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Abstract

There are many business situations in which investments by a supplier and a producer ("coinvestments") are both necessary for either of them to grasp a business opportunity. For instance, better quality tanks are needed to manufacture reliable hydrogen-powered vehicles. One of these two firms, typically the one facing a lower cost, may be more willing to invest, but the cautionary attitude of the other delays the coinvestment. We model supply-chain interactions in a classical tractable way to derive the firms' net present values (NPVs) upon coinvestment and determine their Nash equilibrium investment (timing) strategies. Firms coinvest when the real options of the weaker firm is 'deep in the money.' These business situations are likely to be affected by evolving market circumstances, in particular due to changes in the demand dynamics or endogenous decision (by, say, the supplier) to conduct research and development (R&D). We investigate related model extensions, which confirm the robustness of our key result.

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1 Introduction

The need to reduce carbon emissions to combat climate change is now well accepted. Today, the transportation sector accounts for around one fifth of global carbon dioxide (CO_2) emissions (see Pales et al., 2021). Of these emissions, road travel accounts for about three quarters (45.1% for daily commutes and 29.4% for truck transportation). Alternatives to Internal Combustion Engines (ICEs) powered by fossil fuels have been proposed to reduce carbon emissions, some of them with decent economic prospects. A standard alternative are electric vehicles (EVs), but these tend to be heavy and less appropriate for larger vehicles as the charging time becomes prohibitive. Hydrogen-powered vehicles (HPVs) are promising because hydrogen combustion does not generate CO_2 but water vapor (Sammel, 2006). This technology also offers a high energy density by mass (119.7 MJ/kg vs 44.8 MJ/kg for gasoline according to Yip et al., 2019), a property that gives an edge to HPVs (compared to EVs) for applications in which weight plays a crucial role, for example, in aviation or other long-distance transportation (Koffler and Rohde-Brandenburger, 2010). Another advantage of HPVs is the refueling time, of a couple of minutes versus hours for EVs (Kostopoulos et al., 2020). On the supply side, hydrogen can be produced from various renewables (Ajanovic et al., 2022), allowing a transition to green fuels (Sharma et al., 2023). Given these advantages, the European Commission considers hydrogen a viable option for next-generation vehicles and has announced that, by 2030, refueling stations will be available every 200 km in Europe (Vălean, 2023).

Unfortunately, the economic equation for HPVs requires overcoming two main technical hurdles:

1. Improving *energy efficiency* during all stages from hydrogen generation to storage and use. About 27% of the energy (expressed in watts) is wasted during electrolysis, while 38% is lost during conversion to fuel cells. Energy waste (6%) is also an issue during high pressure compression in 350-700 bar cylinders. Eventually, only 29% of the initial energy can be used for vehicle propulsion (Pesonen and Alakunnas, 2017).

2. Developing safe *storage* technologies. Due to its diffusivity, hydrogen passes through most conventional sealing materials used for tanks, making a leakage of highly inflammable gas plausible (Berry et al., 1996). New materials and container designs are needed (e.g., storage in a solid state or at lower pressure).

These challenges testify to the need to form "smart coalitions for sustainable mobility," with massive investments required throughout the supply chain ("coinvestments"), not only at one echelon. In particular, suppliers and Original Equipment Manufacturers (OEMs) have traditionally partnered in the automotive industry. In fact, companies often lack internal resources to take over certain upstream or downstream activities, making them at the mercy of other parties throughout the supply chain. This interdependence has been central to the supply chain literature, e.g., in works discussing supply chain coordination (e.g., Li and Kouvelis, 1999; Cachon, 2003; Kouvelis and Zhao, 2015) or disruption (see, e.g., Tomlin, 2006; Swinney and Netessine, 2009; Federgruen and Yang, 2009; Gao et al., 2019). Motivated by the above problem, we develop in this paper a stylized model to study investment decisions across a supply chain with one producer and one supplier, which are all necessary to set up and sell a product using a novel technology. These situations, which we call "coinvestment games," differ from other situations considered in the literature on noncooperative real options games (see Chevalier-Roignant and Trigeorgis, 2012; Chevalier-Roignant et al., 2011; Balter et al., 2022, for surveys). Indeed, this literature focuses on determining the equilibrium times for games involving two firms in horizontal competition which can gain an advantage from moving first or second. In contrast, we focus on vertical supply-chain interactions with the need for an upstream firm ("supplier") and a downstream firm ("producer") to coinvest. First, we leverage a classical *simple* model to characterize the terms of the relationship between the two firms, determining their profits and net present values (NPVs) upon coinvestment. We then determine the firms' equilibrium coinvestment (timing) decisions, which hinge on these NPVs and the firms' respective investment costs. A supply chain that fails to coordinate investment decisions underperforms a vertically integrated firm that decides on the product price and the investment time. One echelon may have an inventive to subsidize the other echelon, to alleviate the negative effect of a lack of coordination on its own value. Evolving market conditions are likely to impact these business scenarios, especially when there are shifts in demand dynamics or when a firm makes a follow-up decision that affects the other firm. As extensions, we allow for the possibility of a paradigm shift at an exponentially distributed time and for an endogenous decision by the supplier to invest in R&D to reduce production costs, and consider how these changes affect the equilibrium decisions.

Our stylized model helps us derive key economic insights:

1. The interests of the firms may not be aligned across the supply chain, for instance, because the firms face differential investment costs. This misalignment may make one of them more eager to invest.

2. Neither firm has an incentive to invest before the other because, if it does so, it incurs costs earlier, but receives benefits later. Investing before the other is thus suboptimal because of the effect of the time value of money. The coinvestment game thus has the flavor of a game of attrition, exhibiting multiple Nash equilibria. For all of them, the firms invest at the same time, if the firms' real options, considered in isolation of strategic interactions ("myopic"), are both "deep in the money." Among these equilibria, the firms are collectively better off to coinvest as soon as both real options are "deep in the money."

3. A supply chain that does not coordinate its decisions is inefficient, in the sense that a vertically integrated firm achieves a profit larger than the sum of the equilibrium profits of a supplier and a producer in a noncoordinated relationship. Consequently, a vertically integrated firm invests earlier than the supply-chain parties would in equilibrium. Such integration is not always realistic. The echelon which extracts is more eager to invest has an incentive to subsidize the other echelon.

Yet, such a mechanism may be difficult to contract or is unlawful.

4. Allowing for the possibility of a paradigm shift (modeled as a change in the growth rate of demand) does not affect the structure of the solution significantly. Firms are still waiting before coinvesting for the weaker firm to be willing to pursue the investment. In this case, the firms will invest earlier if they experience stronger demand growth after the paradigm shift compared to the benchmark with a constant growth rate.

5. We model the possibility for the supplier to reduce the variable production costs following successful R&D activities. The supplier's (myopic) problem is framed as a compound option problem. The producer in a sense freerides as it will benefit from this improvement while not paying for it, thanks to more attractive terms in the input market. The benefit from an endogenous reduction in production costs makes the option values of the (myopic) firms more valuable. Again, the coinvestment takes place if the (compound) option of the weakest of the two supply chain parties is deep in the money.

2 Related concepts and examples

Again, our paper relates to the literature on options games reviewed in Chevalier-Roignant and Trigeorgis (2012), Chevalier-Roignant et al. (2011) and Balter et al. (2022). We stress several key differences. First, most papers in this stream consider cases in which an invested firm receives a payoff up until the other firm has invested. This, for instance, is the case in preemption games (Riedel and Steg, 2017; Décamps et al., 2022). In very few works, it is necessary for several agents to make a timing decision for all of them to receive a payoff. Exceptions include Morellec and Zhdanov (2005) and Hackbarth and Miao (2012), in which a bidder and a seller must agree on the timing of a M&A deal for it to materialize. Second, Chevalier-Roignant et al. (2011, Section 3.2) considered the lack of consideration to vertical interactions across the supply chain as a significant gap in the literature. A focus of our research is to derive the firms' equilibrium investment (timing) decisions after we specified the terms of a relationship between a supplier and a producer. Trigeorgis and Tsekrekos (2018) review recent real options works dealing with issues related to supply chain and logistics (Theme D), some of which include strategic interactions (see Section 3.4.2). Of these, Moon et al. (2011) discuss a problem of supply-chain interactions, but the focus is on the bilateral negotiation in a supply contract where the buyer's revenue and the seller's cost are uncertain, while our focus is on the timings of large upfront investments by both firms. Billette De Villemeur et al. (2014) consider the impact of vertical relationships on investment decisions, with no explicit mention of the underlying market model driving the reduced-form buyer's profit function. The downstream firm decides on the timing of an investment requiring a productive equipment. The upstream firm infers the demand of the downstream firm and decides on the price of that productive equipment by solving a *static* optimization problem.[1] In contrast, we consider the effect of vertical relationships on (i) the terms of the trade (input price, quantities, and profits) if both firms are invested and (ii) the firms' decisions about the times at which to co-invest as solution to a dynamic *timing game*. The upstream firm is not the supplier of an equipment, but the supplier of more classical input sold at a markup.

Third, contrary to its first appearance, our game of investment timing has features reminiscent of wars of attrition/chicken games, a subject less researched in the literature on noncooperative real games with the exception of Hoppe (2000), Murto (2004), Steg and Thijssen (2015), and Décamps et al. (2022). Indeed, as neither firm wants to incur a fixed investment cost before being able to reap the benefits of its investment, i.e., before the other firm has invested as well, there effectively is a second-mover advantage, which drives many of the results.

[1] Billette De Villemeur et al. (2014) observe in Section 2 that the upstream firm charges the equipment at a markup, which leads to a delay on the investment (due to double marginalization) compared to an integrated benchmark. Section 3 explore alternative contractual arrangements for the supply of the productive asset, while Section 4 considers the impact of downstream (duopolistic) competition on the upstream pricing decision.

Finally, another noted difference is that our game involves more than one source of uncertainty, considering not only a stochastic demand factor, but also uncertainty with respect to the arrivals of certain events, in particular of a paradigm shift and of a technological breakthrough by the supplier. Similar techniques have been used to complement the literature on real options by Farzin et al. (1998) and Ye et al. (2024) but in a different context.

