

THE ECONOMICS OF PARTISAN GERRYMANDERING

ANTON KOLOTILIN

School of Economics, UNSW Business School

ALEXANDER WOLITZKY

Department of Economics, MIT

We study the problem of a partisan gerrymanderer who assigns voters to equipopulous districts to maximize his party’s expected seat share. The designer faces both aggregate, district-level uncertainty (how many votes his party will receive) and idiosyncratic, voter-level uncertainty (which voters will vote for his party). *Pack-and-pair districting*, where weaker districts are “packed” with a single type of voter, while stronger districts contain two voter types, is optimal for the gerrymanderer. The optimal form of pack-and-pair districting depends on the designer’s popularity and the relative amounts of aggregate and idiosyncratic uncertainty. When idiosyncratic uncertainty dominates, a designer with majority support pairs all voters, while a designer with minority support packs opposing voters and pairs more favorable voters; these plans resemble uniform districting and “packing-and-cracking,” respectively. When aggregate uncertainty dominates, the designer packs moderate voters and pairs extreme voters; this “matching slices” plan has received some attention in the literature. Estimating the model using precinct-level returns from recent US House elections shows that, in practice, idiosyncratic uncertainty dominates. We discuss implications for redistricting reform, political polarization, and detecting gerrymandering. Methodologically, we exploit a formal connection between gerrymandering—partitioning voters into districts—and information design—partitioning states of the world into signals.

KEYWORDS: Gerrymandering, pack-and-crack, pack-and-pair, information design.

1. INTRODUCTION

Legislative district boundaries are drawn by political partisans under many electoral systems (Bickerstaff, 2020). In the United States, the significance of partisan districting has grown with the rise of computer-assisted districting (Newkirk, 2017), together with intense partisan efforts to gain and exploit control of the districting process. These trends culminated in “The Great Gerrymander of 2012” (McGhee, 2020), where the Republican party’s Redistricting Majority Project (REDMAP), having previously targeted state-level elections that would give Republicans control of redistricting, aggressively redistricted several states, including Michigan, Ohio, Pennsylvania, and Wisconsin. The resulting districting plans are widely viewed as contributing to the outcome of the 2012 general election, where Republican congressional candidates won a 33-seat majority in the House of Representatives with 49.4% of the two-party vote (McGann, Smith, Latner, and Keena, 2016). In light of these developments—along with the Supreme Court ruling in *Rucho v. Common Cause* (2019) that partisan gerrymanders are not judiciable in federal court and the continued prominence of gerrymandering in the 2020 US redistricting

Anton Kolotilin: akolotilin@gmail.com

Alexander Wolitzky: wolitzky@mit.com

We thank Nikhil Agarwal, Garance Genicot, Ben Golub, Richard Holden, Gary King, Hongyi Li, Nolan McCarty, Stephen Morris, Ben Olken, Ken Shotts, and Leeat Yariv, as well as many seminar and conference participants, for helpful comments. We thank Eitan Sapiro-Gheiler and Nancy Wang for excellent research assistance. Anton Kolotilin acknowledges support from the Australian Research Council Discovery Early Career Research Award DE160100964 and MIT Sloan’s Program on Innovation in Markets and Organizations. Alexander Wolitzky acknowledges support from NSF CAREER Award 1555071 and Sloan Foundation Fellowship 2017-9633.

cycle (Rakich and Mejia, 2022)—partisan gerrymandering looks likely to remain an important feature of American politics for some time.

This paper studies the problem of a partisan gerrymanderer (the “designer”) who assigns voters to a large number of equipopulous districts so as to maximize his party’s expected seat share.¹ This problem approximates the one facing many partisan gerrymanderers in the United States, where the constraint that districts must be equipopulous is strictly enforced.² In practice, gerrymanderers also face additional constraints, such as the federal requirements that districts are contiguous and do not discriminate on the basis of race, and various state-level restrictions such as “compactness” requirements, requirements to respect political sub-divisions such as county lines, requirements to represent racial or ethnic groups or other communities of interest, and so on. While these complex constraints can be important, we believe that often they are not as binding as they might seem, and also that they are more productively considered on a case-by-case basis rather than as part of a general theoretical analysis.³ We therefore follow much of the literature (discussed below) in focusing on the simpler problem with only the equipopulation constraint.

