reasons, we also assume that $q \in \mathcal{Q} = [0, \overline{q}]$ where $\overline{q} < +\infty$.[6]

Technology and Information. When producing q units, the seller incurs a variable cost θq. The marginal cost parameter θ is his private information. This parameter is drawn from a common knowledge cumulative distribution function F that is assumed to be atomless and has a positive density function f on the support $\Theta = [\underline{\theta}, \overline{\theta}]$ (we sometimes denote by $\Delta = \overline{\theta} - \underline{\theta}$ the spread of cost uncertainty).

Following Bagnoli and Bergstrom (2005) and most of the screening literature, we

[6]Alternatively, q can be viewed as a quality index in which case, the buyer only needs at most one unit of service. Although they might apply in different economic contexts, those two interpretations of the model are of course equivalent.

assume that the *Monotone Hazard Rate Property* holds, i.e.,

$$\text{(MHRP)} \quad \frac{F-\kappa}{f} \text{ is non-decreasing for } \kappa \in \{0,1\}.$$

On top of its marginal cost, the seller also incurs a fixed cost K for providing the service. This fixed cost is common knowledge.

Moral Hazard. By exerting an effort e, the seller can reduce this fixed cost by an amount B; an event which arises only with probability $e \in [0,1]$. Whether a reduction in fixed cost occurs or not is an observable event.[7]

When exerting effort e, the seller incurs a disutility $\psi(e)$ which is increasing ($\psi' \geq 0$) and strictly convex ($\psi'' > 0$). More precisely, we assume that ψ is three times continuously differentiable with $\psi(0) = 0$, and $\lim_{e \to 1} \psi'(e) = +\infty$ so as to ensure interior solutions to all incentive problems that are considered below. For future reference, we define $\varphi = \psi'^{-1}$ and the seller's liability rent as $\mathcal{R}(e) = e\psi'(e) - \psi(e)$. Observe that $\mathcal{R}(e)$ is itself non-negative, increasing with $\mathcal{R}'(0) = 0$ and convex when $\psi''' \geq 0$; an assumption that is maintained throughout.

Payoffs. The buyer's and the seller's payoffs are respectively written as

$$V = S(q) - \tilde{t} + e\tilde{w},$$

and

$$U = \tilde{t} - \theta q - K + e(B - \tilde{w}) - \psi(e).$$

In the above formulas, $\tilde{t}$ stands for a base-payment for delivery of the good while $\tilde{w}$ is a rebate made to the buyer in case a fixed cost reduction is observed. Observe that not contracting yields zero reservation payoff to both players. For ease of presentation, it might be useful to define the seller's net payments and a (non-negative) bonus for a fixed cost reduction respectively as $t = \tilde{t} - K$ and $w = B - \tilde{w}$. With these pieces of notations at hands, we may rewrite the buyer's and the seller's payoffs in a more compact form respectively as

$$V = S(q) - K - t + e(B - w),$$

and

$$U = t - \theta q + ew - \psi(e).$$

Contracts. The buyer has all bargaining power and makes a take-it-or-leave-it offer to the seller. Following the Revelation Principle (Myerson (1982)), there is no loss of generality in focusing on direct and truthful revelation mechanisms. Such a mechanism

[7] The fact that effort induces only two outcomes (cost-reduction or not) is not really restrictive. To see why, suppose that there are n possible outcomes (n finite), we know that, with limited liability, the agent should receive a positive bonus only for those states having the highest likelihood ratio and should not be paid otherwise. Such contract would thus have a similar bang-bang structure.

is a triplet $(q(\hat{\theta}), t(\hat{\theta}), w(\hat{\theta}))$ which stipulates an output, a base-payment and a bonus in terms of the agent's report $\hat{\theta}$ on his type.

Timing. The game unfolds as follows. First, the agent learns his cost parameter θ. Second, the principal proposes a contract. Third, the agent accepts or refuses this offer, in which case the game ends with zero payoff to all players. Fourth, the agent chooses an output (or reports his type in the direct mechanism version of the game) and exerts an effort. Finally, payments and bonuses are made according to the realized output (or report) and whether the fixed cost has been reduced or not.

The Complete Information Scenario. Suppose, as a benchmark, that both θ and e are observable and verifiable. The buyer can dictate which effort the seller should exert and can use a forcing contract to induce the requested quantity. Because the buyer has all bargaining power, she can reduce the base payment for the seller's services till she captures all of his surplus

$$U^{fb}(\theta) = 0.$$

The buyer thus chooses production and effort so as to maximize the following expression of the overall surplus which, of course, exhibits a separability between its output and effort components

$$S(q) - \theta q - K + eB - \psi(e).$$

The first-best output and effort $(q^{fb}(\theta), e^{fb})$ are then readily characterized. The buyer's marginal benefit of output should be equal to the seller's marginal cost

$$S'(q^{fb}(\theta)) = \theta \quad \forall \theta \in \Theta, \tag{2.1}$$

while the marginal benefit of effort in terms of reduced fixed cost should be equal to its marginal disutility

$$B = \psi'(e^{fb}) \tag{2.2}$$

which leads to an expression of first-best effort independent of the cost parameter; as expected in this framework with separability between effort and production.

For simplicity, we shall also assume that gains from trade are large enough to always warrant production. This condition puts an upper bound on possible values of the fixed cost, namely

$$S(q^{fb}(\bar{\theta})) - \bar{\theta} q^{fb}(\bar{\theta}) + \mathcal{R}(e^{fb}) \geq K. \tag{2.3}$$

In other words, the overall surplus of production and effort should exceed the fixed cost even in the worst scenario where the seller has the highest possible marginal cost and the net surplus from output is at its minimum.

3. INCENTIVE FEASIBILITY

Preliminaries. Our contracting environment entails both adverse selection and moral hazard. We follow Laffont and Martimort (2002) (Chapter 7) in reducing this so called mixed model into a simpler setting with pure screening only. To do so, we use an indirect utility approach; first looking for the optimal effort for a given pair $(\theta, \hat{\theta})$ and, second, computing indirect payoffs accordingly. Incentive compatibility constraints are then written solely in terms of this indirect utility function.

Fix the bonus $w(\hat{\theta})$ that the seller receives when the additional benefit of trade has been realized and he has reported being of type $\hat{\theta}$. His optimal effort is readily obtained as maximizing with respect to e the following expression

$$t(\hat{\theta}) - \theta q(\hat{\theta}) + e w(\hat{\theta}) - \psi(e).$$

Because of strict concavity of this objective with respect to e and thanks to the Inada condition on ψ, the optimal effort $e(\hat{\theta})$, that only depends on the non-negative bonus, is also non-negative and given by the following first-order condition

$$e(\hat{\theta}) = \varphi(w(\hat{\theta})). \tag{3.1}$$

Accordingly, we may rewrite the seller's payoff for an arbitrary choice within the offered menu $(t(\hat{\theta}), q(\hat{\theta}), w(\hat{\theta}))_{\hat{\theta}\in\Theta}$ as

$$\mathcal{R}(\varphi(w(\hat{\theta}))) + t(\hat{\theta}) - \theta q(\hat{\theta}).$$

Incentive Compatibility. We now define the seller's information rent as his equilibrium payoff when truthtelling, i.e.,

$$U(\theta) = \mathcal{R}(\varphi(w(\theta))) + t(\theta) - \theta q(\theta). \tag{3.2}$$

This expression showcases that the seller's information rent is the addition of a liability rent $\mathcal{R}(\varphi(w(\theta)))$ associated to the moral hazard problem and the more standard rent $t(\theta) - \theta q(\theta)$ which is familiar from pure screening environments.

From the Revelation Principle, we must have

$$U(\theta) = \max_{\hat{\theta}\in\Theta} \mathcal{R}(\varphi(w(\hat{\theta}))) + t(\hat{\theta}) - \theta q(\hat{\theta}) \quad \forall \theta \in \Theta. \tag{3.3}$$

In addition, we may rewrite the moral hazard incentive constraint (3.1) when the seller adopts a truthful strategy as

$$e(\theta) = \varphi(w(\theta)) \quad \forall \theta \in \Theta. \tag{3.4}$$

As usual, it is useful to recast the contracting problem in terms of the allocation $(U(\theta), e(\theta), q(\theta))$ that is induced by an incentive compatible contract $(t(\hat{\theta}), w(\hat{\theta}), q(\hat{\theta}))_{\hat{\theta}\in\Theta}$. The next fundamental Lemma characterizes such incentive compatible allocations.

Lemma 1 *The seller's information rent $U(\theta)$ is convex, absolutely continuous and satisfies the following integral representation*

$$U(\theta) = U(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} q(\tilde{\theta}) d\tilde{\theta}. \tag{3.5}$$

Modulo the addition of the effort specification in (3.4), Lemma 1 is a fundamental result which is standard in the mechanism design literature. It relates any non-increasing output profile q that the buyer may want to implement with the seller's payoff U that it induces. This Lemma is the source of the infamous trade-off between efficiency and rent extraction that we will repeatedly encountered under various degrees of complexity in the rest of our analysis.

We may rewrite the integral representation (3.5) using an envelope condition. U being absolutely continuous, it is actually almost everywhere differentiable with the following condition holding at any point of differentiability

$$\dot{U}(\theta) = -q(\theta). \tag{3.6}$$

To understand the envelope condition (3.6), it is useful to consider the benefits that a seller with cost θ gets when he claims having a marginally higher cost $\hat{\theta} = \theta + d\theta$. Doing so means that this seller can produce the requested output $q(\theta + d\theta)$ at a slightly lower cost and accordingly save $q(\theta + d\theta)d\theta \approx q(\theta)d\theta$. To induce information revelation, a type θ must thus receive an extra marginal rent $q(\theta)d\theta$ beyond what is already left to a type $\theta + d\theta$. The integral representation (3.5) shows how those marginal rents are compounded.

Remark. The convexity requirement can also be written as a familiar monotonicity condition on output, namely

$$q \text{ non-increasing.} \tag{3.7}$$

As we will see below, and it is a striking fact of our environment, this monotonicity constraint generally cannot be omitted when looking for an optimal contract.

Participation Constraints. The seller participates whenever he makes a non-negative profit

$$U(\theta) \geq 0 \quad \forall \theta \in \Theta. \tag{3.8}$$

When taken in tandem with the incentive compatibility requirement (3.6), it is straightforward to check that this participation constraint holds whenever it holds for the seller having the highest possible cost, i.e.,

$$U(\overline{\theta}) \geq 0. \tag{3.9}$$

Limited Liability Constraint. The seller is protected by limited liability. Even in the worst scenario where his effort has not induced any fixed cost reduction, he cannot be punished beyond the value of his existing assets. Assuming, for simplicity, that the seller has no assets to start with, this limited liability requirement can be written as

$$t(\theta) - \theta q(\theta) \geq 0 \quad \forall \theta \in \Theta.[8] \tag{3.10}$$

[8]This condition can also be viewed as an ex post participation constraint that would prevent the seller to walk away from contracting once he has already completed his effort. On this interpretation of the liquidity constraint, see Ollier and Thomas (2013) and Krähmer and Strausz (2024).

Using (3.2), a more compact expression is then obtained as

$$U(\theta) \geq \mathcal{R}(e(\theta)) \quad \forall \theta \in \Theta. \tag{3.11}$$

Notice that the right-hand side of (3.11) is non-decreasing with the agent's effort $e(\theta)$, and thus indirectly so with the bonus $w(\theta)$ that implements this effort.[9]

It turns out that this limited liability constraint can be rewritten by means of an extra control variable as

$$U(\theta) = \mathcal{R}(e(\theta)) + z(\theta) \text{ where } z(\theta) \geq 0 \quad \forall \theta \in \Theta.$$

The new non-negative control variable z so defined can be viewed as the mere share of the overall information rent of the seller that accrues to his private information on his cost parameter.

Moving forward, and in view of preparing for the optimization below, we may define effort in terms of the rent profile as

$$e(\theta) = E(U(\theta) - z(\theta)) \quad \forall \theta \in \Theta \tag{3.12}$$

where $E = \mathcal{R}^{-1}$.[10]

Principal's Problem. The buyer wants to maximize, over all incentive-feasible allocations, her expected payoff, namely

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(S(q(\theta)) - K - t(\theta) + e(\theta)(B - w(\theta))\right) f(\theta) d\theta;$$

an expression that we may rewrite as

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(S(q(\theta)) - \theta q(\theta) - K + e(\theta)B - \psi(e(\theta)) - U(\theta)\right) f(\theta) d\theta.$$

The above integrand stands for the overall expected social surplus less the seller's information rent. This expression of the buyer's expected payoff highlights a trade-off between efficiency and rent extraction which is familiar from the screening and moral hazard literatures. This optimization problem is denoted as $(\mathcal{P})$ in the sequel.

4. Two Polar Scenarios

This section considers two polar scenarios. Each of those entails only one kind of incentive problem as a source of agency frictions; either moral hazard or adverse selection. The important take-away hereafter is that, thanks to the separability of the effort and production technologies, incentive distortions on one side of the problem never trickle down to the other. A strong form of incentive dichotomy always arises when only one side of the agency problem matters.

[9] Indeed, $\mathcal{R}'(e) = e\psi''(e) \geq 0$ and $e(\theta)$ is itself increasing with the bonus $w(\theta)$ from (3.4) and $\varphi' > 0$.

[10] Observe that $E' > 0$, $E'(0) = +\infty$ and $E'' \leq 0$ because of the assumptions made on ψ.

4.1. *Moral Hazard*

Suppose now that effort is non-observable while the seller's cost parameter remains common knowledge. The sole contracting problem for the principal is now to induce effort. The solution to the buyer's contracting problem is thus straightforward.

Proposition 1 *Under pure moral hazard, the optimal allocation* $(U^{mh}(\theta), e^{mh}(\theta), q^{mh}(\theta), z^{mh}(\theta))$ *entails the following features.*

- *The optimal output is always efficient*

$$q^{mh}(\theta) = q^{fb}(\theta) \quad \forall \theta \in \Theta. \tag{4.1}$$

- *Effort is downward distorted below the first-best level and it remains independent of types*

$$B = \psi'(e^{mh}) + \mathcal{R}'(e^{mh}). \tag{4.2}$$

- *The seller gets a non-negative liability rent worth*

$$U^{mh}(\theta) = \mathcal{R}(e^{mh}) \Leftrightarrow z^{mh}(\theta) = 0 \quad \forall \theta \in \Theta. \tag{4.3}$$

- *Contracting is always valuable when*

$$S(q^{fb}(\overline{\theta})) - \overline{\theta} q^{fb}(\overline{\theta}) + e^{mh} R'(e^{mh}) \geq K. \tag{4.4}$$

It is useful to interpret the moral hazard distortion in terms of the shadow cost of the limited liability constraint (3.11). Because transferring one unit of util to the seller to relax this constraint also costs one util to the buyer, this shadow cost is equal to one. On the other hand, reducing the effort to $e - de$ relaxes this limited liability constraint by $\mathcal{R}'(e^{mh})de$. This marginal change of effort has thus a benefit $1 \times \mathcal{R}'(e^{mh})de$ while the cost for the buyer in terms of lost expected surplus is $(B - \psi'(e^{mh}))de$. At the optimum, the cost and benefit of such a perturbation should be the same; which yields (4.2).