3 Supply chain model

In the context set by our motivating example, consider as a producer the manufacturer of HPVs and as a supplier the manufacturer of hydrogen tanks. If both firms coinvested, we assume for tractability that the producer wields monopoly power, while the supplier and producer are in a monopsonic relationship. We adopt a classical model setup with linear tariffs (see, for instance, Spengler, 1950 or pp. 174-175 in Tirole, 1988). Specifically, we make the following assumptions about the product market.

End demand function. End customers have a demand for the end product/HPVs characterized by an inverse demand function of the form

$$P(x,q) := xq^{-\delta}, \quad \delta \in (0,1), \tag{3.1}$$

where x denotes the realization of an exogenous demand shock (observed by the producer and the supplier) and q denotes the producer's output/end consumer demand.[2] The demand shock captures changes in income levels or changes in the market's size. We note that this model specification suggests a constant price elasticity of demand, given by $\frac{dq/q}{dP/P} = -\frac{1}{\delta}$. Adopting constant passthrough specifications (which nest constant elasticity demand models) are increasingly com-

[2] The case with linear inverse demand function $q \mapsto x - q$ is studied by Tirole (1988, pp. 174-175). Yet, this case involves a constraint on the demand intercept x, which must exceed the supplier's marginal cost, κ in our setting. If this constraint is not satisfied, no firm produces. If one assumes the demand intercept follows a GBM, the equilibrium profits would have an explicit solution, but expressed piecewise. While the investment problems remain tractable, the computations become more cumbersome (see, e.g., Chevalier-Roignant et al., 2011, for a related problem).

mon in the literature (see Weyl and Fabinger, 2013).

Cost functions. The producer needs to buy q units of input (e.g., hydrogen tank) to produce and sell q units of output to the end consumers. We assume linear tariffs, in the sense that the supplier sets a wholesale price $c \geq 0$ per unit, which is independent of the total purchase order. For simplicity, we assume that the producer incurs no variable cost, but the input price. The supplier faces a linear cost κq, where $\kappa \geq 0$ is some exogenous constant (e.g., determined by the choice of technology to produce and store hydrogen). Later, we consider the possibility for the cost κ to change over time as the result of a decision by the supplier to invest in R&D.

Equilibrium. Suppose both firms have invested. Given an input price c, the producer selects an output level q_P^* that maximizes the profit $qP(x,q) - cq = xq^{1-\delta} - cq$. The first-order condition (FOC) gives the optimal order quantity

$$x(1-\delta)\left(q_P^*\right)^{-\delta} = c. \tag{3.2}$$

Because the supplier has only one customer, namely the producer, it infers its demand from the above mechanism, selecting the input price that solves $\max_{c\geq 0}\{(c-\kappa)q_P^\star(c)\}$. The optimum is attained at $c^\star = \frac{\kappa}{1-\delta}$.[3] The supplier sells the input at a markup as $c^\star > \kappa$. From this, we infer (after simplifications) the supplier's and the producer's reduced-form profits given by

$$\pi_S(x) = \pi_S(x,\kappa) = \delta(1-\delta)x\left[\frac{x(1-\delta)^2}{\kappa}\right]^{\frac{1-\delta}{\delta}} \text{ and } \pi_P(x) = \pi_P(x,\kappa) = (1-\delta)\pi_S(x), \tag{3.3}$$

respectively. It appears that firms' interests are aligned in the sense that their profits are proportional to one another. Note that the reduced-form profits would also be proportional to another another in the case of a linear demand (see §4.2.2 in Tirole, 1988). Because we ignore fixed costs

[3] As the producer is in a monopoly situation, a model whereby the producer decides on the product price (rather than its output) would yield the same result. Similarly, a model in which the supplier sets its output (rather than the wholesale price) would yield the same result.

and allow the firms to wield market power (i.e., to set a product price above the marginal cost), firm profits are positive. Both profits $\pi_S(\cdot)$ and $\pi_P(\cdot)$ are strictly convex in the demand state x (because of the power $1/\delta > 1$). This is because the output price in eq. (3.1) grows linearly in x and the firms react to more favorable demand shocks x by raising production.

4 Baseline coinvestment game under uncertainty

The demand for HPVs (and thus hydrogen tanks) is uncertain, but positive. Given this, we find it natural to model its dynamics as a geometric Brownian motion (GBM),

$$dX_t = X_t\big(\mu\, dt + \sigma\, dW_t\big) \quad \text{and} \quad X_0 = x \text{ almost surely,} \tag{4.1}$$

where $\sigma > 0$ is a volatility parameter and $(W_t)_t$ a standard Brownian motion. This assumption also ensures that the price in eq. (3.1) remains positive. Both the supplier and the producer are assumed to be risk neutral, discounting at the same discount rate $r > 0$. We use $\mathbb{E}_x$ as a conditional expectation operator depending on the initial value x of the stochastic process.

4.1 Static game of coinvestment

For now, we disregard the possibility for either firm to delay its investment and discuss whether, given the current market conditions (indexed by x), the firms should make the now-or-never decision to coinvest. For that purpose, we first need to assess the values of operating in this market.

4.1.1 Present values

After coinvestment by both firms, the firms earn the reduced-form profits in eq. (3.3), with the factor X affecting the price in eq. (3.1) following the dynamics of eq. (4.1). The **supplier**'s present value (PV) is given by

$$\Pi_S(x) := \mathbb{E}_x \left[\int_0^\infty e^{-rs} \pi_S(X_s)\, ds \right]. \tag{4.2}$$

Leveraging known properties of GBM, we are able to express the supplier's PV in a more convenient manner in the next proposition. All proofs are provided in appendix. We recall that the stochastic process $(X_t^\varepsilon)_t$ is a geometric Brownian motion with a growth rate given by

$$m(\varepsilon) := \mu\varepsilon + \frac{1}{2}\varepsilon(\varepsilon - 1)\sigma^2. \tag{4.3}$$

Proposition 4.1 (Supplier's PV)**.** *If $r > m(\frac{1}{\delta})$, then the supplier's PV in eq. (4.2) is of the form*

$$\Pi_S(x) = \beta x^{\frac{1}{\delta}}, \;\; \textit{where } \beta = \beta(\kappa, \mu) := \frac{\delta(1-\delta)}{r - m(\frac{1}{\delta})} \left[\frac{(1-\delta)^2}{\kappa} \right]^{\frac{1-\delta}{\delta}} > 0.$$

Otherwise, it is infinite/undefined.

The supplier's PV in Proposition 4.1 increases in the state of demand x in the output market. Intuitively, the increase in the demand for hydrogen-powered vehicles in the output market implies a greater demand for hydrogen tanks in the input market. Again, both the producer and the supplier can scale up their production as demand (indexed by x) materializes, leading to convexity in x.

Looking at the other echelon of the supply chain, we recall the **producer**'s (proportional) reduced-form profit given in eq. (3.3). The producer's PV is thus also proportional, given by

$$\Pi_P(x) := \mathbb{E}_x \left[\int_0^\infty e^{-rs} \pi_P(X_s)\, ds \right] = (1-\delta)\Pi_S(x), \quad \forall x > 0. \tag{4.4}$$

The producer's PV also increases convexly the state of demand x.

These expressions are based on the assumption that both companies are invested. We want to study each firm's incentive to invest and determine an equilibrium point for a static game of coinvestment.

4.1.2 Matrix game

The launch of the product requires substantial nonrecoupable investments from both firms. One firm (e.g., the supplier or producer) wishing to invest can only benefit from its investment if the other firm also makes the decision to invest.[4] The sunk investment costs of the firms may differ, with $I_S > 0$ and $I_P > 0$ denoting the cost of the supplier and producer, respectively.

From Proposition 4.1, the supplier's net payoff, if it decides to invest at time 0 and if the upstream firm has already invested, is $\beta x^{\frac{1}{\delta}} - I_S$. From eq. (4.4), the producer's payoff is $(1-\delta)\beta x^{\frac{1}{\delta}} - I_P$. We depict the NPVs of the firms in the matrix game of Figure 1. Regardless of strategic interactions,

$$\text{the NPV thresholds are given by } \left(\frac{I_S}{\beta}\right)^{\delta} \text{ for the supplier and } \left(\frac{I_S}{[1-\delta]\beta}\right)^{\delta} \text{ for the producer.} \tag{4.5}$$

The supplier has a weakly dominant strategy not to invest if $x < (I_S/\beta)^{\delta}$, but a weakly dominant strategy to invest if $x \geq (I_S/\beta)^{\delta}$. Similarly, the producer has a weakly dominant strategy not to invest if $x < (I_P/[1-\delta]\beta)^{\delta}$, but to invest if $x > (I_P/[1-\delta]\beta)^{\delta}$. The only scenarios in which the firms coinvest are when they both have a weakly dominant strategy to invest, i.e., if

$$x \geq \max\left\{\left(\frac{I_S}{\beta}\right)^{\delta}; \left(\frac{I_P}{(1-\delta)\beta}\right)^{\delta}\right\} = \left(\frac{\max\left\{I_S; \frac{I_P}{1-\delta}\right\}}{\beta}\right)^{\delta}. \tag{4.6}$$

In the jargon of option theory, this situation can be described as one in which the real options of both firms are “in the money."

		Producer	
		Invest	Not invest
Supplier	Invest	$\{(1-\delta)\beta x^{\frac{1}{\delta}} - I_P, \beta x^{\frac{1}{\delta}} - I_S\}$	$\{0,0\}$
	Not invest	$\{0,0\}$	$\{0,0\}$

Figure 1: **Two-by-two matrix game of coinvestment**

[4] This situation shares an analogy with hold-up problems (see Rogerson, 1992) known in contract theory, but here the relationship between the supplier and producer are assumed not to be ruled by a long-term contract.

To avoid dealing with numerous cases, we now make the following assumption:

Assumption 4.1 (Ranking of the firms' investment costs)**.** *The producer's and supplier's investment costs are such that* $I_P < (1-\delta)I_S$.