When the designer has perfect information, the solution to this problem is well-known. If the designer’s party is supported by a minority of voters of size $m < 1/2$, he “packs” $1 - 2m$ opposing voters in districts where he receives zero votes and “cracks” the remaining $2m$ voters in districts which he wins with 50% of the vote. If the designer has majority support, he can win all districts by making them identical. Thus, under perfect information, *pack-and-crack* is optimal for a designer with minority support, while *uniform districting* is optimal for a designer with majority support. We instead consider the more general and realistic case where the designer must allocate a variety of types of voters (or, more realistically, groups of voters such as census blocks or precincts) under uncertainty. The goal of this paper is to characterize optimal partisan gerrymandering in this setting and to draw implications for some broader legal and political economy issues surrounding gerrymandering.

In outline, our model and results are as follows. We assume that the designer faces both aggregate, district-level uncertainty (how many votes his party will receive) and idiosyncratic, voter-level uncertainty (which voters will vote for his party). Aggregate uncertainty is parameterized by a one-dimensional aggregate shock, while voters are parameterized by a one-dimensional type that determines a voter’s probability of voting for the designer’s party for each value of the aggregate shock. We assume that the distributions of the aggregate and idiosyncratic shocks are symmetric and unimodal with log-concave densities drawn from the same location-scale family. This assumption lets us cleanly compare the “amounts” of aggregate and idiosyncratic uncertainty.

Our first result is that optimal districting takes a *pack-and-pair* form. Under pack-and-pair districting, the designer creates weaker districts that contain a single voter type (which are analogous to the packed districts under pack-and-crack) and stronger districts that contain two voter types (which are analogous to the cracked districts under pack-and-crack). The class of

¹We hasten to add that studying this problem does not endorse gerrymandering, any more than studying monopolistic behavior endorses monopoly.

²In *Karcher v. Daggett* (1983), the Supreme Court rejected a districting plan in New Jersey with less than a 1% deviation from population equality, finding that “there are no *de minimus* population variations, which could practically be avoided, but which nonetheless meet the standard of Article I, Section 2 [of the U.S. Constitution] without justification.”

³An exception is the requirement to respect county lines, which we address in Section 6.2. See Friedman and Holden (2008) for discussion of the other constraints. For example, contiguity is not as severe a constraint as it might seem, because contiguous districts can have highly irregular shapes. The title of this paper, typeset in Gerry font (<https://www.uglygerry.com/>), contains many examples of irregularly shaped districts.

pack-and-pair plans admits a tight characterization but is rich enough to cover a variety of districting plans, including refinements of all of the main plans proposed in the prior literature. The optimality of pack-and-pair districting is thus a key organizing result.

We then turn to our main substantive results, which characterize optimal districting as a function of the designer's popularity and the relative amounts of idiosyncratic and aggregate uncertainty. First, we show that if the designer has strong support from all voter types, then a *negative assortative districting (NAD)* plan is optimal, where extreme left and right-wing voters are paired together. Conversely, if the designer has weak support from all voter types and idiosyncratic uncertainty is larger than aggregate uncertainty, then a *segregation* plan is optimal, where each district contains only a single voter type. Second, if aggregate uncertainty is very small, optimal districting for a designer with majority support approximates NAD, while optimal districting for a designer with minority support approximates a *pack-opponents-and-pair (POP)* plan, where unfavorable voters are segregated and more favorable voters are paired in a negatively assortative manner. The former result is analogous to the optimality of uniform districting for a designer with majority support without uncertainty, because NAD plans are versions of uniform districting that pair voter types rather than pooling all types together; similarly, the latter result is analogous to the optimality of pack-and-crack districting for a designer with minority support without uncertainty, because POP plans are versions of pack-and-crack districting that segregate unfavorable voters and pair more favorable voter types rather than pooling them. Indeed, while exactly optimal districting with small aggregate uncertainty approximates NAD or POP, much simpler uniform districting or pack-and-crack plans are approximately optimal (for the cases of majority and minority designer support, respectively). Third, if idiosyncratic uncertainty is very small, optimal districting approximates NAD with a 50-50 voter type split in each district.⁴ Fourth, in the intermediate region where both the designer's support among voters and the ratio of aggregate and idiosyncratic uncertainty are balanced, mixed plans can be optimal (as well as a *pack-moderates-and-pair* plan, where moderate voters are segregated and extreme left and right-wing voters are paired), and we can numerically trace out the boundaries of the parameter regions where each type of plan is optimal.