When the cost parameter is observable, the principal can force the agent to trade efficiently. The separability between the effort and the production technologies now implies that output has no consequences whatsoever on the moral hazard problem. On the other hand and because he is protected by limited liability, the buyer can only be rewarded when his effort pays off and the fixed cost is reduced but he cannot be punished otherwise. The seller must thus give up some costly liability rent $\mathcal{R}(e)$ to induce a positive effort e. Reducing this liability rent calls for implementing a lower effort by means of a lower bonus

$$w^{mh} = B - \mathcal{R}'(e^{mh}) < B.$$

The buyer's payment in this pure moral hazard scenario just covers the seller's observable cost

$$t^{mh}(\theta) = \theta q^{fb}(\theta) \Leftrightarrow z^{mh}(\theta) = 0.$$

Finally, Condition (4.4) for valuable contracting is now strictly harder to satisfy than its complete information counterpart, namely Condition (2.3). It is so because

$$e^{mh} R'(e^{mh}) = e^{mh} B - \psi(e^{mh}) - \mathcal{R}(e^{mh}) < e^{fb} B - \psi(e^{fb}) = \mathcal{R}(e^{fb}).$$

Intuitively, the seller's liability rent is an extra cost that reduces the surplus of the transaction and makes it more difficult to be carried on than under complete information.

For future reference, we now introduce the following variable

$$\rho(B,e) = \frac{B - \psi'(e)}{\mathcal{R}'(e)}$$

To understand the meaning of this variable, it is useful to view the buyer's expected benefit of the seller's effort, namely $V = eB - \psi(e) - \mathcal{R}(e)$ as a function of the seller's limited liability rent $U = \mathcal{R}(e) \Leftrightarrow e = E(U)$. In other words, the function $V(U)$ so implicitly defined represents the frontier of the set of incentive-feasible payoffs induced by the agent's effort. Over the range $\left[0, U^{mh}\right]$, this function is increasing, reaches its maximum at $U^{mh} = \mathcal{R}(e^{mh})$ and is decreasing thereafter. The slope of this function is $\rho(B, E(U)) - 1$, namely the marginal rate of transformation between the agent's and the principal's payoffs. Accordingly, $\rho(B, E(U)) - 1$ is the shadow cost of the liability constraint: Raising the agent's payoff by dU costs $(\rho(B, E(U)) - 1)dU$ to the principal.[11] Finally, observe that $\rho(B,e)$ is non-decreasing in B and non-increasing in e,[12] with $\rho(B, e^{fb}) = 0$ and $\rho(B, e^{mh}) = 1$.

4.2. *Adverse Selection, Moral Hazard and No Limited Liability*

Suppose now that effort remains non-verifiable but that the seller keeps private information on his marginal cost parameter. We also assume that the firm has no liability constraint. As we will see below, the sole contracting problem for the buyer consists in inducing information revelation of this cost parameter. Moral hazard is not an issue in this context.

PROPOSITION 2 *Under adverse selection, moral hazard and no limited liability, the optimal allocation $(U^{as}(\theta), e^{as}(\theta), q^{as}(\theta))$ entails the following features.*

- *The optimal output is non-increasing, downward distorted with respect to the first best, $q^{as}(\theta) \leq q^{fb}(\theta)$ (for all $\theta \in \Theta$, with equality at $\underline{\theta}$ only) with*

$$S'(q^{as}(\theta)) = \theta + \frac{F(\theta)}{f(\theta)}. \tag{4.5}$$

- *The first-best level of effort is always implemented*

$$e^{as}(\theta) = e^{fb} \quad \forall \theta \in \Theta. \tag{4.6}$$

- *The rent profile is decreasing and convex with*

$$U^{as}(\theta) = \int_{\theta}^{\bar{\theta}} q^{as}(\tilde{\theta}) d\tilde{\theta}. \tag{4.7}$$

---

[11] By the same token, $\rho(B, E(U))$ is also the marginal rate of transformation between the seller's payoff U and overall welfare $U + V(U)$. Another interpretation is also worth to be made. Define the overall surplus from effort as $\gamma(e) = eB - \psi(e)$. With those notations at hands, we may compute $\gamma'(e) = B - \psi'(e)$ and $\mathcal{R}'(e) = -e\gamma''(e)$. In other words, we get $\rho(B,e) = -\frac{\gamma'(e)}{e\gamma''(e)}$, and $\rho(B,e)$ is the inverse elasticity of that surplus with respect to effort.

[12] It is straightforward to prove that $\frac{\partial \rho}{\partial B}(B,e) = \frac{1}{\mathcal{R}'(e)} > 0$ and $\frac{\partial \rho}{\partial e}(B,e) = -\frac{\psi''(e)}{\mathcal{R}'(e)} - \rho(B,e)\frac{\mathcal{R}''(e)}{\mathcal{R}'(e)} \leq 0$ whenever $\rho(B,e) \geq 0$.

• *Contracting is always valuable when*

$$S(q^{as}(\overline{\theta})) - \overline{\theta}q^{as}(\overline{\theta}) + \mathcal{R}(e^{fb}) \geq K. \tag{4.8}$$

Here, the buyer obtains information revelation by leaving some information rent to the seller. This rent is costly and output distortions are needed to reduce this cost. Increasing production $q^{as}(\theta)$ for all types within a small interval $[\theta, \theta + d\theta]$ by a positive amount dq increases expected surplus by an amount

$$(S'(q^{as}(\theta)) - \theta)f(\theta)d\theta dq.$$

On the other hand, making trade more likely also increases the costly information rent left to all infra-marginal types by an amount $F(\theta)d\theta dq$. Hence, the optimal quantity balances those two effects whenever (4.5) holds.

As standard in the screening literature, the *Monotone Hazard Rate Property* ensures that the optimal output q^{as} always satisfies the monotonicity condition (3.7).

Condition (4.8) ensures that production is always valuable for all possible types. Because production is now downward distorted and the overall surplus decreases in comparison with the first best, Condition (4.8) is harder to satisfy than (2.3). Of course, that condition prevails when K is small enough.

In this scenario with no liability constraint, the separability between effort and production technologies implies that the additional expected surplus from effort plays no role whatsoever on the screening side. As a result, the buyer can make the seller residual claimant for the choice of effort with a bonus that perfectly reflects the marginal benefit of the seller's effort

$$w^{fb} = B. \tag{4.9}$$

This bonus allows the seller to capture a rent $\mathcal{R}(e^{fb})$ that is entirely recouped by the buyer if she reduces the base payment for the service precisely by that amount. Everything happens as if the seller was offering upfront a rebate $\mathcal{R}(e^{fb})$ to the buyer. Later, the seller enjoys a bonus $w^{fb} = B$ if the fixed cost has been successfully reduced. From (4.9), this bonus induces a first-best level of effort. Formally, the base payment now satisfies

$$t^{as}(\theta) = \theta q^{as}(\theta) + \int_{\theta}^{\overline{\theta}} q^{as}(\tilde{\theta})d\tilde{\theta} - \mathcal{R}(e^{fb}).$$

Observe that, since (3.9) is binding, (3.10) is violated at $\overline{\theta}$,

$$t^{as}(\overline{\theta}) - \overline{\theta}q^{as}(\overline{\theta}) = -\left(e^{fb}\ B - \psi(e^{fb})\right) < 0.$$

In other words, the seller's liability constraint would always be violated with the proposed solution if his cost is large enough. It suggests that the liability constraint, when taken consideration, will bind on the upper tail of the types distribution.

5. ADVERSE SELECTION, MORAL HAZARD AND LIMITED LIABILITY

Suppose now that adverse selection and moral hazard both impede contracting but that the seller is protected by limited liability. Some preliminary remarks are worth to be made.

First, notice that, whenever a non-negative effort $e(\theta)$ is implemented, the limited liability constraint (3.12) implies the participation constraint (3.8). Therefore, (3.9) is necessarily satisfied and can thus be omitted in the description of the set of incentive-feasible allocations.

Second, notice that, whenever (3.11) is slack (i.e., $z(\theta) > 0$), the seller receives an information rent due to his private knowledge on cost that goes beyond the sole limited liability rent due to moral hazard. When (3.11) is instead binding (i.e., $z(\theta) = 0$), the seller receives no extra rent for his private information. As it will soon become clearer, it does not mean at all that effort is as in the pure moral hazard scenario. Distortions might be needed to extract more information rent from the seller.

5.1. *Solution to the Relaxed Problem*

In a first pass, we consider the so called relaxed problem $(\mathcal{P}^r)$ obtained when omitting the monotonicity condition (3.7).

Quantity and Effort Profiles. We start by analyzing distortions on output and effort.

Proposition 3 *The solution $(U^r(\theta), e^r(\theta), q^r(\theta), z^r(\theta))$ to the relaxed problem $(\mathcal{P}^r)$, entails the following features.*

• *The optimal quantity satisfies*

$$(5.1)\qquad S'(q^r(\theta)) = h(\theta) \quad \forall \theta \in \Theta.$$

where

$$(5.2)\qquad h(\theta) = \theta + \frac{\lambda^r(\theta)}{f(\theta)} \quad \forall \theta \in \Theta$$

and

$$(5.3)\qquad \lambda^r(\theta) = F(\theta) - \int_{\underline{\theta}}^{\theta} \rho\left(B, e^r(\tilde{\theta})\right) f(\tilde{\theta}) d\tilde{\theta} \geq 0 \quad \forall \theta \in \Theta.$$

• *This optimal quantity is distorted downwards with respect to the first-best level and upwards with respect to the pure screening scenario*

$$(5.4)\qquad q^{as}(\theta) \leq q^r(\theta) \leq q^{fb}(\theta) \quad \forall \theta \in \Theta.$$

In particular, the optimal quantity is efficient at both ends of the types distribution

$$(5.5)\qquad q^r(\underline{\theta}) = q^{fb}(\underline{\theta}) = q^{as}(\underline{\theta}) \text{ and } q^{as}(\overline{\theta}) < q^r(\overline{\theta}) = q^{fb}(\overline{\theta}).$$

• *The optimal effort satisfies*

$$(5.6)\qquad e^r(\theta) = \min\left\{e^{fb}, E(U^r(\theta))\right\} \quad \forall \theta \in \Theta$$

where the seller's rent U^r is defined as in (5.9) below.

- *Contracting is always valuable when*

$$S(q^{fb}(\bar{\theta})) - \bar{\theta}q^{fb}(\bar{\theta}) + e^r(\bar{\theta})B - \psi(e^r(\bar{\theta})) - \mathcal{R}(e^r(\bar{\theta})) \geq K. \tag{5.7}$$

The optimality condition (5.1) looks familiar and bears some resemblance with the similar condition when only screening matters (4.5). Here also, the marginal benefit of output must equal its virtual cost $h(\theta)$ at the optimum. Yet, the expression of this virtual cost is novel. It depends on the expression of the costate variable $\lambda(\theta)$ for (3.6), which in turn depends on the profile of rent that is implemented at the optimum.

More precisely, to determine the optimal quantity, it is indeed useful to come back on the marginal argument we made in Section 4.2. Consider again a perturbation consisting in increasing output $q^r(\theta)$ for all types within a small interval $[\theta, \theta + d\theta]$ by a small amount dq. Suppose also for the sake of the argument that the liability constraint is binding everywhere; in other words, $U^r(\theta) = \mathcal{R}(e^r(\theta))$ for all $\theta \in \Theta$. Such a perturbation again increases expected surplus by

$$(S'(q^r(\theta)) - \theta)f(\theta)d\theta dq.$$

As in the pure screening scenario, raising production for type θ again increases the costly information rent of all inframarginal types by an amount

$$F(\theta)d\theta dq.$$

Yet, with this perturbation, payments to all those inframarginal types must also be increased and uniformly so to preserve incentive compatibility. The liability constraint is accordingly relaxed for all those types. Building on the insights developed in Section 4.1, the cumulative social benefit of relaxing this liability constraint for all inframarginal types is thus

$$\int_{\underline{\theta}}^{\theta} (B - \psi'(e^r(\tilde{\theta})))f(\tilde{\theta})de(\tilde{\theta})d\tilde{\theta}$$

where $de(\tilde{\theta})$ is type $\tilde{\theta}$'s effort increment which is induced by a perturbation dq. Using the fact that $\mathcal{R}'(e^r(\tilde{\theta}))de(\tilde{\theta}) = dq$, the overall social benefit of such a perturbation can finally be written as

$$\left(\int_{\underline{\theta}}^{\theta} \rho\left(B, e^r(\tilde{\theta})\right) f(\tilde{\theta})d\tilde{\theta}\right) dq.$$

When both adverse selection and moral hazard are present, output $q^r(\theta)$ is chosen whenever

$$(S'(q^r(\theta)) - \theta)f(\theta) = F(\theta) - \int_{\underline{\theta}}^{\theta} \rho\left(B, e^r(\tilde{\theta})\right) f(\tilde{\theta})d\tilde{\theta}. \tag{5.8}$$

It immediately follows from comparing the right-hand sides of (4.5) and (5.8) that quantity distortions are now mitigated when moral hazard is also present. By increasing output and thus payments, the buyer guarantees more cash to the seller. This cash is used to relax the liability constraint and thus indirectly boosts effort.

Although now mitigated, output distortions are still present. The form taken by those distortions somewhat contrasts with the pure screening setting of Section 4.2. Output is efficient for the most efficient type (i.e, $\theta = \underline{\theta}$); a familiar expression of the so called *"no distortion at the top"* result of the screening literature.[13] Yet, output is also set at the first-best level for the worst type as well (i.e., at $\overline{\theta}$) as it can be seen from (5.5). Because $U^r(\theta) = \mathcal{R}(e^r(\theta))$ around $\overline{\theta}$ (see (5.11)), the seller's rent is positive and, in sharp contrast with the pure screening scenario, rent extraction is no longer the buyer's main concern. If it was, fully extracting the rent of the worst type $\overline{\theta}$ would also mean implementing zero effort; a strategy which entails a significant efficiency loss.

To see why in more details, first observe that keeping the same rent profile as that implemented under pure screening would now mean implementing zero effort for the worst type $\overline{\theta}$ and thus no fixed cost reduction when moral hazard is a concern. Starting from that profile and slightly increasing the worst-type seller's information rent has thus a second-order impact on the virtual surplus from trade enjoyed by the buyer but it as also a first-order effect in terms of the fixed cost reduction that is generated by the seller's effort.

Information and Liability Rents. Of course, the seller's information rent $U^r(\theta)$ again satisfies the integral representation

$$(5.9) \qquad U^r(\theta) = U^r(\overline{\theta}) + \int_\theta^{\overline{\theta}} q^r(\tilde{\theta})d\tilde{\theta}.$$

Proposition 4 provides more properties of the rent profile.