Assumption 4.1 implies that the supplier will require a higher state of demand x to invest compared to be producer. The cost asymmetry specified in Assumption 4.1 makes the supplier less prone to invest, thus constraining the producer about the profitability of its potential investment. Under this assumption, the threshold at the right-hand side (RHS) of eq. (4.6) is $(I_S/\beta)^\delta$, so for the coinvestment to take place, the weaker of the two firms, here the supplier, must have an inventive to invest. On the other hand, if Assumption 4.1 does not hold true , i.e., if the producer's cost satisfies $I_P \geq (1-\delta)I_S$, the RHS of eq. (4.6) becomes $(\frac{I_P}{(1-\delta)\beta})^\delta$: the coinvestment decision is then hampered because that producer is less eager to invest than the supplier.

4.2 Dynamic game of coinvestment

Now suppose that firms can time their investments flexibly. We ignore strategic interactions first and suppose that each firm assumes that the other has invested, a situation called "myopic."

4.2.1 Myopic investments

A "myopic" supplier decides on the best (stopping) time of its investment, i.e.,

$$V_S(x) := \sup_\tau \mathbb{E}_x \left[e^{-r\tau}(\beta X_\tau^{\frac{1}{\delta}} - I_S) \right], \tag{4.7}$$

a standard problem which is solved using classical real options techniques (see Dixit and Pindyck, 1994). It is known that $V_S(\cdot)$ is of the form $V(x) = Cx^\theta$ in the continuation region, where θ is the unique positive solution of equation $m(\theta) = r$ for $m(\cdot)$ defined in eq. (4.3). To determine the threshold x_S above which it is optimal for the supplier to invest, we use the principle of smooth fit: $V_S(x_S) = Cx_S^{\theta} = \beta x_S^{\frac{1}{\delta}} - I_S$ and $x_S V_S'(x_S) = \theta C x_S^{\theta} = \frac{\beta}{\delta} x_S^{\frac{1}{\delta}}$. This eventually leads to:

Proposition 4.2 (Myopic supplier's investment problem)**.** *Assume $r > m(\frac{1}{\delta})$. Eq. (4.7) has an explicit solution, given by*

$$V_S(x) = \begin{cases} \left[\beta x_S^{\frac{1}{\delta}} - I_S\right]\left(\frac{x}{x_S}\right)^{\theta}, & x < x_S := \left(\frac{\theta\delta}{\theta\delta - 1}\frac{I_S}{\beta}\right)^{\delta} \\ \beta x^{\frac{1}{\delta}} - I_S, & x \geq x_S. \end{cases}$$

The myopic supplier decides to invest if the demand state exceeds a cut-off x_S, which exceeds the NPV threshold $(I_S/\beta)^{1/\delta}$ in eq. (4.5) by a factor $[\delta\theta/(\delta\theta - 1)]^{1/\delta} > 1$. Above the level x_S, the cost of waiting outweighs the benefit of waiting, so it is optimal to invest despite the investment cost being non-recoupable. In the jargon of option pricing theory, one says that the (American) option is "deep in the money" for any realization of the underlying above the threshold x_S. The expression for $V_S(\cdot)$ in Proposition 4.2 suggests that, above the demand level x_S, the supplier's value corresponds to the NPV of future profits, but below that level, it is the expected value of the NPV following the investment time, $(1-\delta)\beta x_P^{1/\delta} - I_P$ discounted back to the present using the discount factor $\mathbb{E}_x e^{-rT_{x_P}} = (x/x_P)^{\theta}$, where T_z denotes the first-hitting time from below defined as

$$T_z = \inf\left\{t \geq 0 \middle| X_t \geq z\right\}. \tag{4.8}$$

Conversely, if the producer assumes that the supplier has already invested, than it follows from eq. (4.4) that the producer faces the problem:

$$V_P(x) = (1-\delta) \times \sup_{\tau} \mathbb{E}_x \left[e^{-r\tau}\left(\beta X_{\tau}^{\frac{1}{\delta}} - \frac{I_P}{1-\delta}\right)\right], \tag{4.9}$$

which we solve using similar techniques:

Proposition 4.3 (Myopic producer's investment problem)**.** *Assume $r > m(\frac{1}{\delta})$. The value function $V_P(\cdot)$ in eq. (4.9) is given by*

$$V_P(x) = \begin{cases} \left[(1-\delta)\beta x_P^{\frac{1}{\delta}} - I_P\right]\left(\frac{x}{x_P}\right)^{\theta}, & x < x_P := \left(\frac{\theta\delta}{\theta\delta-1}\frac{I_P}{(1-\delta)\beta}\right)^{\delta}, \\ (1-\delta)\beta x^{\frac{1}{\delta}} - I_P, & x \geq x_P. \end{cases}$$

Following Proposition 4.3, the myopic producer invests if its option to wait is deep in the money, i.e., if demand exceeds x_P. This cut-off level differs from the one in Proposition 4.2 by a factor

$$\frac{x_P}{x_S} = \left(\frac{1}{1-\delta}\frac{I_P}{I_S}\right)^{\delta}. \tag{4.10}$$

An immediate consequence of Propositions 4.2 and 4.3 is that, if $I_P = (1-\delta)I_S$, the two investment thresholds coincide, the two firms investing at the same time and $V_S \equiv (1-\delta)V_P$. Indeed, the producer's profit in eq. (4.4) differs from the supplier's by a factor $(1-\delta)$. Again, Assumption 4.1 suggests that, in a myopic setting, the supplier will invest if demand is higher compared to the producer (because $x_P < x_S$ according to Propositions 4.2 and 4.3). Conversely, if this assumption is not satisfied, then the producer's threshold will be larger, and so the producer invests later than the supplier.

We note:

Remark 4.1. *In Propositions 4.2 and 4.3, we have made the assumption $r > m(\frac{1}{\delta})$, which is equivalent to $\theta\delta > 1$. The latter inequality proves to be key to characterizing a Nash equilibrium in threshold strategies (see Theorem 4.1).*

4.2.2 Equilibrium solution(s)

If one firm decides to invest at time t, it will pay the sunk cost at that time, but will receive a payoff only if the second firm has already invested at or before time t. If this is not the case, the firm has to wait until the investment of the second firm to get a payoff. The supplier would like to improve

$$J_S(x,\tau_S,\tau_P)=\mathbb{E}_x\Big[e^{-r\tau_S}\big(\Pi_S(X_{\tau_S})1\!\!1_{\{\tau_P\le\tau_S\}}-I_S\big)\Big], \tag{4.11}$$

while the producer would like to raise

$$J_P(x,\tau_S,\tau_P)=\mathbb{E}_x\left[e^{-r\tau_P}(\Pi_P(X_{\tau_P})1\!\!1_{\{\tau_S\le\tau_P\}}-I_P)\right]. \tag{4.12}$$

We define this situation as a coinvestment game, which is another type of a nonzero-sum two-player (Dynkin) game of optimal stopping, in which each firm $i\in\{S,P\}$ chooses a stopping time τ_i to improve their payoff. We recall the definition of Nash equilibrium (NE) for stopping games:

Definition 4.1 (Nash equilibrium)**.** *A pair of stopping time $\{\tau_S^*,\tau_P^*\}$ is a NE for non-zero stopping games starting at x if for every stopping time τ, $J_S(x,\tau_S^*,\tau_P^*)\ge J_S(x,\tau,\tau_P^*)$ and $J_P(x,\tau_S^*,\tau_P^*)\ge J_S(x,\tau_S^*,\tau)$.*

If the pair (τ_S^*,τ_P^*) is a NE for all possible starting values x of the game, then it is Markov (subgame) perfect. We aim to provide sufficient conditions for the existence of Markov perfect equilibria (MPEs) of the following threshold type: $\tau_S^*=T_{z_S}$ and $\tau_P^*=T_{z_P}$, with T_z defined in eq. (4.8).

When the level of demand at time 0 is higher than the supplier's investment threshold x_S, the two firms invest optimally at time 0. Thus, the interesting case is when the current demand is less than x_S. The game arising in this particular case has a flavor of a war of attrition/chicken game (Hoppe, 2000; Murto, 2004; Steg and Thijssen, 2015; Décamps et al., 2022). Indeed, no firm has an incentive to invest before its rival because doing so would imply making no profit for a while but having to incur the rental cost rI_i, $i\in\{P,S\}$, of its productive assets. In other words, the first investor would not reap any benefits until the other firm also invests, but it would incur a discounted cost, I_je^{-rt} for $j\in\{P,S\}$, which decreases as time passes. So there effectively is no first-mover advantage, which makes both firms wait until the weaker party has an incentive to invest, from a myopic viewpoint. We next prove the existence of a MPE of threshold type:

Theorem 4.1 (Producer's best-reply investment strategy.)**.** *We take $r > m(1/\delta)$ and make Assumption 4.1. The best reply for the producer who anticipates that the supplier will invest above the threshold $\xi \geq x_S$ is to invest above the threshold ξ as well. The pair $\{T_\xi, T_\xi\}$ is thus a subgame perfect Nash equilibrium of threshold type for the coinvestment game between the supplier and the producer for every threshold $\xi \geq x_S$. There is an infinite number of equilibria in the class of threshold types. Among them, the sum of values achieved by the two players in case of the Nash equilibrium $\{T_{x_S}, T_{x_S}\}$ is the largest.*

An interesting consequence of this theorem is that the firms should coordinate on the Nash equilibrium $\{T_{x_S}, T_{x_S}\}$, where x_S is the optimal threshold of the supplier's myopic problem. The firm with the highest investment cost imposes its investment strategy on its partner in the supply chain. This result is not surprising given the interpretation of the situation as analogous to a war of attrition. In equilibrium, the firms wait until the weaker firm decides to invest, at a threshold determined myopically, namely, when the weaker firm's real option is deep in the money.

The particular forms of the NPVs received at exercise in eqs. (4.7) and (4.9) are sufficient, but not necessary, to obtain threshold solutions. Indeed, threshold policies arise for a larger class of NPVs (Villeneuve, 2007). Effectively, Theorem 4.1 holds provided the firms' myopic problems have threshold solutions.