As we discuss in Section 6, the form of optimal partisan districting has significant implications for several political and legal issues, including redistricting reform, intra- and inter-district political polarization, and measuring gerrymandering. Since our results show that the ratio of idiosyncratic and aggregate uncertainty is a key determinant of the form of optimal districting, it is therefore important to understand whether idiosyncratic or aggregate uncertainty is larger in practice. We answer this question using precinct-level returns from the 2016, 2018, and 2020 US House elections. The data clearly show that idiosyncratic uncertainty is much larger than aggregate uncertainty. Intuitively, this finding results from the simple observation that, in practice, most precinct vote splits are much closer to 50-50 (the vote split under high idiosyncratic uncertainty) than 100-0 or 0-100 (the vote splits under high aggregate uncertainty).⁵ Therefore, for the realistic parameter range, exactly optimal districting approximates NAD (for a designer with majority voter support) or POP (for a designer with minority support), while uniform districting or pack-and-crack is approximately optimal. This finding can help explain why actual gerrymandering usually resembles uniform districting or pack-and-crack.

Methodologically, we establish a formal connection between gerrymandering—partitioning voters into districts—and information design—partitioning states of the world into signals.

⁴This result refines the main result of [Friedman and Holden \(2008\)](#).

⁵This observation also implies that simpler models with only two types of voters or precincts (e.g., [Owen and Grofman 1988](#)) cannot closely approximate the problem facing actual gerrymanderers, who must assign many different types of precincts.

The partisan gerrymandering problem we study is mathematically equivalent to a non-linear Bayesian persuasion problem with a one-dimensional state, a one-dimensional action for the receiver, and state-independent sender preferences. Our results are novel in the context of this persuasion problem, so this paper directly contributes to information design as well as gerrymandering. More importantly, we establish a strong connection between these topics.⁶

Related Literature. The closest prior papers on optimal partisan gerrymandering are [Owen and Grofman \(1988\)](#), [Friedman and Holden \(2008\)](#), and [Gul and Pesendorfer \(2010\)](#). [Owen and Grofman](#)'s model is equivalent to the special case of our model with two voter types. [Gul and Pesendorfer](#) study competition between two designers who each control districting in some area and aim to win a majority of seats.⁷ A simplified version of their model with a single designer is equivalent to the special case of our model with uniform idiosyncratic shocks. [Friedman and Holden](#) consider a model similar to ours (although with finitely many districts, rather than a continuum as in our model and [Gul and Pesendorfer](#)), but their main results concern the case where idiosyncratic uncertainty is much smaller than aggregate uncertainty. In contrast, we do not restrict the relative amounts of aggregate and idiosyncratic uncertainty, and we show empirically that the practically relevant case is the one where idiosyncratic uncertainty dominates (i.e., the opposite of the case emphasized by [Friedman and Holden](#)).

The broader literature on gerrymandering and redistricting addresses a wide range of issues, including geographic constraints on gerrymandering ([Sherstyuk, 1998](#), [Shotts, 2001](#), [Puppe and Tasnádi, 2009](#)), gerrymandering with heterogeneous voter turnout ([Bouton, Genicot, Castanheira, and Stashko, 2023](#), [Gomberg, Pancs, and Sharma, 2024](#)), socially optimal districting ([Gilligan and Matsusaka, 2006](#), [Coate and Knight, 2007](#), [Bracco, 2013](#)), measuring district compactness ([Chambers and Miller, 2010](#), [Fryer and Holden, 2011](#), [Ely, 2022](#)), the interaction of redistricting and policy choices ([Shotts, 2002](#), [Besley and Preston, 2007](#), [Groll and O'Halloran, 2024](#)), measuring gerrymandering ([King and Browning, 1987](#), [McGhee, 2014](#), [Stephanopoulos and McGhee, 2015](#), [Deford, Duchin, and Solomon, 2021](#), [Gomberg, Pancs, and Sharma, 2023](#)), and assessing the consequences of redistricting (among many: [Gelman and King, 1994b](#), [McCarty, Poole, and Rosenthal, 2009](#), [Hayes and McKee, 2009](#), [Jeong and Shenoy, 2022](#), [Sabet and Yuchtman, 2024](#)). As the partisan gerrymandering problem interacts with many of these issues, our analysis may facilitate future research in these areas.

Outline. The paper is organized as follows: Section 2 presents the model. Section 3 establishes general properties of optimal districting plans that hold regardless of the designer's popularity or the amount of aggregate and idiosyncratic uncertainty. Section 4 contains our main theoretical and numerical results, which characterize optimal districting as a function of these parameters. Section 5 contains our empirical results, which estimate which parameters are the practically relevant ones. Section 6 discusses policy implications. Section 7 concludes. All proofs are deferred to the appendix.