Proposition 4 *The seller's information rent $U^r(\theta)$ satisfies the following properties.*

- $U^r(\overline{\theta}) > 0$ *is implicitly defined by the condition*

$$(5.10) \qquad 1 = \int_{\underline{\theta}}^{\overline{\theta}} \rho\left(B, E\left(\min\left\{U^r(\overline{\theta}) + \int_\theta^{\overline{\theta}} q^r(\tilde{\theta})d\tilde{\theta}; \mathcal{R}(e^{fb})\right\}\right)\right) f(\theta)d\theta.$$

- *The seller's information rent exceeds his liability rent only for the most efficient types $\theta < \hat{\theta}^r$*

$$(5.11) \qquad z^r(\theta) = \max\left\{U^r(\theta) - \mathcal{R}(e^{fb}), 0\right\}.$$

where $\hat{\theta}^r$ is defined as

$$(5.12) \qquad U^r(\hat{\theta}^r) = \mathcal{R}(e^{fb})$$

when $\hat{\theta}^r \in (\underline{\theta}, \overline{\theta}]$ and $\hat{\theta}^r = \underline{\theta}$ when (5.12) has no solution.

- *In particular, a seller with a sufficiently high cost parameter gets less rent than in the scenario with pure moral hazard since*

$$(5.13) \qquad U^r(\overline{\theta}) < \mathcal{R}(e^{fb}).$$

[13]See for instance Laffont and Martimort (2002), Chapter 2.

The solution to this relaxed problem can be best understood by observing that increasing output, beyond the pure screening scenario, yields profits that can be used to relax the seller's limited liability constraint and thus facilitate effort implementation.

In the more extreme scenario, inducing cost revelation from the seller requires to give him enough information rent to pay for the liability rent that is necessary to induce first-best effort even if effort is here no longer verifiable. The dichotomy between output distortions and effort provision already highlighted in Section 4.2 holds true; at least when the seller's cost parameter is below a threshold $\hat{\theta}^r$ and the adverse selection rent is sufficiently large. Effort remains at the first-best level.

Unfortunately, this possibility no longer arises when the seller's cost parameter is beyond the threshold $\hat{\theta}^r$. Extracting the seller's information rent when he has a low cost indeed requires making unattractive allocations targeted to less efficient types. Therefore, output is reduced with those types and, accordingly, their payments diminish. For those types, the limited liability constraint is thus now binding. Relaxing this constraint calls for reducing the bonus and implementing an effort below the first-best level. The dichotomy between output distortion and effort provision now fails. The buyer can no longer extract information rent without worsening the moral hazard problem. In particular, as the cost parameter θ increases, less of the seller's rent comes from his private information on cost and thus more effort distortion is needed. This last point is illustrated by Condition (5.13) which shows that the worst types of the seller necessarily get less than the first-best level of liability rent $\mathcal{R}(e^{fb})$ needed to implement first-best effort.

Bonus and Effort. In contrast with the pure moral hazard scenario of Section 4.1, bonuses and efforts are now type-dependent, and more precisely non-increasing with the seller's type. When the seller has private information on his cost, inducing information revelation is facilitated by giving high (resp. low) rewards for the most (resp. least) efficient types. When the limited liability constraint is binding, which arises on the interval $[\hat{\theta}^r, \overline{\theta}]$, greater bonuses are offered to the most efficient types within this interval while lower bonuses target highest types. Low-cost types are thus induced to exert more effort than in the pure moral hazard scenario while high-cost types exert less. This result is formalized in next Proposition.

Proposition 5 *There exists $\theta^{mh} \in (\hat{\theta}^r, \overline{\theta})$ such that*

$$e^r(\theta) \geq e^{mh} \Leftrightarrow w^r(\theta) \geq w^{mh} \Leftrightarrow \theta \leq \theta^{mh}. \tag{5.14}$$

5.2. *Sufficient Conditions for Monotonicity*

Next proposition provides sufficient conditions so that the omitted output monotonicity condition (3.7) holds for the solution to the relaxed problem $(\mathcal{P}^r)$.

Proposition 6 *Suppose that $\dot{f}(\theta) \leq 0$ for all $\theta \in \Theta$. Then, q^r is non-increasing if*

$$U^r(\overline{\theta}) \geq U^i = \mathcal{R}(e^i) \tag{5.15}$$

where e^i is such that

$$\rho(B, e^i) = 2. \tag{5.16}$$

Figures 1a and 1b below represent the rent profile respectively when (5.15) holds and when it does not. In this latter scenario, U^r is first convex before being concave for θ such that $U^r(\theta) \leq U^i$.

[Insert Figure 1a here]

[Insert Figure 1b here]

RUNNING EXAMPLE. Suppose that F is uniform on Θ (with $\overline{\theta} = \underline{\theta} + 1$, i.e., $\Delta = 1$ and thus $F(\theta) = \theta - \underline{\theta}$), that the demand function is linear, $D(p) = a - bp$, with $a, b > 0$ and that the disutility of effort is quadratic, $\psi(e) = \frac{e^2}{2}$ so that $\mathcal{R}(e) = \frac{e^2}{2}$, $e^{fb} = B$, $e^{mh} = \frac{B}{2}$, $e^i = \frac{B}{3}$, $U^{mh} = \frac{B^2}{8}$ and $U^i = \frac{B^2}{18}$.[14]

Those specifications allow us to refine our previous findings and provide a sufficient condition for monotonicity which is almost explicit.

PROPOSITION 7 *For the specifications of our* RUNNING EXAMPLE*, q^r is non-increasing (i.e., U^r is convex) if*

(5.17) $\Gamma(U^i) > 1$

where Γ is defined through

(5.18)
$$\Gamma(U) = \int_U^{\underline{\mathcal{U}}(U)} \frac{d\tilde{U}}{\sqrt{D^2(\overline{\theta}) + 2(\Omega(\tilde{U}) - \Omega(U))}}, \quad \Omega(U) = b\int_0^U \min\left\{2, 3 - \frac{B}{\sqrt{2\tilde{U}}}\right\} d\tilde{U}$$

and $\underline{\mathcal{U}}$ satisfies

(5.19) $$D(\underline{\theta}) = \sqrt{D^2(\overline{\theta}) + 2(\Omega(\underline{\mathcal{U}}(U)) - \Omega(U))}.$$

Figure 1c illustrates the shape of the optimal (relaxed) solution when $B = 1, b = \frac{1}{7}, \bar{\theta} = 1$, and $a = \frac{2}{7}$. Indeed in this parameters' configuration, the relaxed solution is monotonic as $\Gamma(U^i) = 1.004 > 1$. Quantity profiles are depicted as

[Insert Figure 1c here]

5.3. *Ironing*

It is well known, at least from the work of Myerson (1981) and Guesnerie and Laffont (1984), that the solution to the relaxed problem might not always solve the output monotonicity condition. To illustrate, it can be readily verified that, when B is small, U^i converges to zero and $\Omega(U) \approx 2bU$ so that $\underline{\mathcal{U}}(0) = \frac{1}{4b}(D^2(\underline{\theta}) - D^2(\overline{\theta}))$. Then, $\Gamma(0) \approx \frac{1}{2}$ and Condition (5.17) cannot hold. More generally, this monotonicity failure always arises

[14] The reader might have noticed that ψ does not satisfy the Inada assumption at $e = 1$. We shall consider parameter values, especially B, such that the optimal effort remains in $[0, 1]$ under all circumstances below.

when the solution to the relaxed problem entails a very low level of utility for the worst-type seller $\overline{\theta}$, namely $U^r(\overline{\theta}) < U^i$.[15] Ironing techniques are then needed to characterize the optimal monotonic output. This section develops those techniques.

Quantity and Effort Distortions. The ironed solution, that respects the monotonicity constraint (3.7), remains rather simple in our context. Provided the virtual cost is conveniently modified, it remains true that, at the optimum the marginal benefit of output is equal to this ironed virtual cost as stated in next proposition.

Proposition 8 *The optimal allocation $(U^m(\theta), e^m(\theta), q^m(\theta), z^m(\theta))$ entails the following features.*

- *The optimal output satisfies*

$$(5.20) \quad S'(q^m(\theta)) = \overline{h}(\theta) \quad \forall \theta \in \Theta.$$

where

$$(5.21) \quad \overline{h}(\theta) = \dot{\overline{Z}}(F(\theta)), \overline{Z}(F(\theta)) = \text{co}\left(\int_{\underline{\theta}}^{\theta} f(\tilde{\theta})h(\tilde{\theta})d\tilde{\theta}\right), h(\theta) = \theta + \frac{\lambda^m(\theta)}{f(\theta)},$$[16]

and

$$(5.22) \quad \lambda^m(\theta) = F(\theta) - \int_{\underline{\theta}}^{\theta} \rho\left(B, e^m(\tilde{\theta})\right) f(\tilde{\theta})d\tilde{\theta} \geq 0 \quad \forall \theta \in \Theta.$$

- *The optimal effort satisfies*

$$(5.23) \quad e^m(\theta) = \min\left\{e^{fb}; E\left(U^m(\theta)\right)\right\} \quad \forall \theta \in \Theta$$

where the rent profile satisfies

$$(5.24) \quad U^m(\theta) = U^m(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} q^m(\tilde{\theta})d\tilde{\theta} \quad \forall \theta \in \Theta.$$

Ironed virtual cost and bunching. To understand the main features of the solution, it is useful to come back on the ironing techniques. A first step of the analysis consists in ironing the virtual cost $h(\theta)$ to obtain a non-decreasing version of this virtual cost $\overline{h}(\theta)$. This first step is familiar from the work of Myerson (1981) and Toikka (2011). This ironed virtual cost is the non-decreasing function that comes *"closer"* to the possibly non-monotonic virtual cost; where the notion of proximity is in terms of the buyer's expected payment.

In the contracting/auction contexts analyzed by Myerson (1981) and Toikka (2011), the virtual cost is entirely determined by the hazard rate of the type distribution and its ironed version inherits this property. In our more complex model, the expression of the

[15] To see why, observe that $\dot{h}(\overline{\theta}) = 2 - \rho(B, E(U^r(\overline{\theta})))$. Therefore, h is decreasing in the neighborhood of $\overline{\theta}$ whenever $U^r(\overline{\theta}) < U^i$.

[16] We define co(f), the convexification of f as the highest convex function below f. Formally, standard techniques from convexity theory can be used to prove that $\text{co}(f)(x) = \max_y \{yx + \{\min_z f(z) - yz\}\}$.

virtual cost encapsulates the interaction between adverse selection and moral hazard. More precisely, the virtual cost now depends on the costate variable λ^m for (3.6), which in turn depends on the rent profile U^m that is implemented at the optimum. Ironing the virtual cost thus requires to take carefully into account that this rent profile should remain convex to satisfy output monotonicity, which eventually entails flat parts on bunching areas where output is kept constant. Since output is kept constant when the ironed virtual cost is itself constant, the whole process of finding a bunching area relies on solving a complex fixed-point problem.

Running Example. Remember that here $F(\theta) = \theta - \underline{\theta}$. From there, we may first define

$$Z(\theta - \underline{\theta}) = \int_{\underline{\theta}}^{\theta} h(\tilde{\theta})d\tilde{\theta}.$$

Z so defined is thus first strictly convex and then concave when h is non-monotonic. In particular, at a point θ^i where $\ddot{Z}(\theta^i - \underline{\theta}) = 0$, we thus have $\dot{h}(\theta^i) = 0$ and, finally, $U^m(\theta^i) = U^i$.

Henceforth, $\overline{Z} = \text{co}(Z)$ is first strictly convex on an interval $[\underline{\theta}, \theta_0]$ and then linear on $\left[\theta_0, \overline{\theta}\right]$ where necessarily $\theta_0 < \theta^i$.[17] Because Z is continuously differentiable, it must thus be that

$$\dot{Z}(\theta_0 - \underline{\theta}) = \frac{Z(1) - Z(\theta_0 - \underline{\theta})}{\overline{\theta} - \theta_0}.$$

This condition can also be rewritten as

$$h(\theta_0) = \frac{1}{\overline{\theta} - \theta_0} \int_{\theta_0}^{\overline{\theta}} h(\theta)d\theta. \tag{5.25}$$

From there, it also follows that

$$\overline{h}(\theta) = \begin{cases} h(\theta) & \text{for } \theta \in [\underline{\theta}, \theta_0], \\ h(\theta_0) & \text{for } \theta \in \left[\theta_0, \overline{\theta}\right]. \end{cases}$$

We are now ready to state the following result.

Proposition 9 *For the specifications of our* Running Example, U^m *is strictly convex on* $[\underline{\theta}, \theta_0]$ *and linear on* $\left[\theta_0, \overline{\theta}\right]$ *of the form*

$$U^m(\theta) = U^m(\overline{\theta}) + q_0(\overline{\theta} - \theta) \quad \forall \theta \in \left[\theta_0, \overline{\theta}\right] \tag{5.26}$$

where $q_0 = D(h(\theta_0))$, $U^m(\overline{\theta})$, $U^m(\underline{\theta})$ *and* θ_0 *altogether solve*

$$\Omega(U^m(\overline{\theta}) + q_0(\overline{\theta} - \theta_0)) = \frac{1}{\overline{\theta} - \theta_0} \int_{\theta_0}^{\overline{\theta}} \Omega(U^m(\overline{\theta}) + q_0(\overline{\theta} - \theta))d\theta, \tag{5.27}$$

$$S'(q_0) = \overline{\theta} - \Omega(U^m(\overline{\theta})) + \Omega(U^m(\overline{\theta}) + q_0(\overline{\theta} - \theta_0)), \tag{5.28}$$

[17] Eventually, the corner solution $\theta_0 = \overline{\theta}$ arises when h is everywhere non-decreasing.

$$(5.29)\quad D(\underline{\theta}) = \sqrt{q_0^2 + 2(\Omega(U^m(\underline{\theta})) - \Omega(U^m(\overline{\theta}) + q_0(\overline{\theta} - \theta_0)))},$$

$$(5.30)\quad \int_{U^m(\theta_0)}^{U^m(\underline{\theta})} \frac{dU}{\sqrt{q_0^2 + 2(\Omega(U) - \Omega(U^m(\overline{\theta}) + q_0(\overline{\theta} - \theta_0)))}} = \theta_0 - \underline{\theta}.$$

The nonlinear system (5.26) to (5.30) can only be solved numerically. To illustrate, let us take $B = 1, b = \frac{1}{2}, \bar{\theta} = 1$, and $a = 2b$, then the relaxed solution is such that $\hat{\theta}^r = 0.102$ and it is non-monotonic.[18] Indeed $\Gamma\left(U^i\right) = \Gamma\left(\frac{1}{18}\right) = 0.69 < 1$. The ironed solution entails[19]

$$\begin{aligned}
q_0 &= 0.376 > q^r(\theta^i) = 0.375 \\
U^m(\overline{\theta}) &= 0.0105 < U^i = \frac{1}{18} \simeq 0.05 \\
U^m(\underline{\theta}) &= 0.6 > U^r(\underline{\theta}) = 0.59 > U^{as}(\underline{\theta}) = \frac{1}{2} \\
\theta_0 &= 0.799 \Rightarrow U^m(\theta_0) = 0.08 \\
\hat{\theta}^m &= 0.106 > \hat{\theta}^r = 0.102
\end{aligned}$$

For $B = \frac{1}{2}, b = \frac{1}{7}, \bar{\theta} = 1$, and $a = \frac{2}{7}$, with $\Gamma(U^i) = 0.67 < 1$, we have

$$\begin{aligned}
q_0 &= 0.131 > q^r(\theta^i) = 0.1 \\
U^m(\overline{\theta}) &= 0.04 > U^i = \frac{1}{72} \simeq 0.0138 \\
U^m(\underline{\theta}) &= 0.22 > U^r(\underline{\theta}) = 0.167 > U^{as}(\underline{\theta}) = \frac{1}{7} \\
\theta_0 &= 0.52 \Rightarrow U^m(\theta_0) = 0.11 \\
\hat{\theta}^m &= \underline{\theta} = 0 < \hat{\theta}^r = 0.16
\end{aligned}$$

Figures 2a, 2b, 2c below represent this construction and the corresponding output and rent profiles when bunching arises.