4.3 Comparison with the vertically integrated solution

We previously assumed that the decision making (affecting both the price-setting mechanism and the coinvestment timing) was decentralized, e.g., because a vertical merger is forbidden by antitrust authorities or because vertical integration is prohibitively costly. For the sake of argument, we now compare the result in Theorem 4.1 with the optimum achieved in a situation where one firm is vertically integrated, which is equivalent to the Pareto optimum for the two firms. Now, the integrated problem, given by[5]

$$
\begin{aligned}
W(x) &:= \sup_{\tau} \mathbb{E}_x \left[\int_{\tau}^{\infty} e^{-rt} \pi_C(X_t) dt + e^{-r\tau}(I_S + I_P) \right], \\
\text{where } \pi_C(x) &:= \sup_{q \geq 0} \left\{ x q^{1-\delta} - \kappa q \right\} = (1-\delta)^{-\frac{1}{\delta}} \pi_S(x),
\end{aligned}
\tag{4.13}
$$

has a solution given in:

Proposition 4.4 (Solution of the integrated problem in eq. (4.13))**.** *The integrated profit in eq.* (4.13) *satisfies $\pi_C(x) > \pi_S(x) + \pi_P(x)$. The integrated solution is, if $r > m(1/\delta)$, to invest if demand exceeds*

$$
x_C := (1-\delta) \left(1 + \frac{I_P}{I_S} \right)^{\delta} x_S,
$$

for x_S given in Proposition 4.2. Accordingly, the value of the integrated firm is given by

$$
W(x) = \begin{cases} \left[\beta (x_C/[1-\delta])^{\frac{1}{\delta}} - I_S - I_P \right] \left(\frac{x}{x_C} \right)^{\theta}, & x < x_C, \\ \beta (x/[1-\delta])^{\frac{1}{\delta}} - I_S - I_P, & x \geq x_C. \end{cases}
$$

Following Proposition 4.4, the firms would extract more profit for themselves if they were to coordinate their decisions (e.g., if either firm vertically integrates the other). Furthermore, the integrated solution is to invest if demand exceeds the level x_C. As a function of δ, the ratio $\frac{x_C}{x_S}$ is decreasing on $(0, 1)$ from 1 to 0, which implies $x_C \leq x_S$. The latter inequality proves that the vertically integrated investment takes place earlier than the coinvestment under any of the possible MPE in Theorem 4.1. This result is in line with the paper by Billette De Villemeur et al. (2014).

4.4 Producer's cost-sharing

[5] For convenience, we assume that there is no improvement in fixed costs for the vertically integrated firm. If there were a reduction in the fixed cost, eq. (4.13) would provide an upper bound for the value function of the new problem.

We consider now whether the producer may be willing to subsidize the supplier's fixed cost. There indeed may be a tradeoff for the producer: an increase in its cost (due to the subsidizing) results in an earlier time at which the (equilibrium) coinvestment takes place, which indirectly benefits it. Following Theorem 4.1 (which holds under Assumption 4.1), the coinvestment takes place at the first time the supplier's threshold x_S is reached. Suppose the producer can commit at time 0 to paying a subsidy $\epsilon \geq 0$ reducing the supplier's investment cost. The supplier will always be inclined to accept such a subsidy, but under what conditions would it be in the producer's interest to make such a proposal?

The supplier's threshold (given in Proposition 4.2) is continuous in the investment cost I_S and remains above the producer's myopic threshold x_P (in Proposition 4.3) for any subsidy $\epsilon \geq 0$ satisfying $(1 - \delta)(I_S - \epsilon) \geq (I_P + \epsilon)$, which is equivalent to ϵ belonging to a compact set $[0, \bar{\epsilon}]$. Similarly to the arguments used to establish Theorem 4.1, we can prove that the coinvestment takes place in equilibrium at the supplier's myopic threshold adjusted for the subsidy paid by the producer, namely

$$x_S(\epsilon) := \left(\frac{\theta\delta}{\theta\delta - 1} \frac{I_S - \epsilon}{\beta} \right)^{\delta}.$$

Given this, the producer value in equilibrium (for a parameter ϵ) is given by

$$F(\epsilon) = \left[(1 - \delta)\beta x_S(\epsilon)^{\frac{1}{\delta}} - (I_P + \epsilon) \right] \left(\frac{x}{x_S(\epsilon)} \right)^{\theta}.$$

We prove that the maximizer of $F(\cdot)$ over the compact set $[0, \bar{\epsilon}]$ is strictly positive, by performing a first-order Taylor series expansion around zero. We have

$$x_S(\epsilon)^{-\theta} = \Big(\frac{\theta\delta}{\theta\delta - 1} \frac{I_S}{\beta} \Big)^{-\theta\delta} \Big(1 + \frac{\theta\delta}{I_S} \epsilon \Big) + o(\epsilon).$$

Then,

$$F(\epsilon) = F(0) + x^{\theta}\Big(\frac{\theta\delta}{\theta\delta - 1}\frac{I_S}{\beta}\Big)^{-\theta\delta}\left[(1-\delta-\frac{I_P}{I_S})\theta\delta - 1\right]\epsilon + o(\epsilon).$$

Therefore, a sufficient condition for a subsidy ϵ in $(0, \bar{\epsilon}]$ by the producer to be optimal is

$$\theta\delta\big[(1-\delta)I_S - I_P\big] > I_S. \tag{4.14}$$

We have by Assumption 4.1 that $1 - \delta - \frac{I_P}{I_S} > 0$ and know from Remark 4.1 that $\delta\theta > 1$. So, ineq. (4.14) is a further restriction on the cost parameters I_P and I_S compared to Assumption 4.1.

5 Evolving market circumstances

The previous economic insights were derived for a simple model. However, in many coinvestment games, market conditions evolve. This section investigates such changes: Section 5.1 studies a setup in which the demand dynamics are subject to exogenous changes, while, in Section 5.2, one firm in the supply chain takes an action that affects the other echelon.

5.1 Impact of a likely exogenous paradigm shift

We revisit the above setup to allow for a paradigm shift, with the factor affecting demand now given by

$$d\xi_t = \xi_t(\mu_t\, dt + \sigma\, dW_t) \quad \text{and} \quad \xi_0 = \xi \text{ almost surely.} \tag{5.1a}$$

instead of eq. (4.1), the main difference being the random drift of the demand process. The arrival of a paradigm shift is also uncertain, modeled with a suitably measurable random variable T capturing the arrival date. The random variation T could be the time at which the ban on ICEs will become effective. While deadlines have been announced in the EU and in the US, industry participants are not confident whether the underlying infrastructure (in terms of charging stations, transmission lines, etc.) and electricity generation capacities will be sufficient for all demand for transportation services to be met by the announced deadline. This makes the date of the effective paradigm shift a random variable. We assume that companies have homogeneous expectations about its probability distribution. The paradigm shift affects the rate at which demand grows, with the drift μ_t in eq. (5.1a) assumed to follow a single-jump process given by

$$\mu_t = \mu_0 \mathbb{1}_{\{t<T\}} + \mu_1 \mathbb{1}_{\{t\geq T\}}, \quad \text{with } \mu_1 > \mu_0 > 0. \tag{5.1b}$$

Unlike Farzin et al. (1998), we assume that, conditionally on the two firms having invested, T is exponentially distributed with intensity $\lambda > 0$. The larger the rate λ, the faster the paradigm shift tends to materialize (because the expected time of arrival, conditional on both firms being invested, is $\mathbb{E}[T] = 1/\lambda$). Because of the assumption $\mu_1 > \mu_0 > 0$ in eq. (5.1b), the paradigm shift causes an increase in the demand drift. Once firms are invested, the arrival of a paradigm shift is exogenous to their decisions. We now introduce $m_i(\varepsilon) := \mu_i\varepsilon + \frac{\sigma^2}{2}\varepsilon(\varepsilon-1)$ for $i \in \{0,1\}$. Because of the inequality $\mu_1 > \mu_0$, we get $m_0(\varepsilon) < m_1(\varepsilon)$ for every $\varepsilon > 0$.

In this case, Proposition 4.1, which established the supplier's NPV, is revised to

Proposition 5.1 (Supplier's PV)**.** *If $r > m_1(\frac{1}{\delta})$, then the supplier's PV is given by*

$$\mathbb{E}_x\left[\int_0^\infty e^{-rs}\pi_S(\xi_s)\,ds\right] = B\xi^{\frac{1}{\delta}}, \text{ where } B := \frac{1}{r+\lambda-m_0(\frac{1}{\delta})}\left(1+\frac{\lambda}{r-m_1(\frac{1}{\delta})}\right)\delta(1-\delta)\left[\frac{(1-\delta)^2}{\kappa}\right]^{\frac{1-\delta}{\delta}} > 0.$$

As a function of the intensity parameter λ, $B(\lambda)$ rises from $B(0) = \beta_0 := \beta(\kappa,\mu_0)$ to $\lim_{\lambda\to+\infty} B(\lambda) = \beta_1 := \beta(\kappa,\mu_1)$, for $\beta(\kappa,\mu)$ defined in Proposition 4.1.

The relation $\beta_0 < B < \beta_1$ is intuitive. The demand dynamics in eq. (5.1) with $\mu_1 > \mu_0$ implies stronger (resp., weaker) growth after (resp., before) the paradigm shift compared the benchmark with constant demand growth $\mu = \mu_0$ (resp., $\mu = \mu_1$) in Proposition 4.1. Consequently, the present value is increased (resp., reduced) compared to the benchmark with $\mu = \mu_0$ (resp., $\mu = \mu_1$). Again, if such a paradigm shift is not likely (resp., is sure) to happen, i.e., $\lambda = 0$ (resp., $\lambda = \infty$), the parameter B equals β_0 (resp., β_1).

The line of arguments used earlier to determine the equilibrium investment behaviors carries over: in the new case with the demand dynamics of eq. (5.1), the MPE (which achieves the largest sum of values for the players) is for the firms to invest above the cut-off level $\tilde{x}_S := \left(\frac{\theta\delta}{\theta\delta-1}\frac{I_S}{B}\right)^{\delta}$.
Denoting by $x_{S,i} := \left(\frac{\theta\delta}{\theta\delta-1}\frac{I_S}{\beta_i}\right)^{\delta}$ the threshold of Proposition 4.2 for $\mu = \mu_i$, $i \in \{1, 2\}$, it follows from Proposition 5.1 that $x_{S,1} < \tilde{x}_S < x_{S,0}$. Intuitively, in this equilibrium, the firms will invest earlier if they experience stronger (resp., weaker) demand growth after (resp., before) the paradigm shift compared to the benchmark with constant growth rate $\mu = \mu_0$ (resp., $\mu = \mu_1$).