2. MODEL

We consider a standard electoral model with one-dimensional voter types (parameterizing voter partisanship) and one-dimensional aggregate uncertainty in each district-level race (parameterizing the vote share for the designer's party).

⁶Contemporaneous papers by [Lagarde and Tomala \(2021\)](#) and [Gomberg, Pancs, and Sharma \(2023\)](#) also emphasize connections between gerrymandering and information design, albeit in less general models: [Lagarde and Tomala](#) assume two voter types, while [Gomberg, Pancs, and Sharma](#) assume no aggregate uncertainty. The closest paper in the persuasion literature is our companion paper, [Kolotilin, Corrao, and Wolitzky \(2024\)](#), which we discuss later on.

⁷[Friedman and Holden \(2020\)](#) study designer competition in the model of their 2008 paper.

Voters and Vote Shares. There is a continuum of voters. A voter votes for the designer’s party (for short, “votes for the designer”) iff

$$s - r - t \geq 0,$$

where

- $s \in [\underline{s}, \bar{s}]$, with $\underline{s} < \bar{s}$, is the voter’s type, which is observed by the designer and is the object of districting. The population distribution of s is denoted by F .
- $r \in \mathbb{R}$ is the aggregate shock in the voter’s district, which realizes after districting. The distribution of r in each district is denoted by G .⁸
- $t \in \mathbb{R}$ is an idiosyncratic, voter-specific “taste shock,” which also realizes after districting. The distribution of t is denoted by Q .

Thus, the share of type- s voters who vote for the designer in a district where the aggregate shock takes value r equals $Q(s - r)$.

Note that the designer faces two kinds of uncertainty at the time of districting: aggregate, district-level uncertainty, r , and idiosyncratic, voter-level uncertainty, t . Many of our results turn on a comparison of the “amount” of each kind of uncertainty. To facilitate this comparison, we assume that G and Q have the same shape, in that there exists $\eta > 0$ such that $G(r) = Q(\eta r)$.⁹ We also define $\gamma = \eta^2 / (1 + \eta^2) \in (0, 1)$, so the ratio of the variances of r and t is $(1 - \gamma) / \gamma$. The parameter γ thus captures the *share of idiosyncratic uncertainty*. We say that aggregate uncertainty is *larger* than idiosyncratic uncertainty if $\gamma < 0.5$, while idiosyncratic uncertainty is larger if $\gamma > 0.5$.

The model is now fully parameterized by the distributions F and Q and the parameter $\gamma \in (0, 1)$. We assume that F and Q are four-times differentiable and that the corresponding densities f and q are strictly positive. We also assume that q is symmetric about 0 and strictly log-concave:

$$\frac{d^2}{dt^2} \ln(q(t)) < 0 \quad \text{for all } t.$$

This implies that Q is strictly convex below 0 and strictly concave above 0, with $Q(0) = 1/2$.

Log-concavity of q is a key assumption. This standard property is satisfied by many distributions (Bagnoli and Bergstrom, 2005) and is similar to Friedman and Holden’s (2008) “Informative Signal Property” assumption. Substantively, it captures the realistic feature that moderate voters are more sensitive to the aggregate shock than more extreme voters.¹⁰

Districting Plans. The designer assigns voters to a continuum of equipopulous districts based on their types s , and thus determines the distribution P of s in each district.¹¹ A district is characterized by the distribution P of voter types s it contains. Thus, a *districting plan*—which specifies the measure of districts with each voter type distribution P —is a distribution

⁸The correlation among district-level aggregate shocks is irrelevant for our analysis. However, we do estimate it empirically.

⁹Mathematically, this says that G and Q lie in the same location-scale family. An earlier version of this paper, Kolotilin and Wolitzky (2020), contains additional results where G and Q have different shapes.

¹⁰As Nathaniel Rakich and Nate Silver put it in describing the “elasticity scores” in FiveThirtyEight.com’s forecasting model, “Voters at the extreme end of the spectrum—those who have close to a 0 percent or a 100 percent chance of voting for one of the parties—don’t swing as much as those in the middle,” (Rakich and Silver, 2018).