[Insert Figure 2a here]

[Insert Figure 2b here]

[Insert Figure 2c here]

The following figures depict our two simulation cases. For $B = 1, b = \frac{1}{2}, \bar{\theta} = 1$, and $a = 1$, the output profiles are as follows

[Insert Figure 3a here]

For $B = \frac{1}{2}, b = \frac{1}{7}, \bar{\theta} = 1$, and $a = \frac{2}{7}$, output profiles are depicted as

[Insert Figure 3b here]

[18] Quantity profiles for theses simulations can be found at the end of the Appendix

[19] The threshold type $\hat{\theta}^m$ is the ironed counterpart of $\hat{\theta}^r$ and defined in the next proposition.

Information and Liability Rents. For completeness, next proposition characterizes the profiles of information rent in this context with bunching. It echoes our earlier findings in the scenario without bunching.

Proposition 10 *The optimal allocation* $(U^m(\theta), e^m(\theta), q^m(\theta), z^m(\theta))$ *that solves problem* $(\mathcal{P})$ *entails the following features.*

• $U^m(\overline{\theta})$ *is implicitly defined by the condition*

$$1 = \int_{\underline{\theta}}^{\overline{\theta}} \rho\left(B, e^m(\theta)\right) f(\theta) d\theta. \tag{5.31}$$

• *The seller's information rent exceeds his liability rent only for the most efficient types* $\theta < \hat{\theta}^m$ *only*

$$z^m(\theta) = \max\left\{U^m(\theta) - \mathcal{R}(e^{fb}); 0\right\} \tag{5.32}$$

where

$$U^m(\hat{\theta}^m) = \mathcal{R}(e^{fb}) \tag{5.33}$$

when $\hat{\theta}^m \in (\underline{\theta}, \overline{\theta}]$ *and* $\hat{\theta}^m = \underline{\theta}$ *otherwise.*

6. THE SHAPE OF PAYMENTS AND BONUSES

Focusing on the relaxed solution for simplicity, the expression of the seller's payment is readily obtained as

$$t^r(\theta) - \theta q^r(\theta) = z^r(\theta) = \begin{cases} U^r(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} q^r(\tilde{\theta}) d\tilde{\theta} - \mathcal{R}(e^r(\theta)) > 0 & \text{if } \theta \in [\underline{\theta}, \hat{\theta}^r], \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the seller is reimbursed for the cost of operating the service when $\theta \geq \hat{\theta}^r$. The liability constraint is binding and, in that case, effort is below the first-best level. This is similar to what would arise with a cost-plus contract even though, in our context, costs are not observable. Following the seminal works of Laffont and Tirole (1986), Rogerson (2003), Chu and Sappington (2007), Bajari and Tadelis (2001) and Garrett (2014) (among others) the incentive regulation literature has already pointed out the virtues of fixed-price contracts to incentivize cost reduction when effort impacts marginal costs and while. It has also recognized the benefits of using cost-plus contracts to better extract information rent. Interestingly, in our framework, the same qualitative features that would arise with fixed-price and cost-plus contracts are also present even though costs are not observable and effort affects fixed costs and not marginal costs.

For types $\theta \leq \hat{\theta}^r$, the effort is first best and the bonus is thus $w^r(\theta) = B$. This feature would also be found had costs been observable and a fixed price contract had been offered. For types $\theta \geq \hat{\theta}^r$, this bonus is determined as $w^r(\theta) = \psi'(E(U^r(\theta)))$ and, taking derivatives, we obtain

$$\dot{w}^r(\theta) = -\frac{q^r(\theta)}{e^r(\theta)}.$$

The bonus, and the induced effort, are thus decreasing functions of θ. Rewards for effort performances covary with output and less incentives for fixed cost reduction are provided as the seller's marginal cost increases.

7. OTHER APPLICATIONS

This section illustrates the broad applicability of our framework in a number of economic environments of interest for applied researchers. Most often, we can directly import our previous findings to shed new lights on contracting issues in those environments. In a few cases below, simple extensions are needed; in which case, we present some key results without providing formal proofs.

7.1. *Regulation and Investment*

In many regulated sectors, the regulated firm may enjoy the stream of stable profits on its core activities and use those profits to invest in adjacent markets or various add-ons. To illustrate, a highway franchisee may have to use toll revenues to finance additional infrastructures (bridges, highway ramps, and the like). Similar issues arise also for water provision, train transportation, energy, and telecommunications. A key question in those contexts is how to modify contracts on core activities to facilitate financing.

To illustrate those issues, we now suppose that the firm, which is regulated on its core activities, may also need to undertake an investment I to provide an add-on. This add-on generates some extra profit worth $B > 0$ with probability e and zero otherwise. Effort is non-verifiable. The public authority in charge provides funding and asks for repayments $\overline{R}$ and $\underline{R}$ depending on whether the add-on is a success or not. Of course, this principal still represents consumers for the core activities. The public authority has thus the following objective function

$$V = S(q) - \tilde{t} + e\overline{R} + (1-e)\underline{R} - I$$

where $\tilde{t}$ is the payment for core activities. Assuming that there is no fixed cost, the firm's profit can now be written as

$$U = \tilde{t} - \theta q + e(B - \overline{R}) - (1-e)\underline{R} - \psi(e).$$

Following Arve and Martimort (2024) who analyze procurement contracts in a related context, we now assume that profits on core activities can be used as collateral for financing the add-on. Accordingly, we shall write the firm's limited liability constraint as

$$\tilde{t} - \theta q - \underline{R} \geq 0.$$

This condition becomes (3.11) when expressed in terms of the firm's information rent and effort. Formally, our analysis then applies *mutatis mutandis* provided we define the bonus as $w = B - (\overline{R} - \underline{R})$ and the payments on the base service as $t = \tilde{t} - \underline{R}$. Our findings in Section 6 then suggest that the regulation of core activities should be tilt towards cost-plus contracts and that repayments in case of a successful add-on should be larger for those firms with less efficient technologies than for more efficient firms.

7.2. *Financial Contracting*

Consider a borrower (the agent) with no liquidity who needs funding for a first project with stable returns. Following Freixas and Laffont (1990), we assume that the project returns $\theta g(k)$ (with $g' > 0$, $g'' < 0$ and the Inada conditions $g'(0) = +\infty$, $g(0) = 0$)

depend not only on the loan size k but also on a technology shock θ which is private information to the borrower. Financiers (the principal) provide funding and, in return, ask for a repayment t on this stable project.

The borrower is also willing to undertake another investment that requires some extra funding K from financiers and whose return is risky. As in Section 7.1 above and much in spirit of Innes (1990), this risky project generates some extra profit $B > 0$ with probability e and zero otherwise. Effort is non-verifiable. Financiers ask for repayments $\overline{R}$ and $\underline{R}$ depending on whether the project succeeds or fails. Financiers have thus the following objective function

$$V = t + e\overline{R} + (1-e)\underline{R} - r(k+K)$$

where r is the rental rate of capital.

The borrower's payoff can instead be written as

$$U = \theta g(k) - t + e(B - \overline{R}) - (1-e)\underline{R} - \psi(e).$$

Under complete information, the first-best effort level is still given by (2.2). The optimal loan size $k^{fb}(\theta)$ trades off its marginal return against the rental rate of capital r and we obtain

$$\theta g'(k^{fb}(\theta)) = r.$$

The techniques that were developed throughout this paper can again be readily applied to this framework. The main difference here is that incentive compatibility now implies that the rent profile is increasing; a borrower who has a high productivity shock is willing to report a low shock to save on repayments. Standard techniques allow us to obtain the derivative of the borrower's information rent as

$$\dot{U}(\theta) = g(k(\theta)).$$

In the pure screening environment, the loan size is thus distorted downward for all types except for the top $\overline{\theta}$. Those distortions undermine the borrower's incentives to under-report the productivity shock. The optimal loan size depends on a virtual productivity shock by means of the following familiar expression

$$\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) g'(k^{as}(\theta)) = r.$$ [20]

The borrower's limited liability constraint now writes as

$$\theta g(k) - t - \underline{R} \geq 0.$$

From there, it is easy to replicate our previous approach when both adverse selection and moral hazard are at play. The solution to the relaxed problem $(\mathcal{P}^r)$ entails a distortion in the loan size that is expressed as

$$\left(\theta - \frac{\lambda^r(\theta)}{f(\theta)}\right) g'(k^r(\theta)) = r$$

[20] We assume that $\underline{\theta}$ is large enough so that a positive loan is always valuable even under asymmetric information The condition $\underline{\theta} - \frac{1}{f(\underline{\theta})} > 0$ is sufficient in this respect provided that the assumption MHRP holds. Assumption MHRP also ensures that k^{as} is non-decreasing; a monotonicity condition required for incentive compatibility.

where

$$\lambda^r(\theta) = 1 - F(\theta) - \int_{\theta}^{\overline{\theta}} \rho\left(B, e^r(\tilde{\theta})\right) f(\tilde{\theta})d\tilde{\theta} \geq 0 \quad \forall \theta \in \Theta \text{ with } \lambda^r(\underline{\theta}) = 0 \tag{7.1}$$

and e^r still satisfies (5.6).

In particular, the stable project is financed with loans of a greater size than under pure screening; $k^r(\theta) \geq k^{as}(\theta)$. This upward distortion relaxes the borrower's liability constraint and facilitates funding of the risky project. Interestingly, when the monotonicity condition (that now requires k non-decreasing) binds and bunching also arises, the optimal financial contract imposes a minimal loan size on the stable project.

Finally, comparing to a scenario of pure moral hazard as in Innes (1990), the addition of private information implies that credit is more (resp. less) costly for borrowers with low (resp. high) return opportunities on stable projects.

7.3. *Price Discrimination and Quality*

Consider a firm (the principal) that price discriminates her customers (the agent/the tenant) with respect to quality in a context *à la* Mussa and Rosen (1978). This seller rents to those customers one unit of a good of quality q at a price t. A customer's willingness to pay for one unit of quality q is θq where θ is a preference parameter. Supplying one unit of the good of quality q costs $c(q)$ to the seller (with $c' \geq 0$, $c'' > 0$ and the Inada conditions $c'(0) = c(0) = 0$). The tenant must exert effort e to return the good in proper shape at the end of the rental period; an event that arises with probability e. With probability $1 - e$, the good is damaged and the seller incurs a loss worth B. Let P be the fine paid by the customer in that event.

The firm's and the customer's expected payoffs respectively write as

$$V = t - c(q) + (1 - e)(P - B)$$

and

$$U = \theta q - t - (1 - e)P - \psi(e).$$

The first-best effort $e^{fb}(\theta)$ here again satisfies (2.2). The first-best quality level $q^{fb}(\theta)$ is (assuming an interior solution) easily obtained as

$$c'(q^{fb}(\theta)) = \theta.$$

With pure screening, the quality level entails a familiar downward distortion where, again, the preference parameter has been replaced by a virtual type of a lower magnitude

$$c'(q^{as}(\theta)) = \theta - \frac{1 - F(\theta)}{f(\theta)}.$$ [21]

[21] Here, we also assume that $\underline{\theta}$ is large enough, namely $\underline{\theta} - \frac{1}{f(\underline{\theta})} > 0$, to avoid a corner solution at zero quality. Assumption MHRP again ensures that q^{as} remains non-decreasing, as requested for incentive compatibility.

Consider now the scenario with both adverse selection and moral hazard. The customers' limited liability constraint now writes as

$$\theta q - t - P \geq 0.$$

This condition can again be expressed in terms of the customers' information rent and effort as (3.11).

It can be readily checked that the quality level q^r that solves the relaxed problem $(\mathcal{P}^r)$ is now defined as

$$c'(q^r(\theta)) = \theta - \frac{\lambda^r(\theta)}{f(\theta)}.$$

where λ^r and e^r still satisfy (7.1) and (5.6).

One key lesson of this model is that quality distortions are less significant when moral hazard is a concern since $q^{as}(\theta) \leq q^r(\theta)$. Moreover, the possibility of bunching may induce a minimal quality standard and thus limits the spectrum of quality levels offered by the seller.

7.4. *Quality of Service*

Procurement contracts commonly include targets for quality of service.[22] Among the most important such targets are on-time delivery requirements specifying the exact date at which the good or service procured should be delivered. Obviously, on-time delivery imposes efforts to the regulated firm.

To bring those issues into our framework, suppose that the good or service is only consumed at date 2 if a delay in delivery arises. Denoting by δ the discount factor, everything happens as if society was thus incurring a disutility of consumption $(1-\delta)S(q)$ whenever a delay arises; an event that occurs with probability $1-e$ where e is the firm's effort in diligence. The public authority has thus an objective function that we write as

$$V = (e + \delta(1-e))(S(q) - t) - ew$$

where w now stands for some extra reward in case of earlier completion while t is the basic payment. Assuming again that there is no fixed cost, the firm's profit can now be written as

$$U = (e + \delta(1-e))(t - \theta q) + ew - \psi(e).$$

Observe that, absent a bonus, the firm's marginal benefit of early completion consists of getting profit early.

It can be readily checked that, at the first best, the buyer's marginal benefit of output takes the by-now familiar expression (2.1). Turning to effort, the optimal effort balances the social benefit of enjoying early consumption against the marginal disutility

$$(1-\delta)(S(q^{fb}(\theta)) - \theta q^{fb}(\theta)) = \psi'(e^{fb}), \quad \forall \theta \in \Theta.$$

[22] See for instance Lewis and Sappington (1992), Laffont and Tirole (1993) Chapter 4, and Weisman (2005).

When effort is non-verifiable, the marginal benefit for early delivery entails not only the opportunity benefit of getting profit earlier but also the bonus. We have

$$w(\theta) + (1-\delta)(t(\theta) - \theta q(\theta)) = \psi'(e(\theta)).$$

From there, we can easily check that the liability constraint (3.12) can here also be expressed in terms of the firm's information rent and effort as (3.11). Relaxing the liability constraint (when binding) again calls for further reductions in effort.