5.2 Impact of an endogenous technological breakthough

In the baseline model (and its extension in Section 5.1), the firms were not trying to improve the performance of the technology. In our setup, product performance can be indexed by the variable production cost κ, which affects firm profits in eq. (3.3). Again, the supplier sells the input at a markup compared to the production cost, with an equilibrium input price $\hat{c} = \kappa/[1-\delta] \geq \kappa$. We now consider an extension whereby the supplier can make an investment that allows it to reduce the production cost from κ_0 to κ_1 following a *technological breakthough*. An industry implication is that the producer will benefit from this production cost reduction at no cost (via a reduced input price), with the supplier not able to internalize all the positive externalities of its own investment.

Supplier's perspective. The supplier is now assumed to have leeway in the time τ at which to initiate R&D activities. Specifically, from that time until a *technological breakthrough* is achieved, the supplier engages a team of engineers (e.g., consultants or postdoctoral students) at a cost $k \geq 0$ per unit of time. The duration $\hat{T}$ of the period until a technological breakthrough is achieved is considered a random variable. Both parties have homogeneous beliefs about the law of distribution of this random variable, which is assumed to be exponentially distributed with intensity $\hat{\lambda}$. The variable production cost evolves according to

$$\kappa^{\tau}(t) = \kappa_0 1\!\!\mathrm{I}_{(0,\tau+\hat{T})}(t) + \kappa_1 1\!\!\mathrm{I}_{[\tau+\hat{T},\infty)}(t), \quad \text{where } \kappa_0 > \kappa_1. \tag{5.2}$$

The short-term profits of the supplier and producer in eq. (3.3) become time-dependent because they depend on the state $\kappa^{\tau}(t)$ of the variable cost. In this model extension, the supplier's present value is itself the solution of a real option problem related to the investment problem in R&D, namely

$$\hat{\Pi}_S(x) = \sup_{\tau \geq 0} \mathbb{E}_x \left[\int_0^{\infty} e^{-rt} \pi_S\big(X_t, \kappa^{\tau}(t)\big) dt - \int_{\tau}^{\tau+\hat{T}} e^{-rt} k dt \right], \tag{5.3}$$

while the investment problem of a myopic supplier is now given by

$$\hat{V}_S(x) := \sup_{\vartheta} \mathbb{E}_x \left[e^{-r\vartheta} \left(\hat{\Pi}_S(X_{\vartheta}) - I_S \right) \right]. \tag{5.4}$$

In contrast to the supplier's NPV in eq. (4.2), the NPV in eq. (5.3) accounts for an endogenous decision to initiate R&D activities with the goal of reducing production cost. The first integral term on the RHS of eq. (5.3) corresponds to the supplier's discounted profits given a time-dependent production cost $\kappa^{\tau}(t)$ in eq. (5.2). Eq. (5.4) represents a compound option problem. Let $\beta_i = \beta(\kappa_i, \mu)$ for β defined in Proposition 4.1. One readily sees that $\hat{\Pi}_S(\cdot)$ in eq. (5.3) satisfies $\hat{\Pi}_S(x) > \beta_0 x^{1/\delta}$ for all $x > 0$, because the value function in Proposition 4.1 embeds an upside improvement by investing in R&D to reduce production costs (simply take $\tau = \infty$ in eq. (5.3)). As a consequence, $\hat{V}_S \geq V_S$ for V_S given in Proposition 4.3 (and computed at β_0). The intuition for $\hat{V}_S \geq V_S$ is that the payoff of the compound option comprises an option term that has a positive value. We establish:

Proposition 5.2 (Supplier's myopic strategy in case of endogenous R&D activities.)**.** *Assume* $r > m(1/\delta)$*. Then, eq.* (5.3) *admits an explicit expression, namely*

$$\hat{\Pi}_S(x) = \beta_0 x^{\frac{1}{\delta}} + \begin{cases} \left[b\hat{x}_S^{\frac{1}{\delta}} - K\right]\left(\frac{x}{\hat{x}_S}\right)^{\theta}, & x < \hat{x}_S := \left(\frac{\theta\delta}{\theta\delta-1}\frac{K}{b}\right)^{\delta} \\ bx^{\frac{1}{\delta}} - K, & x \geq \hat{x}_S. \end{cases} \tag{5.5}$$

for

$$b := \frac{\hat{\lambda}}{r+\hat{\lambda}-m(\frac{1}{\delta})}\left[\left(\frac{\kappa_0}{\kappa_1}\right)^{\frac{1-\delta}{\delta}} - 1\right]\beta_0 > 0 \quad \text{and} \quad K := \frac{k}{r+\hat{\lambda}}. \tag{5.6}$$

The supplier's optimal time at which to initiate R&D activities is $\hat{\tau}_S := \inf\left\{t \geq 0 \middle| X_t^x \geq \hat{x}_S\right\}$*. Furthermore, there exists a unique* $\bar{x}_S$ *obtained by smooth fit such that* $\hat{V}_S$ *in eq.* (5.4) *has an explicit solution of the form*

$$\hat{V}_S(x) = \begin{cases} \left(\frac{x}{\bar{x}_S}\right)^{\theta}\left[\hat{\Pi}_S(\bar{x}_S) - I_S\right], & 0 < x < \bar{x}_S, \\ \hat{\Pi}_S(x) - I_S, & 0x \geq \bar{x}_S. \end{cases}$$

The first term in eq. (5.5) corresponds to the perpetuity of the supplier's profit under the old cost structure κ_0, while the second (piecewise) term embeds the option value to reduce the variable cost from κ_0 to κ_1. The parameter b, which drives the upside option value, becomes larger if the cost reduction, $\kappa_0 - \kappa_1$, is larger. The expression for the threshold $\bar{x}_S$ is not neat because of the nonlinearity of the payoff function in eq. (5.5) (with respect to $x^{1/\delta}$) and is therefore omitted.

Our next result looks at the impact of the opportunity to reduce costs on the investment thresholds:

Proposition 5.3 (Comparison of supplier's thresholds)**.** *The optimal investment thresholds* x_S *and* $\bar{x}_S$ *given in Propositions 4.2 and 5.2, respectively, satisfy* $\bar{x}_S \leq x_S$*.*

Following Proposition 5.3, the possibility of reducing variable costs thanks to successful R&D activities leads the myopic supplier to invest earlier compared to the benchmark without this possibility (as $\bar{x}_S \leq x_S$). This is because the myopic supplier benefits more from the underlying demand x thanks to the (compound) option to conduct R&D, analogously to having a larger slope in case of an affine payoff function.

Producer's perspective. As soon as the engineers' team makes a technological breakthrough, the supplier will benefit from a cost reduction. This cost reduction will impact equilibrium conditions throughout the supply chain, affecting the wholesale price and the price to the end customers. The producer does not subsidize the supplier here, but receives some financial benefits once the technological breakthrough takes place. The producer's profit is also stochastic and time-dependent given by the stochastic process $\big((1-\delta)\pi_S(X_t, \kappa^{\hat{\tau}_S}(t))\big)_t$ where π_S is given in eq. (3.3) and $\hat{\tau}_S$ is specified in Proposition 5.2. The producer's myopic problem is to determine the time at which to invest:

$$\hat{V}_P(x) := \sup_{\theta} \mathbb{E}_x \left[e^{-r\theta} \big(\hat{\Pi}_P(X_\theta) - I_P \big) \right], \text{ where } \hat{\Pi}_P(x) = (1-\delta)\mathbb{E}_x \left[\int_0^\infty e^{-rt} \pi_S(X_t, \kappa^{\hat{\tau}_S}(t)) dt \right]. \quad (5.7)$$

The problem in eq. (5.7) differs from the one in eq. (4.9) because the producer now benefits from the positive externalities of the supplier's R&D activities on its profits. We obtain:

Proposition 5.4 (Producer's NPV in case of endogenous R&D activities by the supplier)**.** *Assume $r > m(1/\delta)$. The producer's NPV in eq. (5.7) is given by*

$$\frac{\hat{\Pi}_P(x)}{1-\delta} = \beta_0 x^{\frac{1}{\delta}} + \begin{cases} b\hat{x}_S^{\frac{1}{\delta}} \left(\frac{x}{\hat{x}_S} \right)^{\theta}, & 0 < x < \hat{x}_S \\ b x^{\frac{1}{\delta}}, & x \geq \hat{x}_S. \end{cases} \quad (5.8)$$

for $\hat{x}_S$ given in eq. (5.5) and the parameter b defined in eq. (5.6). The producer's payoff $\hat{\Pi}_P$ is continuous, but it has a concave kink at the point $\hat{x}_S$ above which the supplier decides to initiate R&D activities *(to reduce the production cost). The producer's myopic problem in eq. (5.7) has a solution, characterized by threshold denoted $\bar{x}_P$.*

In both eqs. (5.5) and (5.8), the second term corresponds to the value of benefitting from an upside thanks to an R&D investment. Yet, the supplier needs to pay the discounted cost of hiring engineers K in eq. (5.5), while the producer in a sense freerides benefitting from a cost reduction which is passed through by the supplier as in eq. (5.8).

Interestingly, the kink for $\hat{\Pi}_P(\cdot)$ does not take place when the technological breakthrough happens at time T but at the threshold $\hat{x}_S$ above which the supplier decides to initiate R&D activities. This is because the producer has rational expectations and thus prices in the technological breakthrough that will eventually take place given the observed supplier's R&D policy. This kink is driven by the property that the option term in the producer's payoff in eq. (5.8) grows as $x \mapsto x^{\theta}$ with the demand state x prior to the supplier's investment threshold $\bar{x}_S$, but as $x \mapsto x^{\frac{1}{\delta}}$ after that threshold. But because $\theta > 1/\delta$ (see Remark 4.1), the growth is stronger in the first period than in the second. The economic force driving this result is that, in a sense, the producer benefits from the optionality of an investment by the supplier to eventually reduce the production cost, so the producer's payoff grows in a "more convex" manner in the first region (in the sense that $\theta(\theta - 1) > \frac{1}{\delta}(\frac{1}{\delta} - 1) > 0$). We also note that there is no smooth fit at $\hat{x}_S$ because the choice of the threshold $\hat{x}_S$ is at the discretion of the supplier.