To see this point in more details, we again focus on the solution to the relaxed problem $(\mathcal{P}^r)$. The optimal output now satisfies

$$(7.2)\qquad (e^r(\theta) + \delta(1-e^r(\theta)))(S'(q^r(\theta)) - \theta) = \frac{\lambda^r(\theta)}{f(\theta)}, \quad \forall\theta \in \Theta$$

where e^r and U^r still satisfy respectively (5.6) and (5.9) while λ^r now solves

$$\lambda^r(\theta) = F(\theta) - \int_{\underline{\theta}}^{\theta} \rho\left((1-\delta)(S(q^r(\tilde{\theta})) - \tilde{\theta} q^r(\tilde{\theta})), e^r(\tilde{\theta})\right) f(\tilde{\theta})d\tilde{\theta} \geq 0 \quad \forall\theta \in \Theta \text{ with } \lambda^r(\overline{\theta}) = 0.$$

This formula is again familiar provided that one is ready to now view the marginal benefit of effort as the gain from earlier completion in terms of social surplus. Qualitatively, we observe that pushing output further away from the first-best level decreases this marginal benefit of effort, increases λ^r and thus contributes to reduce output further through (7.2). In other words, low output is expected in such setting.

7.5. *Environmental Regulation*

Laffont (1995) and Hiriart and Thomas (2017) have studied the potential conflict between cost minimization and safety care in the context of major environmental risks. A regulated firm may undertake a non-verifiable effort to reduce the probability of an environmental damage. Yet, doing so increases the marginal disutility of a cost-reducing effort and, as in Laffont and Tirole (1993), costs are observable. The intricacy of having moral hazard, adverse selection and limited liability elements altogether in this model with cost observability forces these authors to focus on the discrete scenario where safety care effort is a $0-1$ decision.

Our model, which assumes away cost observability and follows Baron and Myerson (1982)'s seminal model of regulation in this respect, is much more tractable. To see why, consider the following simple variation of our model. A regulator (who buys the firm's services or good) maximizes a consumer surplus and takes into account the risk of an environmental disaster for society. Formally, we state her objective function as

$$V = S(q) - t - (1-e)Bq - ew$$

where, for simplicity, the regulator gives no weight to the firm's profit and Bq stands for an environmental damage that arises with probability $1-e$. Observe that this damage depends on the scale of production. With this formulation, the firm receives $t+w$ for its services in case no damage occurs while its payment is reduced to t otherwise.

To capture the fact that inducing safety care increases production costs, we posit that the disutility of effort is counted per unit of output so that the firm's perceived cost of production encapsulates non-monetary elements and writes now $(\theta+\psi(e))q$. For simplicity, we also assume away any fixed cost. The firm's payoff can thus be written as

$$U = t - (\theta + \psi(e))q + ew.$$

At the first best, output is set so that marginal surplus, now including the marginal damage is equal to the firm's marginal cost which now includes its disutility of effort

$$S'(q^{fb}(\theta)) - B(1-e^{fb}) = \theta + \psi(e^{fb}), \quad \forall \theta \in \Theta. \tag{7.3}$$

Turning to effort, the marginal benefit of safety care in terms of reduced expected damage is again equal to marginal disutility of effort but now both are counted per unit of output. This leads to the familiar expression of the first-best effort (2.2). The optimality condition (7.3) showcases a fundamental substitutability between effort and output. More safety care effort increases the firm's marginal cost of production and thus reduces production.[23]

When effort in safety care is non-verifiable, the bonus $w(\theta)$ is used to providing incentives. The corresponding incentive constraint writes as

$$w(\theta) = \psi'(e(\theta))q(\theta).$$

The firm's liability constraint still writes as in (3.12); a condition that can be now expressed in terms of the firm's information rent, output and effort as

$$U(\theta) \geq q(\theta)\mathcal{R}(e(\theta)). \tag{7.4}$$

Although details differ, the analysis of this scenario would bear much resemblance with our main analysis. The new expression (7.4) shows that relaxing the binding liability constraint would call for reductions in both effort provision and output. Yet, reducing output makes it less attractive to reduce effort (and reciprocally).

To see this point in more details, we again focus on the solution to the relaxed problem $(\mathcal{P}^r)$. Proceeding as in our main analysis, it is straightforward to show that the optimal output now satisfies

$$S'(q^r(\theta)) - B(1-e^r(\theta)) = \theta + \frac{\lambda^r(\theta)}{f(\theta)} + \psi(e^r(\theta)) + \rho\left(B, e^r(\theta)\right)\frac{U^r(\theta)}{q^r(\theta)}, \quad \forall \theta \in \Theta \tag{7.5}$$

where λ^r and U^r again satisfy (5.3) and (5.9) while e^r now solves

$$e^r(\theta) = \min\left\{E\left(\frac{U^r(\theta)}{q^r(\theta)}\right), e^{fb}\right\}, \quad \forall \theta \in \Theta. \tag{7.6}$$

The output distortion (7.5) not only comes from a by-now familiar addition of a virtual cost but also from a novel extra cost $\rho\left(B, e^r(\theta)\right)\frac{U^r(\theta)}{q^r(\theta)}$ which stems for the (positive) shadow cost of the liability constraint (7.4). In this respect, decreasing further output becomes more attractive than in our basic model. In other words, with limited liability and asymmetric information, the substitutability between effort and output is exacerbated. Providing more incentives on safety care also means a significantly more depressed output and thus higher prices for consumers.

[23] Finally, note that the gains from trade are always large enough to warrant production in this context when

$$S(q^{fb}(\underline{\theta})) - (B+\theta)q^{fb}(\underline{\theta}) + \mathcal{R}(e^{fb})q^{fb}(\underline{\theta}) \geq 0.$$

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APPENDIX: PROOFS OF THE MAIN RESULTS

Proof of Lemma 1: Although, as stated in the text, Lemma 1 provides only necessary conditions. There is a sufficiency part also that we present below.

Necessity. First, observe that U defined in (3.3) is convex as a maximum of linear functions of θ. Second, it immediately follows from Theorem 2 and Corollary 1 in Milgrom and Segal (2002), that U is absolutely (in fact Lipschitz) continuous and almost everywhere differentiable with (3.6) holding at any point of differentiability. From there, the integral representation (3.5) follows.

Sufficiency. Reciprocally, any allocation $(U(\theta), e(\theta), q(\theta))$ such that $U(\theta)$ is absolutely continuous and convex, with (3.2), $-q(\theta) \in \partial U(\theta)$[24] and (3.4) being satisfied, is such that (3.3) holds.

To prove this, consider any pair $(\theta, \hat{\theta}) \in \Theta^2$. We may then rewrite the integral representation (3.5) as

$$U(\theta) = U(\hat{\theta}) + \int_{\theta}^{\hat{\theta}} q(\tilde{\theta}) d\tilde{\theta}.$$

[24] $\partial U(\theta)$ denotes the subdifferential of the convex function U at θ, namely $\partial U(\theta) = \Big\{ -q \text{ such that } U(\hat{\theta}) - U(\theta) \geq -q(\hat{\theta} - \theta) \quad \forall \hat{\theta} \in \Theta \Big\}$.

Because U is convex, it admits a sub-differential ∂U and, since $-q(\hat{\theta}) \in \partial U(\hat{\theta})$, we have

$$U(\theta) \geq U(\hat{\theta}) - q(\hat{\theta})(\theta - \hat{\theta}).$$

From there and the definition of U as in (3.2), (3.3) follows. Lastly, (3.4) implies

$$\mathcal{R}(\varphi(w(\hat{\theta}))) \geq ew(\hat{\theta}) - \psi(e), \quad e \in [0, 1]\,.$$

Q.E.D.

Proofs of Propositions 1 and 2: We notice that the set of incentive-feasible allocations under pure screening is

$$\mathcal{A}^{as} = \{(U(\theta), e(\theta), q(\theta)) \text{ s.t. } q(\theta) \in \mathcal{Q}, (3.6), (3.7), (3.9)\}\,.$$

Under pure moral hazard, the set of incentive-feasible allocations is instead

$$\mathcal{A}^{mh} = \{(U(\theta), e(\theta), q(\theta), z(\theta)) \text{ s.t. } q(\theta) \in \mathcal{Q}, z(\theta) \geq 0, (3.6), (3.7), (3.12)\}\,.$$

The proofs consist in maximizing the buyer's expected payoff over those sets of incentive-feasible allocations. They are routine and thus omitted.

There are two points we want to point out here. First, q^{as} is non-increasing when Assumption MHRP holds. Second, condition for contracting to be valuable for Proposition 1. Given that the productive surplus is lowest at $\overline{\theta}$, this condition writes as

$$S(q^{fb}(\overline{\theta})) - \overline{\theta}q^{fb}(\overline{\theta}) + e^{mh}B - \psi(e^{mh}) - \mathcal{R}(e^{mh}) \geq K,$$

that can be simplified as (4.4) using (4.2).

Q.E.D.

Proofs of Propositions 3 and 4: We now define the set of incentive-feasible allocations $\mathcal{A}$ under both adverse selection and moral hazard as

$$\mathcal{A} = \{(U(\theta), e(\theta), q(\theta), z(\theta)) \text{ s.t. } q(\theta) \in \mathcal{Q}, z(\theta) \geq 0, (3.4), (3.6), (3.7), (3.9), (3.12)\}\,.$$

In the sequel, we first consider a larger space of incentive-feasible allocations $\mathcal{A}^r$ which is obtained by omitting the monotonicity condition (3.7), namely

$$\mathcal{A}^r = \{(U(\theta), e(\theta), q(\theta), z(\theta)) \text{ s.t. } q(\theta) \in \mathcal{Q}, z(\theta) \geq 0, (3.4), (3.6), (3.9), (3.12)\}\,.$$

Consider thus the maximization problem $(\mathcal{P}^r)$ which consists in maximizing the buyer's expected payoff over $\mathcal{A}^r$. We denote by λ^r the co-state variable for (3.6), we can now write the Hamiltonian for this optimization problem as

$$\mathcal{H}(U, q, z, \lambda^r, \theta) = (S(q) - \theta q - K + BE(U - z) - \psi(E(U - z)) - U)\, f(\theta) - \lambda^r q \tag{A.1}$$

Let $(U^r(\theta), q^r(\theta), z^r(\theta))$ denote an optimal arc.

Necessity. We use Pontriagyn Principle to get necessary conditions for optimality of such an arc. (See Chapter 2, Theorem 2 in Seierstad and Sydsaeter (1986).) Those conditions are listed below.

- *Costate variable.* There exists λ^r, continuous and piecewise differentiable, such that

$$-\dot{\lambda}^r(\theta) = f(\theta)\,(\rho(B, E(U^r(\theta) - z^r(\theta))) - 1) \tag{A.2}$$

- *Transversality conditions.* Because there is no boundary condition on U at both $\underline{\theta}$ and $\overline{\theta}$ (remember that (3.9) is automatically satisfied at a non-negative effort level), the transversality conditions are given by

$$\lambda^r(\underline{\theta}) = \lambda^r(\overline{\theta}) = 0. \tag{A.3}$$

- *Optimality condition with respect to z.* We find

$$z^r(\theta)\begin{cases} > 0 & \text{iff } B = \psi'(E(U^r(\theta) - z^r(\theta))) \Leftrightarrow e^r(\theta) = e^{fb} \\ 0 & \text{if } B > \psi'(E(U^r(\theta))) \Leftrightarrow e^r(\theta) < e^{fb}. \end{cases} \tag{A.4}$$

- *Optimality condition with respect to q.* $\mathcal{H}(U, q, z, \lambda^r, \theta)$ being separable in q, we immediately find (5.1).
- *End-point condition at $\overline{\theta}$.*

$$\left(S(q^r(\overline{\theta})) - \overline{\theta}q^r(\overline{\theta}) - K + Be^r(\overline{\theta}) - \psi(e^r(\overline{\theta})) - \mathcal{R}(e^r(\overline{\theta}))\right) f(\overline{\theta}) - \lambda^r(\overline{\theta})q^r(\overline{\theta}) \geq 0. \tag{A.5}$$

Sufficiency. Those necessary conditions are also sufficient. To prove this, we apply Arrow's Sufficiency Conditions. (See Chapter 2, Theorem 5 in Seierstad and Sydsaeter (1986).) We construct the maximized Hamiltonian as

$$\mathcal{H}^*(U, \lambda^r(\theta), \theta) = \max_{q \in \mathcal{Q}, z \geq 0} \mathcal{H}(U, q, z, \lambda^r(\theta), \theta).$$

Observe that the maximizer $z^{**}(U)$ satisfies

$$z^{**}(U) = \max\left\{U - \mathcal{R}(e^{fb}), 0\right\}. \tag{A.6}$$

We may compute

$$\mathcal{H}^*(U, q, \lambda^r(\theta), \theta) = (S(q) - \theta q - K + \zeta(U))f(\theta) - \lambda^r(\theta)q$$

where

$$\zeta(U) = \max\left\{\mathcal{R}(e^{fb}); BE(U) - \psi(E(U))\right\} - U.$$

It is straightforward to observe that $\mathcal{H}^*(U, \lambda^r(\theta), \theta)$ is twice continuously differentiable in U, flat for $U \geq \mathcal{R}(e^{fb})$ and strictly concave for $U < \mathcal{R}(e^{fb})$ since then

$$\frac{\partial^2 \mathcal{H}^*}{\partial U^2}(U, \lambda^r(\theta), \theta) = f(\theta)\frac{\frac{\partial \rho}{\partial E}(B, E(U))}{\mathcal{R}'(E(U))} < 0.$$

where the last inequality follows from $\frac{\partial \rho}{\partial E} < 0$ for $E < e^{fb}$ (as shown in Footnote 12). From this, it follows that the necessary conditions above are also sufficient.

Implications. We now use those necessary and sufficient conditions to derive more specific results.

- The expression (5.9) is nothing more than the integral representation (3.5) expressed for the optimal output profile q^r.

- Condition (5.11) follows from (A.4).

- Integrating (A.2) and using the first boundary condition in (A.3) yields

$$\lambda^r(\theta) = -\int_{\underline{\theta}}^{\theta} \left(\rho(B, E(U^r(\tilde{\theta}) - z^r(\tilde{\theta}))) - 1\right) f(\tilde{\theta}) d\tilde{\theta}.$$

Using (5.11) and integrating, this expression becomes (5.3).

Observe also that λ^r so defined is quasi-concave. Indeed, (A.2) implies

$$\frac{d}{d\theta}\left(\frac{-\dot{\lambda}^r(\theta)}{f(\theta)}\right) = \begin{cases} 0 & \text{if } \theta \in [\underline{\theta}, \hat{\theta}), \\ -\frac{\frac{\partial \rho}{\partial E}(B, E(U^r(\theta)))}{\mathcal{R}'(E(U^r(\theta)))} q^r(\theta) \geq 0 & \text{if } \theta \in (\hat{\theta}^r, \overline{\theta}]. \end{cases}$$

where $\frac{\partial \rho}{\partial E} \leq 0$ is shown in footnote 12.
Because λ^r is quasi-concave and (A.3) holds, we have

$$\lambda^r(\theta) \geq 0 \quad \forall \theta \in \Theta. \tag{A.7}$$

From there, it follows that

$$\theta \leq S'(q^r(\theta)) = \theta + \frac{\lambda^r(\theta)}{f(\theta)} \quad \forall \theta \in \Theta$$

and thus

$$q^r(\theta) \leq q^{fb}(\theta) \quad \forall \theta \in \Theta.$$

From (5.3) and the fact that $e^r(\overline{\theta}) = \mathcal{R}^{-1}(U^r(\overline{\theta})) \in \left[0, e^{fb}\right]$, we deduce that $\rho(B, E(U^r(\theta) - z^r(\theta))) \geq 0$ for all $\theta \in \Theta$. Hence,

$$\lambda^r(\theta) \leq F(\theta) \quad \forall \theta \in \Theta.$$

From there, it follows that

$$S'(q^r(\theta)) \leq \theta + \frac{F(\theta)}{f(\theta)} \quad \forall \theta \in \Theta$$

and thus

$$q^r(\theta) \geq q^{as}(\theta) \quad \forall \theta \in \Theta$$

which ends the proof of (5.4).

- Inserting the transversality conditions (A.3) into (5.1) and taking into account the assumption MHRP yields (5.5). Inserting now the second transversality condition $\lambda^r(\overline{\theta}) = 0$ (see (A.3)) into (5.3) yields

$$1 = \int_{\underline{\theta}}^{\overline{\theta}} \rho(B, E(U^r(\tilde{\theta}) - z^r(\tilde{\theta}))) f(\tilde{\theta}) d\tilde{\theta}.$$

Using (5.11), we find (5.10).

To prove (5.13), suppose to the contrary that $U^r(\overline{\theta}) > \mathcal{R}(e^{fb})$. Then, we get

$$\min\left\{U^r(\theta); \mathcal{R}(e^{fb})\right\} = \mathcal{R}(e^{fb}) \quad \forall \theta \in \Theta$$

and thus

$$\int_{\underline{\theta}}^{\overline{\theta}} \rho\left(B, \min\left\{E(U^r(\theta)); e^{fb}\right\}\right) f(\theta) d\theta = 0;$$

which is a contradiction with (5.10).

- Observe that $\psi'(E(U))$ is non-decreasing in U and $U^r(\theta)$ is non-increasing in θ from (3.6). Hence, if $B > \psi'(E(U^r(\theta)))$, at some $\theta \in \Theta$, it must be that $B > \psi'(E(U^r(\tilde{\theta})))$ for all $\tilde{\theta} \geq \theta$. It implies that $\Omega = \{\theta \in \Theta | z^r(\theta) = 0\}$ is of the form $[\hat{\theta}^r, \overline{\theta}]$ as stipulated in (5.11). Condition (5.6) immediately follows.

- Observe now that $\rho(B, e) \approx \frac{B}{e\psi''(0)} - 1$ when e is sufficiently close to zero. Because $U(\theta) \geq \mathcal{R}(e(\theta))$ with an equality on $[\hat{\theta}^r, \overline{\theta}]$, this approximation implies that the integral $\int_{\underline{\theta}}^{\theta} \rho\left(B, E\left(\min\left\{U^r(\tilde{\theta}); \mathcal{R}(e^{fb})\right\}\right)\right) f(\tilde{\theta})d\tilde{\theta}$ would not converge when $\theta \to \overline{\theta}$ had $U^r(\theta) = \mathcal{R}(e^r(\theta))$ also converges towards $U^r(\overline{\theta}) = 0$ since then, we would have $e^r(\theta) \to 0$. This would mean a contradiction with Condition (5.10). Hence, necessarily $U^r(\overline{\theta}) > 0$.

- Using the boundary condition (A.3) at $\overline{\theta}$ and the fact that $q^r(\overline{\theta}) = q^{fb}(\overline{\theta})$, we rewrite Condition (A.5) as (5.7).

Q.E.D.

Proof of Proposition 5: We first rewrite (5.10) as

$$1 - F(\hat{\theta}^r) = \int_{\hat{\theta}^r}^{\overline{\theta}} \rho(B, E(U^r(\theta)))f(\theta)d\theta. \tag{A.8}$$

Observe that U^r is non-increasing from (3.6) and thus $\rho(B, E(U^r(\theta)))$ is non-decreasing since $\frac{\partial \rho}{\partial E} \leq 0$ for $U^r(\theta) \leq \mathcal{R}(e^{fb})$ when $\theta \geq \hat{\theta}^r$. Moreover, $\rho\left(B, e^{mh}\right) = 1$. Hence, (A.8) implies that there must exist $\theta^{mh} \in (\hat{\theta}^r, \overline{\theta})$ such that

$$U^r(\theta) \geq \mathcal{R}(e^{mh}) \Leftrightarrow \theta \leq \theta^{mh}. \tag{A.9}$$

On $[\hat{\theta}^r, \overline{\theta}]$, $z^r(\theta) = 0$ and thus $U^r(\theta) = \mathcal{R}(e^r(\theta))$. Inserting into (A.9) yields (5.14). *Q.E.D.*

Proof of Proposition 6: Let define

$$h(\theta) = \theta + \frac{\lambda^r(\theta)}{f(\theta)}. \tag{A.10}$$

Equipped with this notation and using (5.1), we may rewrite (3.6) as

$$\dot{U}(\theta) = -D(h(\theta)). \tag{A.11}$$

Hence, U^r is convex whenever $h(\theta)$ is non-decreasing.

Differentiating (A.10) and using (A.2), we obtain

$$\dot{h}(\theta) = 2 - \rho(B, E(U^r(\theta))) - (h(\theta) - \theta)\frac{\dot{f}(\theta)}{f(\theta)}. \tag{A.12}$$

Because of (A.7), we have $h(\theta) \geq \theta$ and the second term on the right-hand side of (A.12) is positive when $\dot{f} \leq 0$. Because $\rho(B, E(U^r(\theta)))$ is non-decreasing and $U^r(\theta)$ is non-increasing, the first term on the right-hand side of (A.12) is positive when

$$\rho(B, E(U^r(\overline{\theta}))) \leq 2 \tag{A.13}$$

which amounts to (5.15), and (5.16) comes from $e^i = E(U^i)$ when (A.13) is an equality. *Q.E.D.*

PROOF OF PROPOSITION 7: With our functional forms, (A.12) simplifies as

$$\dot{h}(\theta) = \omega(U^r(\theta))$$

where $\omega(U) = \min\left\{2, 3 - \frac{B}{\sqrt{2U}}\right\}$. Notice that ω is non-decreasing and concave, and satisfies $\omega(U^i) = 0$ and $\omega(U) = 2$ for $U \geq \mathcal{R}(e^{fb})$. For future reference, let $\Omega(U) = b\int_0^U \min\left\{2, 3 - \frac{B}{\sqrt{2\tilde{U}}}\right\} d\tilde{U}$ be a primitive for ω.

The autonomous system of ordinary differential equations (A.11) and (A.12), together with the boundary conditions (which follow from expressing (A.3) in terms of h)

$$h(\underline{\theta}) = \underline{\theta} \text{ and } h(\overline{\theta}) = \overline{\theta} \tag{A.14}$$

define the pair $(U^r(\theta), h(\theta))$.

Differentiating (A.11) with respect to θ, using $D' = -b$, this system can be transformed into a second-order differential equation in U^r as

$$\ddot{U}^r(\theta) = b\,\omega(U^r(\theta)) \tag{A.15}$$

together with the boundary conditions

$$\dot{U}^r(\underline{\theta}) = -D(\underline{\theta}) \text{ and } \dot{U}^r(\overline{\theta}) = -D(\overline{\theta}). \tag{A.16}$$

Multiplying both sides of (A.15) by $\dot{U}^r(\theta)$ and integrating yields a first quadrature

$$\frac{(\dot{U}^r(\theta))^2}{2} - \frac{(\dot{U}^r(\overline{\theta}))^2}{2} = \Omega(U^r(\theta)) - \Omega(U^r(\overline{\theta})).$$

Taking into account the boundary condition (A.16) at $\overline{\theta}$ and taking the negative (because U^r is non-increasing) root of (A.15) viewed as a second-degree equation in $\dot{U}^r(\theta)$ yields

$$\dot{U}^r(\theta) = -\sqrt{D^2(\overline{\theta}) + 2(\Omega(U^r(\theta)) - \Omega(U^r(\overline{\theta})))}. \tag{A.17}$$

Using the boundary condition (A.16) at $\underline{\theta}$ yields a condition that implicitly defines $U^r(\underline{\theta})$ in terms of $U^r(\overline{\theta})$, namely

$$D(\underline{\theta}) = \sqrt{D^2(\overline{\theta}) + 2(\Omega(U^r(\underline{\theta})) - \Omega(U^r(\overline{\theta})))}. \tag{A.18}$$

Let denote by $U^r(\underline{\theta}) = \underline{\mathcal{U}}(U^r(\overline{\theta}))$ this relationship. Observe that

$$\frac{d\underline{\mathcal{U}}(U^r(\overline{\theta}))}{dU^r(\overline{\theta})} = \frac{\omega(U^r(\overline{\theta}))}{\omega(U^r(\underline{\theta}))} \leq 1 \tag{A.19}$$

because $U^r(\overline{\theta}) < U^r(\underline{\theta})$ and ω is non-decreasing.

We can further integrate the differential equation (A.17) to get $U^r(\theta)$ as an implicit function that solves

$$\int_{U^r(\overline{\theta})}^{U^r(\theta)} \frac{dU}{\sqrt{D^2(\overline{\theta}) + 2(\Omega(U) - \Omega(U^r(\overline{\theta})))}} = \overline{\theta} - \theta.$$

In particular, we have

$$\int_{U^r(\overline{\theta})}^{\underline{\mathcal{U}}(U^r(\overline{\theta}))} \frac{dU}{\sqrt{D^2(\overline{\theta}) + 2(\Omega(U) - \Omega(U^r(\overline{\theta})))}} = 1. \tag{A.20}$$

Using the definition of Γ in (5.18), we can use (A.18) to compute

$$\Gamma'(U^r(\overline{\theta})) = \frac{\omega(U^r(\overline{\theta}))}{D(\underline{\theta})\omega(U^r(\underline{\theta}))} - \frac{1}{D(\overline{\theta})} < 0$$

where the inequality follows from (A.19) and $D(\underline{\theta}) > D(\overline{\theta})$. Hence, (A.20), that writes as

$$\Gamma(U^r(\overline{\theta})) = 1,$$

has at most one solution $U^r(\overline{\theta})$ and this solution is greater than U^i when (5.17) holds In that case, Proposition 6 applies and U^r is convex. *Q.E.D.*

Proofs of Propositions 8 and 10: Consider now the maximization problem $(\mathcal{P})$ where the buyer's expected payoff is maximized over the set $\mathcal{A}$ of incentive-feasible allocations. The Hamiltonian $\mathcal{H}(U, q, z, \lambda^m, \theta)$ for this optimization problem writes again as in (A.1). Yet, a perturbation is now admissible only when U is convex, i.e., when q also satisfies (3.7). Let us now $(U^m(\theta), q^m(\theta), z^m(\theta), e^m(\theta))$ denote an optimal arc.

Fixing the costate variable λ^m that will be soon defined, and following steps similar to those in the Proof of Propositions 3, we now construct the (partially) maximized Hamiltonian (with respect to z only) as

$$\mathcal{H}^{**}(U, q, \lambda^m(\theta), \theta) = \max_{z \geq 0} \mathcal{H}(U, q, z, \lambda^m(\theta), \theta).$$

We may compute

$$\mathcal{H}^{**}(U, q, \lambda^m(\theta), \theta) = (S(q) - \theta q - K + \zeta(U))f(\theta) - \lambda^m(\theta)q.$$

From this, it follows that the sub-gradient of H^{**} in U exists everywhere and, in fact,

$$\partial_U \mathcal{H}^{**}(U^m(\theta), q^m(\theta), \lambda^m(\theta), \theta) = f(\theta)\zeta'(U^m(\theta)) = f(\theta)\left(\rho(B, E(U^m(\theta), z^{**}(U^m(\theta)) - 1))\right).$$

We want to prove the optimality of $(U^m(\theta), e^m(\theta), q^m(\theta), z^m(\theta))$ against any admissible arc $(U(\theta), e(\theta), q(\theta), z(\theta))$ where q satisfies (3.7), U satisfies (3.6) almost everywhere, $e(\theta) = E(U(\theta) - z(\theta))$, and $z(\theta) \geq 0$. To this end, we first define

$$\Phi(U, q, z, \theta) = \mathcal{H}(U, q, z, \lambda^m(\theta), \theta) + \lambda^m(\theta)q \tag{A.21}$$

where the costate variable $\lambda^m(\theta)$ (that will be soon defined) is, at this stage, given.

Second, we form the difference

$$\Delta^m = \int_{\underline{\theta}}^{\overline{\theta}} \left(\Phi(U^m(\theta), q^m(\theta), z^m(\theta), \theta) - \Phi(U(\theta), q(\theta), z(\theta), \theta)\right) d\theta.$$

Using (A.21) yields

$$\Delta^m = \int_{\underline{\theta}}^{\overline{\theta}} \left(\mathcal{H}(U^m(\theta), q^m(\theta), z^m(\theta), \lambda^m(\theta), \theta) - \mathcal{H}(U(\theta), q(\theta), z(\theta), \lambda^m(\theta), \theta)\right) d\theta$$

$$+ \int_{\underline{\theta}}^{\overline{\theta}} \lambda^m(\theta)(q^m(\theta) - q(\theta))d\theta.$$

By definition of $\mathcal{H}^{**}$, we have

$$\mathcal{H}(U^m(\theta), q^m(\theta), z^m(\theta), \lambda^m(\theta), \theta) = \mathcal{H}^{**}(U^m(\theta), q^m(\theta), \lambda^m(\theta), \theta)$$

and

$$\mathcal{H}(U(\theta), q(\theta), z(\theta), \lambda^m(\theta), \theta) \leq \mathcal{H}^{**}(U(\theta), q(\theta), \lambda^m(\theta), \theta).$$

Henceforth, we obtain

$$\text{(A.22)} \qquad \Delta^m \geq \int_{\underline{\theta}}^{\overline{\theta}} \left(\mathcal{H}^{**}(U^m(\theta), q^m(\theta), \lambda^m(\theta), \theta) - \mathcal{H}^{**}(U(\theta), q(\theta), \lambda^m(\theta), \theta)\right) d\theta$$
$$+ \int_{\underline{\theta}}^{\overline{\theta}} \lambda^m(\theta)(q^m(\theta) - q(\theta)) d\theta.$$

To find a lower bound for the right-hand side, several steps are needed.

Convexification. Let us first define Z as

$$\text{(A.23)} \qquad Z(F(\theta)) = \int_{\underline{\theta}}^{\theta} \left(\tilde{\theta} f(\tilde{\theta}) + \lambda^m(\tilde{\theta})\right) d\tilde{\theta}.$$

with

$$h(\theta) = \dot{Z}(F(\theta)) = \theta + \frac{\lambda^m(\theta)}{f(\theta)}.$$

Let the convexification of Z be now denoted as

$$\overline{Z}(y) = \text{co}(Z(y)) \quad \forall y \in [0,1].$$

By definition, we have

$$Z(y) \geq \overline{Z}(y) \quad \forall y \in [0,1]$$

with

$$\text{(A.24)} \qquad Z(0) = \overline{Z}(0) = 0 \text{ and } Z(1) = \overline{Z}(1).$$

Because λ^m (to be defined in (A.25) and (A.26) below) is continuously differentiable, Z is also twice-continuously differentiable and its convexification $\overline{Z}$ inherits this property. Define accordingly the corresponding derivative of $\overline{Z}$ as

$$\overline{h}(\theta) = \dot{\overline{Z}}(F(\theta)).$$

Observe that, whenever $\overline{Z}(y) < Z(y)$ over an interval $\mathcal{I} \subseteq [0,1]$, we necessarily have $\dot{\overline{Z}}(y)$ constant on that interval. Hence, $\overline{h}(\theta)$ remains constant over $F^{-1}(\mathcal{I})$. Moreover, we notice that $\overline{Z}$ convex implies that $\dot{\overline{Z}}$ is non-decreasing. Hence, $\dot{h}(\theta) = f(\theta)\ddot{Z}(F(\theta)) \geq 0$ as requested.