Constructing the optimal stopping strategy here does not follow standard techniques, as those, e.g., in Dixit and Pindyck (1994). This is because the payoff function $\hat{\Pi}_P$ in eq. (5.8) has a concave kink. However, following Villeneuve (2007), we know that the solution is of the threshold type.

Using arguments similar to the ones used to derive Proposition 5.3, we can prove that the thresholds x_P in Proposition 4.3 (valued at $\beta = \beta_0$) and $\bar{x}_P$ in Proposition 5.4 satisfy $\bar{x}_P \leq x_P$. In other words, the perspective of a cost reduction from κ_0 to κ_1 makes the producer invest earlier (compared to the case where the variable production cost remains at the level κ_0). The supplier's endogenous investment in

R&D makes the payoff of both (myopic) firms more valuable, so that they both end up investing earlier. For a particular set of parameter values, the optimal threshold of the myopic producer will be determined by smooth fit. In this case, it is given by

$$\bar{x}_P := \left(\frac{\delta\theta}{\delta\theta - 1} \frac{I_P}{(1-\delta)\left[\beta_0 + b\right]} \right)^{\delta}.$$

If the producer's entry cost is relatively low (i.e., $I_P < \bar{I}_P := (1-\delta)\frac{\beta_0+b}{b}K$), the myopic producer will invest before the supplier has decided to conduct R&D to reduce the production cost (i.e., $\bar{x}_P < \hat{x}_S$). But, if it is larger (i.e., $I_P \geq \bar{I}_P$), the supplier will conduct R&D when the (myopic) producer invests. The solution of this new coinvestment game has a simple form. Following the arguments in the proof of Theorem 4.1, the Markov perfect equilibrium that generates the largest values for both firms is to invest for demand state values above the maximum of $\bar{x}_S$ and $\bar{x}_P$ determined in Propositions 5.2 and 5.4, respectively. Determining which of the two thresholds is larger is a delicate question. In particular, Assumption 4.1 is not sufficient for that purpose.

6 Conclusion

Our stylized model provides important insights. Suppliers and producers may differ in their eagerness to capitalize on business opportunities which requires a coinvestment. This is manifest in particular in the ranking of the firms' investment thresholds. In equilibrium, neither firm benefits from investing first, leading to multiple Nash equilibria where both invest simultaneously when their options are highly valuable. A vertically integrated firm tends to invest earlier and earn higher profits than a noncoordinated supply chain. Even with a paradigm shift affecting demand growth, firms still wait until the weaker party is ready to invest. Lastly, if the supplier reduces production costs through R&D, the producer benefits without bearing the costs, leading to earlier coinvestment once both firms' options are highly valuable.

Our model, like any model, has limitations. In particular, firms typically make choices such as their capacity (Bensoussan and Chevalier-Roignant, 2013; Huisman and Kort, 2015) or product positioning besides their investment timing choices. Furthermore, supply chains are complex ecosystems. We leverage a classical model to characterize the interactions between a supplier and a producer, which is rich, yet sufficiently tractable for solutions to the (baseline) dynamic game to admit explicit expressions. Another demand model for the end market, different characterizations of the firms' cost functions, and more sophisticated negotiation setups would lead to different expressions for the firms' reduced-form profits. Also, there may be more than two echelons, each possibly characterized by different degrees of horizontal competition. For instance, more competition among suppliers will give more relative market power to the producer(s), increase their value(s) and raise their willingness(es) to invest, but the suppliers will earn less and be less prone to investing. More competition at either echelon will affect the investment times for both echelons, e.g., due to preemption (Fudenberg and Tirole, 1985; Thijssen et al., 2012); this further economic force may affect, in equilibrium, the time at which the maximum of the two echelon-specific thresholds is attained. Moreover, we obtained reduced-form profits as solutions to a one-shot game indexed by the demand state; a more realistic price-setting mechanism may involve repeated strategic interactions. These issues are left for future research.

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Online appendix

A Proof of Theorem 4.1

Assume the producer anticipates the supplier will invest at a threshold $\xi \geq x_S$. The first step is to explore whether the producer has an incentive to follow a policy which differs from investing at the same threshold ξ. It follows from standard properties about stopping times and GBMs that, if the producer, who anticipates the supplier stopping strategy invests at a level of demand x, it will receive a normalized payoff given by

$$\phi(x;\xi) = B\xi^{\frac{1}{\delta}} \left(\frac{x}{\xi}\right)^{\theta} \mathbf{1}_{\{x \leq \xi\}} + Bx^{\frac{1}{\delta}} \mathbf{1}_{\{x \geq \xi\}} - I_P,$$

where the first term $B\xi^{\frac{1}{\delta}} \left(\frac{x}{\xi}\right)^{\theta}$ is the expected value arising from the fact that the producer has to wait until the supplier invests at ξ to reap the benefits of its investment.

After the change of variables $Y_t = X_t^{\frac{1}{\delta}}$, the best reply of the producer who anticipates the supplier stopping strategy T_ξ is the optimal stopping problem

$$R(y) = \sup_{\tau} \mathbb{E}_y \left[e^{-r\tau}(BY_\tau \mathbf{1}_{\tau \geq T_\xi} - I_P)\right],$$

where the process $(Y_t = X_t^{\frac{1}{\delta}})_{t \geq 0}$ is a geometric Brownian motion and τ is a stopping time (not necessarily a hitting time). Strong Markov property implies that for every stopping time $\tau \geq T_\xi$, we have

$$\begin{aligned}
\mathbb{E}_y\left[e^{-r\tau}(BY_\tau \mathbf{1}_{\tau\geq T_\xi} - I_P)\right] &= \mathbb{E}_y\left[e^{-r\tau}((BY_\tau - I_P)\mathbf{1}_{\tau\geq T_\xi} - \mathbf{1}_{\tau<T_\xi} I_P))\right] \\
&\leq \mathbb{E}_y\left[e^{-rT_\xi}(B\mathbb{E}\left[e^{-r(\tau-T_\xi)}Y_\tau|\mathcal{F}_{T_\xi}\right] - I_P)\mathbf{1}_{\tau\geq T_\xi}\right] \\
&\leq \mathbb{E}_y\left[e^{-rT_\xi}\mathbf{1}_{\tau\geq T_\xi} V_P(Y_{T_\xi})\right] \\
&\leq \mathbb{E}_y\left[e^{-rT_\xi} V_P(Y^\delta_{T_\xi})\right] \text{ because } V_P > 0 \\
&= \mathbb{E}_y\left[e^{-rT_\xi}(BY_{T_\xi} - I_P)\right] \text{ because } \xi \geq x_S \\
&\leq R(y).
\end{aligned}$$

As a consequence, T_ξ is a best reply for the producer. On the other hand, if the supplier anticipates the producer will invest at a threshold $\xi \geq x_S$, a similar argument proves that the supplier's best reply is to choose the hitting time T_ξ.

Clearly, stopping at the threshold x_S is optimal for the supplier. We will show that the Nash equilibrium (T_{x_S}, T_{x_S}) also provides the largest value for the producer. Indeed, the producer's value function along the Nash equilibrium (T_ξ, T_ξ) is

$$\phi(x;\xi) = B\xi^{\frac{1}{\delta}}\left(\frac{x}{\xi}\right)^\theta \mathbf{1}_{\{x\leq\xi\}} + Bx^{\frac{1}{\delta}}\mathbf{1}_{\{x\geq\xi\}} - I_P,$$

which is a decreasing function with respect to ξ for $\xi \geq x_S$.

If Assumption 4.1 is not satisfied, then it can be proven using similar arguments that $\{T_{x_P}, T_{x_P}\}$ for x_P provided in Proposition 4.3 is the Nash equilibrium providing the largest values for the two players. □

B Proof of Proposition 4.4

Integrated profit. The function $q \mapsto xq^{1-\delta} - \kappa q$ is concave, with $q \mapsto \frac{d}{dq}\left(xq^{1-\delta} - \kappa q\right)$ decreasing from ∞ to $-\kappa$. The optimal quantity q_C solves $(1-\delta)xq^{-\delta} = \kappa$, which is equivalent to $q_C := \left(x\frac{1-\delta}{\kappa}\right)^{\frac{1}{\delta}}$. So,

$$
\begin{aligned}
\pi_C(x) &= \left(x\frac{1-\delta}{\kappa}\right)^{\frac{1}{\delta}} \left\{\frac{\kappa}{1-\delta} - \kappa\right\} \\
&= \frac{\kappa\delta}{1-\delta}\left(x\frac{1-\delta}{\kappa}\right)^{\frac{1}{\delta}} \\
&= \frac{\kappa\delta}{1-\delta}\left(\frac{x(1-\delta)^2}{\kappa}\right)^{\frac{1}{\delta}-1} \frac{x}{\kappa}(1-\delta)^{2-\frac{1}{\delta}} \\
&= x\delta(1-\delta)^{1-\frac{1}{\delta}} \times \pi_S(x)\frac{1}{\delta(1-\delta)x} \\
&= (1-\delta)^{-\frac{1}{\delta}}\pi_S(x).
\end{aligned}
$$

But $\delta \mapsto (1-\delta)^{-\frac{1}{\delta}} - (2-\delta)$ is strictly positive on $(0,1)$, which proves the result $\pi_C(x) > \pi_S(x) + \pi_P(x)$.