Sufficient Conditions. We define an arc $(U^m(\theta), q^m(\theta), z^m(\theta))$ together with a costate variable $\lambda^m(\theta)$ by means of the following conditions.

- *Optimal quantity.* $q^m(\theta)$ satisfies (5.20). Because $\overline{h}$ defined in (5.21) is non-decreasing, q^m is also non-decreasing as requested.

- *Information rent.* The optimal quantity (5.20) induces a absolutely continuous profile of an information rent U^m given by

$$U^m(\theta) = U^m(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} q^m(\tilde{\theta}) d\tilde{\theta}$$

  where $U^m(\overline{\theta})$ is implicitly defined by the condition

$$1 = \int_{\underline{\theta}}^{\overline{\theta}} \rho\left(B, \min\left\{E\left(U^m(\theta)\right); e^{fb}\right\}\right) f(\theta) d\theta,$$

- *Costate variable and transversality conditions.* Let λ^m be the solution to

$$(A.25)\qquad -\dot{\lambda}^m(\theta) = f(\theta)\zeta'(U^m(\theta)) = f(\theta)\left(\rho\left(B, \min\left\{E(U^m(\theta)); e^{fb}\right\}\right) - 1\right)$$

together with the transversality conditions

$$(A.26)\qquad \lambda^m(\underline{\theta}) = \lambda^m(\overline{\theta}) = 0.$$

It is straightforward to check that λ^m is continuously differentiable since U^m is itself continuous. Integrating (A.25) and using the second transversality condition in (A.26) yields (5.22). That λ^m remains non-negative (right-hand side inequality in (5.22)) follows from observing that λ^m is quasi-concave and satisfies (A.26) (by an argument similar to that made in the PROOF OF PROPOSITION 3).

- *Slack.* Using (A.6) yields

$$z^m(\theta) = z^{**}(U^m(\theta)) = \max\left\{U^m(\theta) - \mathcal{R}(e^{fb}), 0\right\}.$$

- *Effort.* We have

$$(A.27)\qquad e^m(\theta) = \min\left\{e^{fb}; E(U^m(\theta))\right\}.$$

- *End-point condition at* $\overline{\theta}$. All types are asked to produce whenever the free-end point condition $\mathcal{H}(U^m(\overline{\theta}), q^m(\overline{\theta}), z^m(\overline{\theta}), \lambda(\overline{\theta}), \overline{\theta}) \geq 0$ holds. We rewrite this condition as

$$\left(S(q^m(\overline{\theta})) - \overline{\theta}q^m(\overline{\theta}) - K + Be^m(\overline{\theta}) - \psi(e^m(\overline{\theta})) - \mathcal{R}(e^m(\overline{\theta}))\right) f(\overline{\theta}) - \lambda^m(\overline{\theta})q^m(\overline{\theta}) \geq 0.$$

Taking into account the second transversality condition in (A.26) finally yields

$$S(q^m(\overline{\theta})) - \overline{\theta}q^m(\overline{\theta}) - K + Be^m(\overline{\theta}) - \psi(e^m(\overline{\theta})) - \mathcal{R}(e^m(\overline{\theta})) \geq 0.$$

As we will check below, those conditions turned out to be sufficient for optimality of the arc $(U^m(\theta), q^m(\theta), z^m(\theta), e^m(\theta))$.

Equipped with our previous notations, we now rewrite the maximized Hamiltonian as

$$\mathcal{H}^{**}(U(\theta), q(\theta), \lambda^m(\theta), \theta) = (S(q(\theta)) - \overline{h}(\theta)q(\theta))f(\theta) + (\overline{h}(\theta) - h(\theta))f(\theta)q(\theta) + f(\theta)\zeta(U(\theta)).$$

Accordingly, we form

$$\mathcal{H}^{**}(U^m(\theta), q^m(\theta), \lambda^m(\theta), \theta) - \mathcal{H}^{**}(U(\theta), q(\theta), \lambda^m(\theta), \theta) = \left[(S(q) - \overline{h}(\theta)q)f(\theta)\right]_{q(\theta)}^{q^m(\theta)}$$

$$+ \left[(\overline{h}(\theta) - h(\theta))f(\theta)q\right]_{q(\theta)}^{q^m(\theta)} + \left[f(\theta)\zeta(U)\right]_{U(\theta)}^{U^m(\theta)}.$$

Integrating over Θ yields

$$(A.28)\qquad \int_{\underline{\theta}}^{\overline{\theta}} \left(\mathcal{H}^{**}(U^m(\theta), q^m(\theta), \lambda^m(\theta), \theta) - \mathcal{H}^{**}(U(\theta), q(\theta), \lambda^m(\theta), \theta)\right) d\theta$$

$$(A.29)\qquad = \int_{\underline{\theta}}^{\overline{\theta}} \left[(S(q) - \overline{h}(\theta)q)f(\theta)\right]_{q(\theta)}^{q^m(\theta)} d\theta$$

$$+\int_{\underline{\theta}}^{\overline{\theta}} \left[(\overline{h}(\theta)-h(\theta))f(\theta)q\right]_{q(\theta)}^{q^m(\theta)} d\theta \tag{A.30}$$

$$+\int_{\underline{\theta}}^{\overline{\theta}} [f(\theta)\zeta(U)]_{U(\theta)}^{U^m(\theta)} d\theta. \tag{A.31}$$

We know evaluate the sign of each term (A.29), (A.30) and (A.31).

- *Fact 1.* Because $S(q)-\overline{h}(\theta)q$ is strictly concave and $q^m(\theta)$ maximizes this expression, we have

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[(S(q)-\overline{h}(\theta)q)f(\theta)\right]_{q(\theta)}^{q^m(\theta)} d\theta \geq 0,$$

  i.e., the term on the right-hand side of (A.29) is non-negative

- *Fact 2.* We also compute

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[(\overline{h}(\theta)-h(\theta))f(\theta)q\right]_{q(\theta)}^{q^m(\theta)} d\theta = \int_{\underline{\theta}}^{\overline{\theta}} (\dot{\overline{Z}}(F(\theta))-\dot{Z}(F(\theta)))(q^m(\theta)-q(\theta))f(\theta)d\theta$$

$$= \left[(\overline{Z}(F(\theta))-Z(F(\theta)))(q^m(\theta)-q(\theta))\right]_{\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} (\overline{Z}(F(\theta))-Z(F(\theta)))(dq^m(\theta)-dq(\theta)) \tag{A.32}$$

  where the equality comes from integrating by parts and where dq^m and dq are negative measures since both q^m and q satisfy (3.7).

  Using (A.24), we observe that the first bracketed term on the right-hand side of (A.32) is zero.
  Turning now to the second term on the right-hand side of (A.32), we notice that it is zero whenever $Z(F(\theta)) = \overline{Z}(F(\theta))$. Denote $\mathcal{I} = \left\{F(\theta) \text{ s.t.} Z(F(\theta)) > \overline{Z}(F(\theta))\right\}$. From a remark above, we necessarily have $\dot{\overline{Z}}(y)$ constant on that interval; which means that $\overline{h}(\theta)$ remains constant over $F^{-1}(\mathcal{I})$. It then follows from (5.20) that q^m is constant on such interval and thus

$$\int_{\underline{\theta}}^{\overline{\theta}} (\overline{Z}(F(\theta))-Z(F(\theta)))(dq^m(\theta)-dq(\theta)) = -\int_{F^{-1}(\mathcal{I})} (\overline{Z}(F(\theta))-Z(F(\theta)))dq(\theta) \leq 0$$

  for any admissible q satisfying (3.7). Hence,

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[(\overline{h}(\theta)-h(\theta))f(\theta)q\right]_{q(\theta)}^{q^m(\theta)} d\theta \geq 0.$$

- *Fact 3.* Because ζ is concave, we also have

$$\zeta(U(\theta))-\zeta(U^m(\theta)) \leq \zeta'(U^m(\theta))(U(\theta)-U^m(\theta)) = -\dot{\lambda}^m(\theta)(U(\theta)-U^m(\theta))$$

  where the last equality uses (A.25).

  Hence, we get

$$\int_{\underline{\theta}}^{\overline{\theta}} [f(\theta)\zeta(U)]_{U(\theta)}^{U^m(\theta)} d\theta + \int_{\underline{\theta}}^{\overline{\theta}} \dot{\lambda}^m(\theta)(U^m(\theta)-U(\theta))d\theta \geq 0.$$

Gathering all *Facts* above and inserting into (A.28) yields

$$\int_{\underline{\theta}}^{\overline{\theta}} (\mathcal{H}^{**}(U^m(\theta), q^m(\theta), \lambda^m(\theta), \theta) - \mathcal{H}^{**}(U(\theta), q(\theta), \lambda^m(\theta), \theta))\, d\theta \tag{A.33}$$

$$\geq - \int_{\underline{\theta}}^{\overline{\theta}} \dot{\lambda}^m(\theta)(U^m(\theta) - U(\theta)))d\theta.$$

Integrating by parts and using the transversality conditions (A.26) the right-hand side of (A.33) can be written as

$$- \int_{\underline{\theta}}^{\overline{\theta}} \lambda^m(\theta)(q^m(\theta) - q(\theta))d\theta. \tag{A.34}$$

Gathering (A.33) and (A.34) and inserting into (A.22) finally yields

$$\Delta^m \geq 0$$

which proves that $(U^m(\theta), q^m(\theta), z^m(\theta), e^m(\theta))$ is an optimal arc.

IMPLICATIONS.

- Integrating (A.25) and using the second transversality condition in (A.26) yields (5.22).

- The seller's information rent $U^m(\theta)$ satisfies

  $$U^m(\theta) = U^m(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} q^m(\tilde{\theta})d\tilde{\theta}$$

  where $U^m(\overline{\theta})$ is implicitly defined by the condition

  $$1 = \int_{\underline{\theta}}^{\overline{\theta}} \rho\left(B, \min\left\{E(U^m(\tilde{\theta})); e^{fb}\right\}\right) f(\tilde{\theta})d\tilde{\theta}.$$

  which is obtained from (5.22) and the first transversality condition in (A.26).

  Following the same steps as in the PROOFS OF PROPOSITIONS 3 AND 4, we also obtain a condition similar to that found in the case of the solution to the relaxed problem

  $$U^m(\overline{\theta}) < \mathcal{R}(e^{fb}).$$

- Finally and again mimicking results obtained in the PROOFS OF PROPOSITION 4, the optimal effort satisfies (A.27) with $U^m(\hat{\theta}^m)$ being defined as in (5.33).

Q.E.D.

PROOF OF PROPOSITION 9: Using (A.10) in the case of a uniform distribution, the virtual cost can be here expressed as

$$h(\theta) = \theta + \lambda^m(\theta). \tag{A.35}$$

Differentiating (A.35) and using (A.2), we obtain

$$\dot{h}(\theta) = \omega(U^m(\theta)) \tag{A.36}$$

to which, we again append the boundary conditions (A.14). It is straightforward to check that $h(\theta)$ so defined is concave. When h is non-monotonic, the only scenario is thus for h to be first increasing before being decreasing in a left-neighborhood of $\overline{\theta}$.

Using (5.20), we may also rewrite (3.6) as

$$\dot{U}^m(\theta) = -D(\overline{h}(\theta)) = \begin{cases} -D(h(\theta)) & \text{for } \theta \in [\underline{\theta}, \theta_0], \\ -D(h(\theta_0)) & \text{for } \theta \in \left[\theta_0, \overline{\theta}\right]. \end{cases}$$

Hence, U^m is linear on $\left[\theta_0, \overline{\theta}\right]$ and thus (5.26) holds.

Using (A.36) and the boundary condition (A.14) at $\overline{\theta}$ thus yields

$$\text{(A.37)} \qquad q_0(\overline{\theta} - h(\theta)) = \Omega(U^m(\overline{\theta})) - \Omega(U^m(\overline{\theta}) + q_0(\overline{\theta} - \theta)) \quad \forall \theta \in \left[\theta_0, \overline{\theta}\right].$$

Inserting into (5.25) and simplifying yields (5.27). Inserting (A.37) into (5.20) yields (5.28).

Proceeding as in the Proof of Proposition 7, we form a first quadrature for U^m and get

$$\dot{U}^m(\theta) = -\sqrt{q_0^2 + 2(\Omega(U^m(\theta)) - \Omega(U^m(\overline{\theta}) + q_0(\overline{\theta} - \theta_0)))} \quad \forall \theta \in [\underline{\theta}, \theta_0].$$

Using the boundary condition (A.14) at $\underline{\theta}$, that we write as $\dot{U}^m(\underline{\theta}) = -D(\underline{\theta})$, we then obtain the following expression for $U^m(\underline{\theta})$ as (5.29).