Investment problem. Furthermore,

$$
\mathbb{E}_x \int_\tau^\infty e^{-rt}\pi_C(X_t)dt = (1-\delta)^{-\frac{1}{\delta}}\beta x^{\frac{1}{\delta}}.
$$

for β defined in Proposition 4.1. It follows that the optimal stopping time of the coordinated problem is T_{x_C}, where

$$
x_C := \left(\frac{\theta\delta}{\theta\delta - 1}\frac{I_S + I_P}{(1-\delta)^{-\frac{1}{\delta}}\beta}\right)^\delta
$$

From Proposition 4.2, we note that

$$
\frac{x_C}{x_S} = (1-\delta)\left(1 + \frac{I_P}{I_S}\right)^\delta.
$$

But we have

$$\frac{\partial}{\partial \delta}\left((1-\delta)\left(1+\frac{I_P}{I_S}\right)^{\delta}\right) = \left(1+\frac{I_P}{I_S}\right)^{\delta}\left\{(1-\delta)\ln\left(1+\frac{I_P}{I_S}\right)-1\right\}$$
$$< \left(1+\frac{I_P}{I_S}\right)^{\delta}\{(1-\delta)\ln(2-\delta)-1\}$$

from Assumption 4.1 and monotonicity of $y \mapsto \ln(1+y)$ on $(0,\infty)$. But $\frac{\partial}{\partial\delta}((1-\delta)\ln(2-\delta)-1) = -\frac{1-\delta}{2-\delta} - \ln(2-\delta) < 0$, so $\delta \mapsto (1-\delta)\ln(2-\delta)-1$ decreases on $(0,1)$ from $\ln 2 - 1 \approx -.3$ to -1. It follows that $\delta \mapsto (1-\delta)\left(1+\frac{I_P}{I_S}\right)^{\delta}$ decreases on $(0,1)$ from 1 to 0 and so that $x_C < x_S$. The expression for $W(\cdot)$ is immediate.

C Proof of Proposition 5.1

Let $H_t = \mathbb{1}_{\{T\le t\}}$ be the single-jump process which jumps when the paradigm shift arises. Because the supplier's NPV will be modified with the arrival of the paradigm shift, we denote by $\Pi(X_t, H_t)$ the supplier's NPV at time t. At time 0, we thus have $\Pi_S(x) = \Pi(x,0)$.

By the Strong Markov property, we have

$$\Pi(x,0) = \mathbb{E}\left[\int_0^T e^{-rs}\pi_S(X_s)\,ds + e^{-rT}\Pi(X_T,1)\right],$$

where, using that $X_s = xe^{\mu_1 s+\sigma W_s - \frac{\sigma^2}{2}s}$ for $s \ge T$,

$$\Pi(x,1) = \mathbb{E}\left[\int_0^\infty e^{-rs}\pi_S(xe^{\mu_1 s+\sigma W_s-\frac{\sigma^2}{2}s})\,ds\right]$$

Observing that π_S in eq. (3.3) writes $\pi_S(x) = Ax^{\frac{1}{\delta}}$, with

$$A = \delta(1-\delta)\left[\frac{(1-\delta)^2}{\kappa}\right]^{\frac{1-\delta}{\delta}} > 0,$$

we obtain

$$\begin{aligned}
\Pi(x,1) &= \mathbb{E}\left[\int_0^\infty e^{-rs} A x^{\frac{1}{\delta}} e^{\frac{1}{\delta}(\mu_1 s + \sigma W_s - \frac{\sigma^2}{2}s)}\, ds\right] \\
&= A x^{\frac{1}{\delta}} \int_0^\infty e^{-rs} e^{\frac{1}{\delta}(\mu_1 s - \frac{\sigma^2}{2}s)} \mathbb{E}[e^{\frac{\sigma}{\delta}W_s}]\, ds \\
&= A x^{\frac{1}{\delta}} \int_0^\infty e^{-(r - m_1(\frac{1}{\delta}))s}\, ds \\
&= \frac{A x^{\frac{1}{\delta}}}{r - m_1(\frac{1}{\delta})}.
\end{aligned}$$

for $m_1(\cdot)$ given in eq. (4.3).

On the one hand, using that $X_s = x e^{\mu_0 s + \sigma W_s - \frac{\sigma^2}{2}s}$ for $s \leq T$,

$$\begin{aligned}
\mathbb{E}\left[\int_0^T e^{-rs} A X_s^{\frac{1}{\delta}}\, ds\right] &= \mathbb{E}\left[\int_0^\infty \left(\int_0^t e^{-rs} A X_s^{\frac{1}{\delta}}\, ds\right) \lambda e^{-\lambda t}\, dt\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-rs} A X_s^{\frac{1}{\delta}} \left(\int_s^\infty \lambda e^{-\lambda t}\, dt\right) ds\right] \\
&= \int_0^\infty e^{-(r+\lambda)s} A \mathbb{E}[X_s^{\frac{1}{\delta}}]\, ds \\
&= \frac{A x^{\frac{1}{\delta}}}{r + \lambda - m_0(\frac{1}{\delta})}.
\end{aligned}$$

We obtain by observing again that $X_s = x e^{\mu_0 s + \sigma W_s - \frac{\sigma^2}{2}s}$ for $s \leq T$,

$$\begin{aligned}
\mathbb{E}\left[e^{-rT}\Pi_1(X_T)\right] &= \int_0^\infty \mathbb{E}\left[e^{-rt}\Pi_1(X_t)\right] \lambda e^{-\lambda t}\, dt \\
&= \frac{\lambda A}{r - m_1(\frac{1}{\delta})} \int_0^\infty e^{-(r+\lambda)t} \mathbb{E}[(X_t)^{\frac{1}{\delta}}]\, dt \\
&= \frac{\lambda A}{r - m_1(\frac{1}{\delta})} \frac{x^{\frac{1}{\delta}}}{r + \lambda - m_0(\frac{1}{\delta})}.
\end{aligned}$$

D Proof of Proposition 5.2

From eqs. (5.2) and (5.3)

$$\begin{aligned}\hat{\Pi}_S(x) =& \sup_{\tau\geq 0}\mathbb{E}_x\left[\int_0^{\tau} e^{-rt}\pi_S\big(X_t,\kappa_0\big)dt + \int_{\tau}^{\tau+\hat{T}} e^{-rt}\big\{\pi_S\big(X_t,\kappa_0\big)-k\big\}dt + \int_{\tau+\hat{T}}^{\infty} e^{-rt}\pi_S\big(X_t,\kappa_1\big)dt\right]\\ =& \sup_{\tau\geq 0}\mathbb{E}_x\left[\int_0^{\tau} e^{-rt}\pi_S\big(X_t,\kappa_0\big)dt + e^{-r\tau}\left\{\int_{\tau}^{\tau+\hat{T}} e^{-r(t-\tau)}\big\{\pi_S\big(X_t,\kappa_0\big)-k\big\}dt + \int_{\tau+\hat{T}}^{\infty} e^{-r(t-\tau)}\pi_S\big(X_t,\kappa_1\big)dt\right\}\right].\end{aligned}$$

Now by the law of iterated expectations and strong Markov property, we have

$$\hat{\Pi}_S(x) = \sup_{\tau\geq 0}\mathbb{E}_x\left[\int_0^{\tau} e^{-rt}\pi_S\big(X_t,\kappa_0\big)dt + e^{-r\tau}\psi(X_\tau)\right],$$

where

$$\begin{aligned}\psi(x) :=& \mathbb{E}_x\left[\int_0^{\hat{T}} e^{-rs}\big\{\pi_S\big(X_s,\kappa_0\big)-k\big\}ds + \int_{\hat{T}}^{\infty} e^{-rs}\pi_S\big(X_s,\kappa_1\big)ds\right]\\ =& \mathbb{E}_x\left[\int_0^{\hat{T}} e^{-rs}\big\{\pi_S\big(X_s,\kappa_0\big)-k\big\}ds + e^{-r\hat{T}}\psi_1(X_{\hat{T}})\right],\end{aligned}$$

with

$$\psi_1(x) = \mathbb{E}_x\int_0^{\infty} e^{-ru}\pi_S\big(X_u,\kappa_1\big)du.$$

It follows from Proposition 4.1 that $\psi_1(x) = \beta_1 x^{\frac{1}{\delta}}$. Hence, by the law of iterated expectations,

$$\begin{aligned}\psi(x) =& \mathbb{E}_x\left[\int_0^{\hat{T}} e^{-rs}\big\{\pi_S\big(X_s,\kappa_0\big)-k\big\}ds + \beta_1 e^{-r\hat{T}}X_{\hat{T}}^{\frac{1}{\delta}}\right]\\ =& \mathbb{E}_x\int_0^{\hat{T}} e^{-rs}\pi_S\big(X_s,\kappa_0\big)ds - \frac{k}{r}\left[1-\mathbb{E}_x[e^{-r\hat{T}}]\right] + \beta_1\mathbb{E}_x e^{-r\hat{T}}X_{\hat{T}}^{\frac{1}{\delta}}.\end{aligned}$$

Now, assume that $\hat{T}$ is exponentially distributed of parameter $\hat{\lambda} > 0$. Then,

$$\begin{aligned}
\mathbb{E}_x\big[e^{-r\hat{T}}X_{\hat{T}}^{\frac{1}{\delta}}\big] =&\hat{\lambda}\int_0^\infty e^{-(r+\hat{\lambda})t}\mathbb{E}_x[X_t^{\frac{1}{\delta}}]dt \\
=&\hat{\lambda}x^{\frac{1}{\delta}}\int_0^\infty e^{-(r+\hat{\lambda}-m(\frac{1}{\delta}))t}dt \text{ by standard properties of GBMs} \\
=&\frac{\hat{\lambda}}{r+\hat{\lambda}-m(\frac{1}{\delta})}x^{\frac{1}{\delta}}
\end{aligned}$$

while

$$\begin{aligned}
1-\mathbb{E}e^{-r\hat{T}} =&1-\int_0^\infty \hat{\lambda}e^{-(r+\hat{\lambda})t} \\
=&\frac{r}{r+\hat{\lambda}},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_x\int_0^{\hat{T}} e^{-rs}\pi_S\big(X_s,\kappa_0\big)ds &= \big[r-m(\tfrac{1}{\delta})\big]\,\beta_0\mathbb{E}_x\int_{s=0}^{\hat{T}} e^{-rs}X_s^{\frac{1}{\delta}}ds \\
&= \big[r-m(\tfrac{1}{\delta})\big]\,\beta_0\int_{t=0}^\infty \hat{\lambda}e^{-\hat{\lambda}t}\Big\{\int_{s=0}^t \mathbb{E}_x e^{-rs}X_s^{\frac{1}{\delta}}ds\Big\}dt \\
&= \big[r-m(\tfrac{1}{\delta})\big]\,\beta_0\int_{s=0}^\infty e^{-rs}\mathbb{E}_x X_s^{\frac{1}{\delta}}\Big\{\int_{t=s}^\infty \hat{\lambda}e^{-\hat{\lambda}t}dt\Big\}ds \\
&= \big[r-m(\tfrac{1}{\delta})\big]\,\beta_0 x^{\frac{1}{\delta}}\int_{s=0}^\infty e^{-(r+\hat{\lambda}-m(\frac{1}{\delta}))s}ds \\
&=\frac{r-m(\frac{1}{\delta})}{r+\hat{\lambda}-m(\frac{1}{\delta})}\beta_0 x^{\frac{1}{\delta}}.
\end{aligned}$$

We now define K as in eq. (5.6) and write

$$\begin{aligned}
\psi(x) &= \left(\frac{r-m(\frac{1}{\delta})}{r+\hat{\lambda}-m(\frac{1}{\delta})}\beta_0 + \frac{\hat{\lambda}}{r+\hat{\lambda}-m(\frac{1}{\delta})}\beta_1\right)x^{\frac{1}{\delta}} - K, \\
&= \frac{r+\hat{\lambda}\left(\frac{\kappa_0}{\kappa_1}\right)^{\frac{1-\delta}{\delta}} - m(\frac{1}{\delta})}{r+\hat{\lambda}-m(\frac{1}{\delta})}x^{\frac{1}{\delta}} - K.
\end{aligned}$$

It follows from the above and Proposition 4.1 that

$$\hat{\Pi}_S(x) = \beta_0 x^{\frac{1}{\delta}} + f(x), \quad \text{where} \quad f(x) := \sup_{\tau \geq 0} \mathbb{E}_x e^{-r\tau} \left[b X_\tau^{\frac{1}{\delta}} - K \right] \tag{D.1}$$

and b is defined as in eq. (5.6). Following the same arguments as for Proposition 4.2 we get the expression for $\hat{\Pi}_S$ in eq. (5.5). The optimal strategy for $\hat{\Pi}_S$ is a threshold policy.

The supplier's value function in eq. (5.4) can be written as

$$\hat{V}_S(x) := \sup_{\vartheta \geq 0} \mathbb{E}_x \left[e^{-r\vartheta} \hat{\chi}(X_\vartheta) \right] \quad \text{where} \quad \hat{\chi}(x) := \hat{\Pi}_S(x) - I_S. \tag{5.4'}$$

We define

$$\mathcal{L} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + \mu x \frac{\partial}{\partial x} - r\mathbf{1}.$$

After standard computations,

$$\mathcal{L}\hat{\chi}(x) = -\left[r - m(\tfrac{1}{\delta})\right] \left[\frac{(1-\delta)^2}{\kappa_0} \right]^{\frac{1-\delta}{\delta}} x^{\frac{1}{\delta}} + rI_S - \left\{ \left(r - m(\tfrac{1}{\delta})\right) b x^{\frac{1}{\delta}} - rK \right\} \mathbb{1}_{(\hat{x}_S, \infty)}(x), \quad \forall x \neq \hat{x}_S.$$

But, from the study of $\theta \mapsto \frac{1}{2}\sigma^2\theta(\theta-1) + \mu\theta - r$, we know that

$$\hat{x}_S > \left(\frac{r}{r - m(\frac{1}{\delta})} K \right)^{\delta} \quad \Longrightarrow \quad \left(r - m(\tfrac{1}{\delta})\right) b x^{\frac{1}{\delta}} - rK > 0, \quad \forall x > \hat{x}_S.$$

It follows from the above and the assumption $r > m(1/\delta)$ that $\mathcal{L}\hat{\chi}$ is continuous and decreasing on $(0, \hat{x}_S)$ and on $(\hat{x}_S, \infty)$. It has a negative jump at $\hat{x}_S$ and has the limits $\mathcal{L}\chi(0) = rI > 0$ and $\mathcal{L}\hat{\chi}(\infty) = -\infty$. Consequently, there exists a point $\xi_S > 0$ such that $\mathcal{L}\hat{\chi} > 0$ on $(0, \xi_S)$ and $\mathcal{L}\hat{\chi} < 0$ on (ξ_S, ∞). According to Villeneuve (2007), the stopping region is thus an interval $[\bar{x}_S, +\infty)$.

E Proof of Proposition 5.3

We define

$$V_S(x) := \sup_{\tau} \mathbb{E}_x \left[e^{-r\tau} \chi(X_\tau) \right], \quad \text{where} \quad \chi(x) := \beta_0 x^{1/\delta} - I_S. \tag{E.1}$$

We made a slight abuse of notations above as V_S defined in eq. (4.7) uses β in lieu of β_0. We recall the problem in eq. (5.4') and note $\hat{\tau}^\star := \inf \left\{ t \geq 0 \middle| \hat{V}_S(X_t) = \hat{\chi}(X_t) \right\}$.

By optimality of $\tau^\star := \inf \left\{ t \geq 0 \middle| V_S(X_t) = \chi(X_t) \right\}$, we have $V_S(x) \geq \mathbb{E}_x e^{-r\hat{\tau}^\star} \chi(X_{\hat{\tau}^\star})$. But, by Dynkin's formula,

$$\mathbb{E}_x e^{-r\hat{\tau}^\star} \chi(X_\tau) - \chi(x) = \mathbb{E}_x \int_0^{\hat{\tau}^\star} e^{-rt} \mathcal{L}\chi(X_t) dt.$$

Hence, $(V_S - \chi)(x) \geq \mathbb{E}_x \int_0^{\hat{\tau}^\star} e^{-rt} \mathcal{L}\chi(X_t) dt$.

From the decomposition in eq. (D.1) and linearity of the operator $\mathcal{L}$, we have $\mathcal{L}\hat{\chi} = \mathcal{L}\chi + \mathcal{L}f$ almost everywhere. But, from the theory of optimal stopping, f is r-excessive. So $\mathcal{L}\hat{\chi} \leq \mathcal{L}\chi$ understood almost everywhere. Hence,

$$(V_S - \chi)(x) \geq \mathbb{E}_x \int_0^{\hat{\tau}^\star} e^{-rt} \mathcal{L}\hat{\chi}(X_t) dt = (\hat{V}_S - \hat{\Pi})(x)$$

by optimality of $\hat{\tau}^\star$ and Dynkin's formula. Because $\hat{V}_S$ dominates the payoff function $\hat{\Pi}$, it follows that $V_S - \chi \geq \hat{V}_S - \hat{\Pi} \geq 0$.

Consequently, a x in $\mathcal{S} := \{V_S = \chi\}$ belongs to $\hat{\mathcal{S}} := \{\hat{V}_S = \hat{\chi}\}$. As we know from Propositions 4.2 and 5.2 that $\mathcal{S} = (x_S, \infty)$ and $\hat{\mathcal{S}} = (\bar{x}_S, \infty)$, we necessarily have $\bar{x}_S \leq x_S$.

F Proof of Proposition 5.4

Recall $\hat{\Pi}_P$ in eq. (5.7). It follows from eq. (5.3) and the definition of $\hat{\tau}_S := \inf\{t \geq 0 \big| X_t \geq \hat{x}_S\}$ that

$$\begin{aligned}
\frac{\hat{\Pi}_P(x)}{1-\delta} &= \hat{\Pi}_S(x) + \mathbb{E}_x\left[\int_{\hat{\tau}_S}^{\hat{\tau}_S+\hat{T}} e^{-rt}k\,dt\right] \\
&= \hat{\Pi}_S(x) + \mathbb{E}_x\Big[e^{-r\hat{\tau}_S}\mathbb{E}\Big[\int_0^{\hat{T}} e^{-rt}k\,dt\Big|\mathcal{F}_{\hat{\tau}_S}\Big]\Big] \text{ by the strong Markov property} \\
&= \hat{\Pi}_S(x) + \frac{k}{r+\hat{\lambda}}\mathbb{E}_x\big[e^{-r\hat{\tau}_S}\big] \\
&= \hat{\Pi}_S(x) + K\mathbb{E}_x\big[e^{-r\hat{\tau}_S}\big]
\end{aligned}$$

by definition of K in eq. (5.6). Therefore, it follows from $\mathbb{E}_x\big[e^{-r\hat{\tau}_S}\big] = (x/[x \vee \hat{x}_S])^\theta$ and eq. (5.5) that

$$\frac{\hat{\Pi}_P(x)}{1-\delta} = \beta_0 x^{\frac{1}{\delta}} + b\hat{x}_S^{\frac{1}{\delta}}\Big(\frac{x}{\hat{x}_S}\Big)^\theta \mathbb{1}_{\{x<\hat{x}_S\}} + bx^{\frac{1}{\delta}}\mathbb{1}_{\{x\geq\hat{x}_S\}}, \tag{F.1}$$

which is eq. (5.8).

We note that

$$\hat{\Pi}_P(\hat{x}_S-) = \hat{\Pi}_P(\hat{x}_S-) \quad \text{and} \quad \hat{\Pi}'_P(\hat{x}_S-) - \hat{\Pi}'_P(\hat{x}_S+) = (1-\delta)b\hat{x}_S^{\frac{1}{\delta}-1}\frac{\theta\delta-1}{\delta},$$

which is strictly positive because $b > 0$ and $\theta\delta > 1$ according to Remark 4.1.

To prove that the optimal strategy is of the threshold type, we use the proof in Villeneuve (2007, Theorem 4.1) by computing $\mathcal{L}(\hat{\Pi}_P - I_P)$ understood in the sense of distributions. $\mathcal{L}(\hat{\Pi}_P - I_P)$ is a negative measure at $\hat{x}_S$ and so the optimal policy is a threshold policy.

This completes the proof.