Finally, a second quadrature yields the following (almost) closed form for $U^m(\theta)$

$$\text{(A.38)} \qquad \int_{U^m(\theta_0)}^{U^m(\theta)} \frac{dU}{\sqrt{q_0^2 + 2(\Omega(U) - \Omega(U^m(\overline{\theta}) + q_0(\overline{\theta} - \theta_0)))}} = \theta_0 - \theta \quad \forall \theta \in [\underline{\theta}, \theta_0]$$

and thus (5.30) holds. Altogether, the system (5.27)-(5.28)-(5.29)-(5.30) determines $(U^m(\overline{\theta}), U^m(\underline{\theta}), q_0, \theta_0)$. *Q.E.D.*

FIGURES

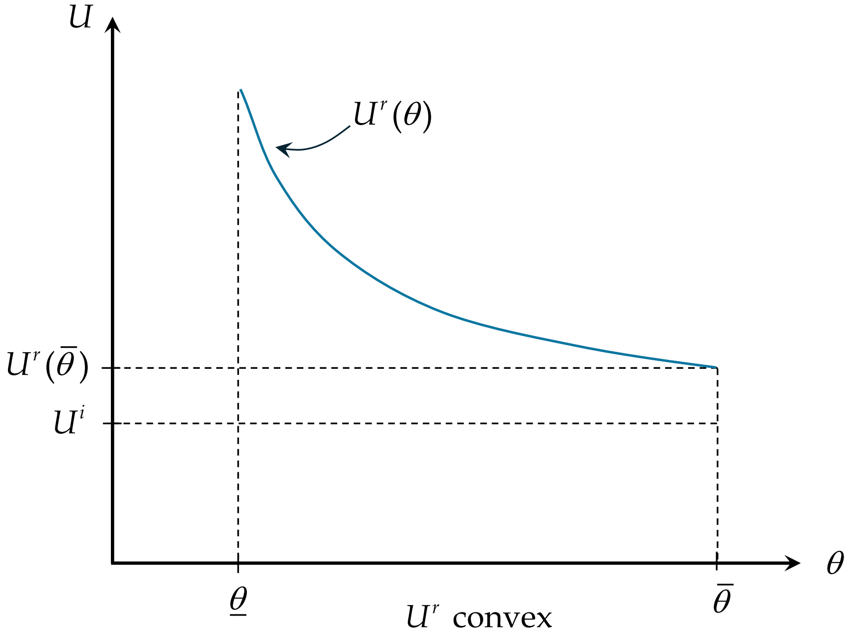

FIGURE 1A.— Running example: Case $U^r(\bar{\theta}) > U^i$

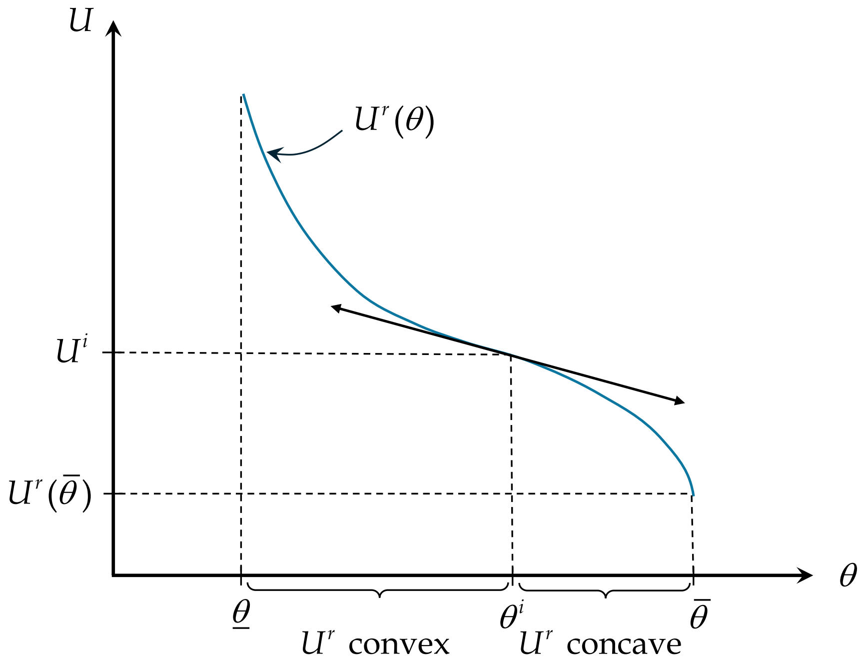

FIGURE 1B.— Running example: Case $U^r(\bar{\theta}) < U^i$

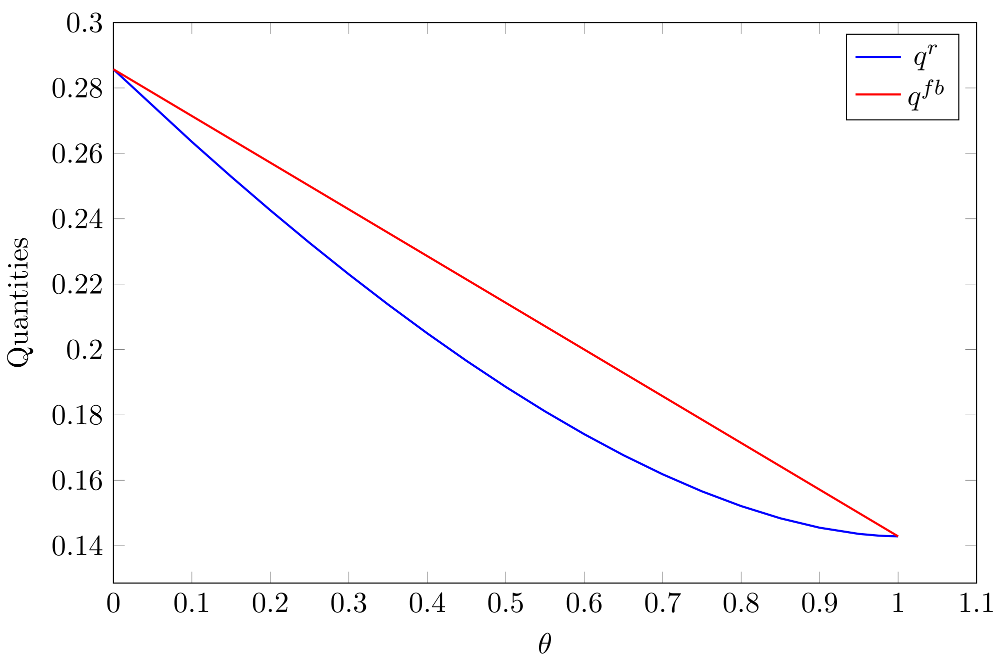

FIGURE 1C.— Illustration of quantity profile

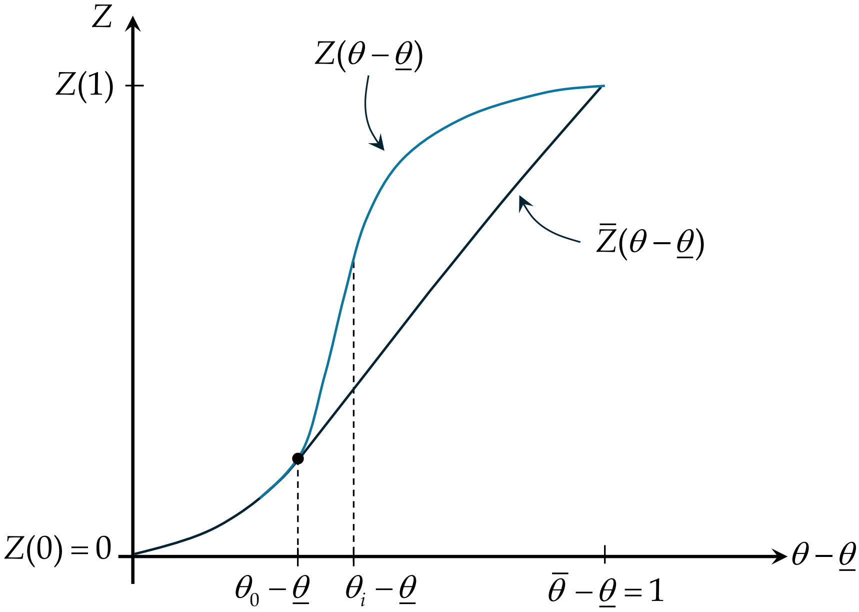

FIGURE 2A.— Profiles for Z and $\bar{Z} = \text{co}(Z)$

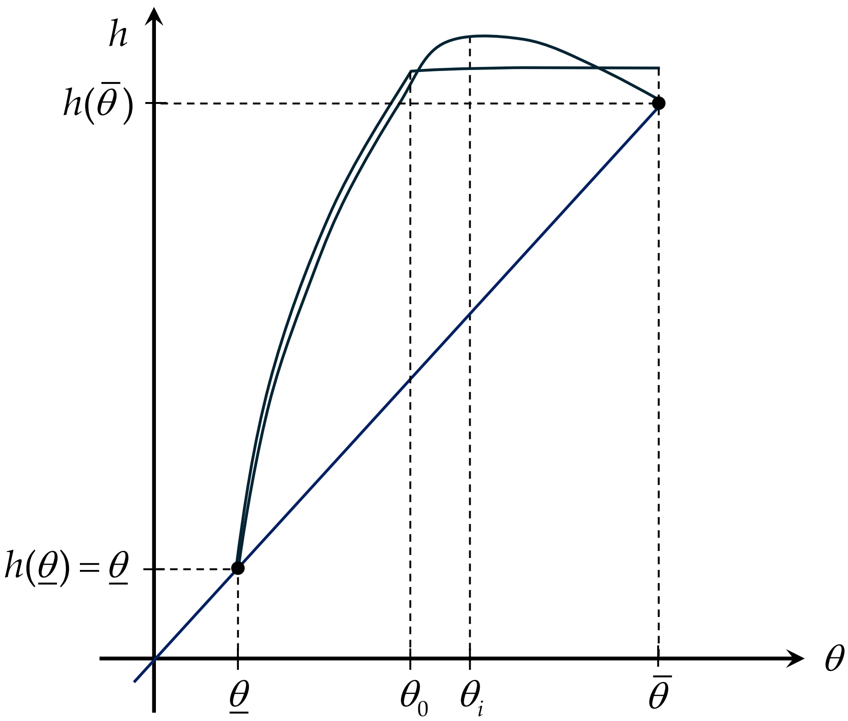

FIGURE 2B.— Profiles for $h = \dot{Z}$ and $\bar{h} = \dot{\bar{Z}}$

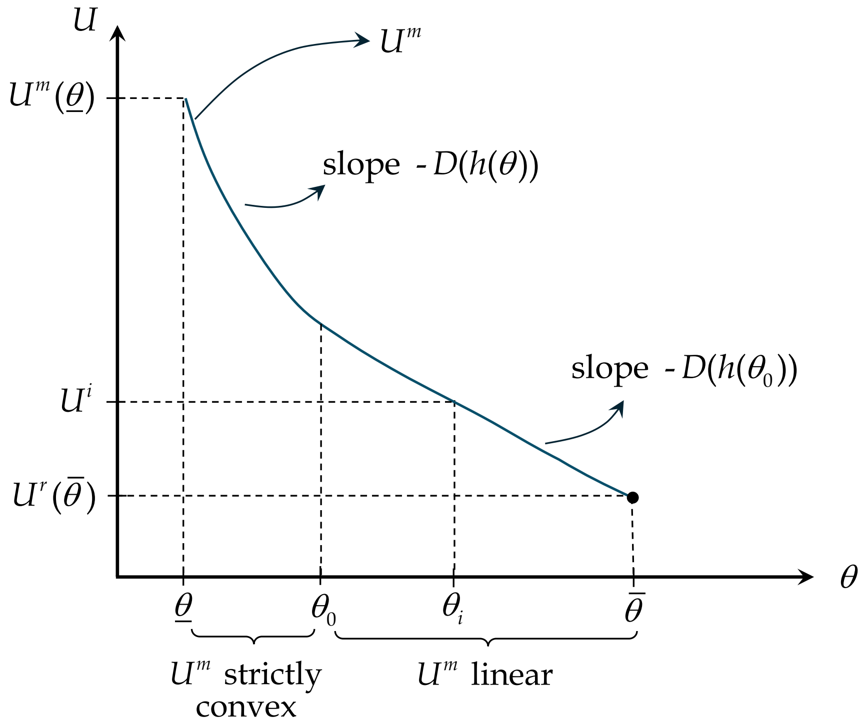

FIGURE 2C.— Ironing: Rents

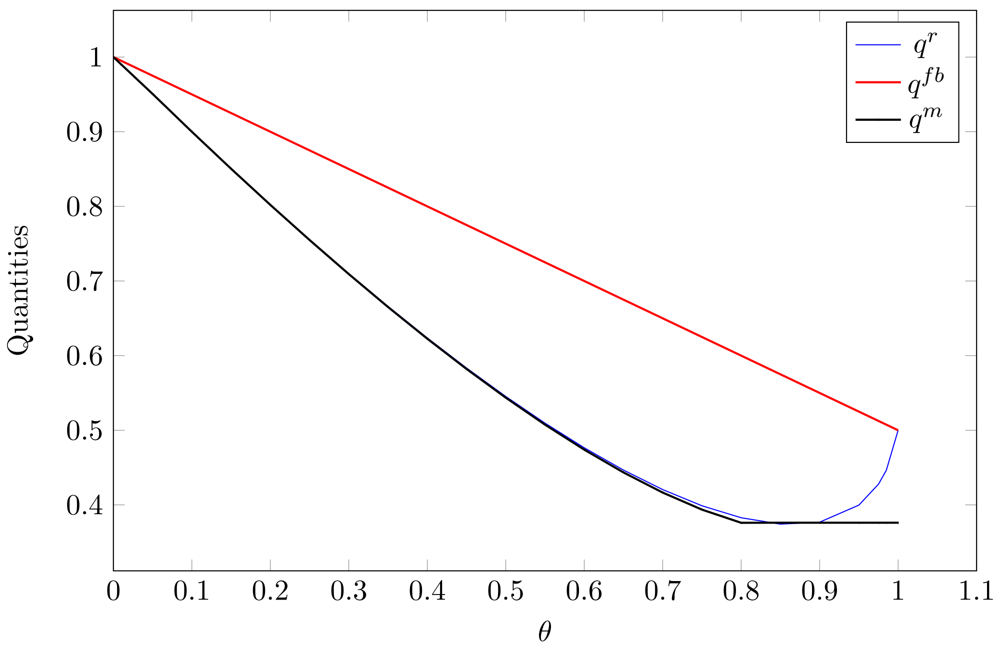

FIGURE 3A.— Quantity profiles: Simulation A

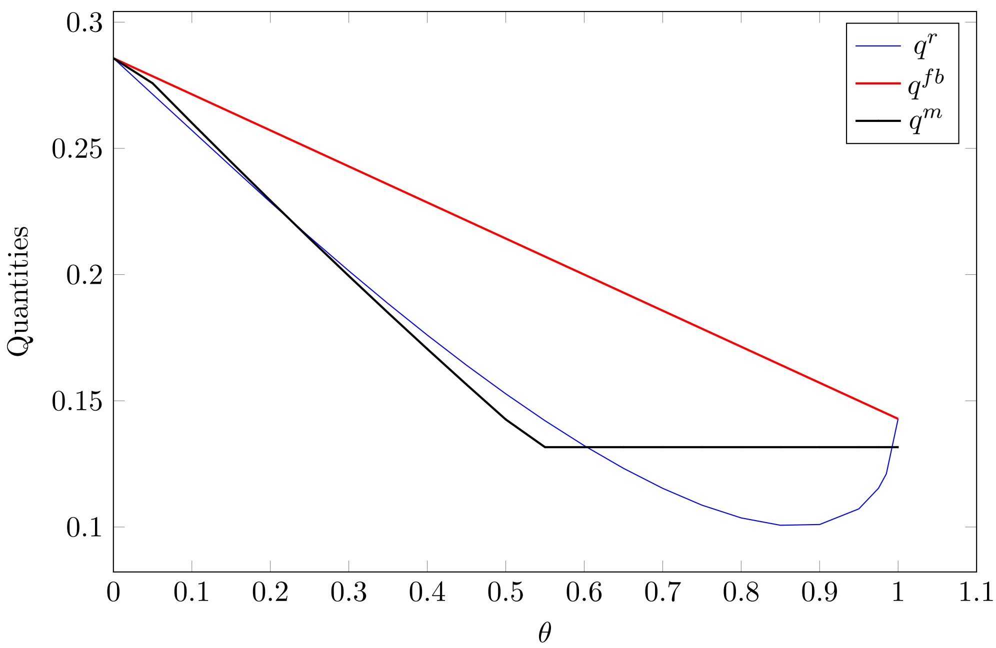

FIGURE 3B.— Quantity profiles: Simulation B