¹¹We follow Gul and Pesendorfer (2010) in assuming a continuum of districts. Since districting plans in the US are drawn at the state level, this implicitly assumes that each state contains a large number of districts. Of course, this is a better approximation for state legislative districts and for congressional districts in large states than it is for congressional districts in small states. Introducing integer constraints on the number of districts, while interesting and realistic, would complicate the analysis and obscure our insights.

\mathcal{H} over distributions P of s , such that the population distribution of s is given by F : that is, $\mathcal{H} \in \Delta\Delta[\underline{s}, \bar{s}]$ and

$$\int P(s)d\mathcal{H}(P) = F(s) \text{ for all } s.^{12}$$

For example, under *uniform districting*, where all districts are the same, \mathcal{H} assigns probability 1 to $P = F$. In the opposite extreme case of *segregation*, where each district consists entirely of one type of voter, every distribution P in the support of \mathcal{H} takes the form $P = \delta_s$ for some $s \in [\underline{s}, \bar{s}]$, where δ_s denotes the degenerate distribution on voter type s . Finally, the best-known districting plan is *pack-and-crack*, where there is a cutoff voter type $s^* \in (\underline{s}, \bar{s})$ such that $\text{supp}(\mathcal{H}) = \{P, P'\}$ and P and P' are the lower and upper truncations of F at s^* .

We say that a districting plan is *pure* if (almost) each voter type s is assigned to only one kind of district (so there is a unique $P \in \text{supp}(\mathcal{H})$ such that $s \in \text{supp}(P)$) and *mixed* otherwise. Since the distribution of voter types F is continuous, it is natural to expect pure districting to be optimal, but we will see that this is not always the case.

Designer's Problem. The designer wins a district iff he receives a majority of the district vote. Thus, the designer wins a district with voter type distribution P (henceforth, "district P ") iff the district's aggregate shock r satisfies $\int Q(s - r)dP(s) \geq 1/2$. Since $Q(s - r)$ is decreasing in r and $Q(0) = 1/2$, the designer wins district P iff

$$r \leq r^*(P) := \left\{ r : \int Q(s - r)dP(s) = Q(0) \right\}.$$

Note that the threshold shock $r^*(P)$ to win a segregated district $P = \delta_s$ is simply s , while in general $r^*(P)$ lies somewhere in the convex hull of $\text{supp}(P)$. We say that a district P is *weaker* than another district P' if $r^*(P) < r^*(P')$. Since the aggregate shock has the same distribution in all districts, the designer wins weaker districts with lower probability.

We assume that the designer maximizes his party's expected seat share.¹³ Thus, the designer's problem is

$$\begin{aligned} & \max_{\mathcal{H} \in \Delta\Delta[\underline{s}, \bar{s}]} \int G(r^*(P))d\mathcal{H}(P) \\ & \text{s.t. } \int Pd\mathcal{H}(P) = F. \end{aligned}$$

This problem is similar to that of [Friedman and Holden \(2008\)](#), which in turn nests [Owen and Grofman \(1988\)](#) and a single-designer version of [Gul and Pesendorfer \(2010\)](#).¹⁴ It is also equivalent to a Bayesian persuasion problem where the designer splits a prior distribution F into posterior distributions P and obtains utility $G(r^*(P))$ from inducing posterior P .¹⁵

¹²Throughout, for any compact metric space X , ΔX denotes the set of Borel probability measures on X , endowed with the weak* topology. For any $\mu \in \Delta X$, its support $\text{supp}(\mu)$ is the smallest compact set of measure one.

¹³See Section 7 and [Kolotilin and Wolitzky \(2020\)](#) for discussion of more general designer objectives.

¹⁴[Friedman and Holden](#) assume a finite number of districts rather than a continuum and do not assume that G and Q have the same shape. [Owen and Grofman](#) assume binary voter types. [Gul and Pesendorfer](#) consider a majoritarian objective with both state-level and district-level aggregate shocks; however, after conditioning on the pivotal value of the state-level shock, their problem reduces to maximizing expected seat share with only district-level shocks.

¹⁵Specifically, the designer's problem lies in the translation-invariant subcase of the *state-independent sender* case of the persuasion problem studied in [Kolotilin, Corrao, and Wolitzky \(2024\)](#), which specializes the general Bayesian persuasion problem of [Kamenica and Gentzkow \(2011\)](#) by assuming that the state and the receiver's action are one-

3. OPTIMAL PARTISAN GERRYMANDERING: GENERAL PROPERTIES

We first establish two general properties of optimal districting plans that hold regardless of the designer’s popularity or the amount of aggregate and idiosyncratic uncertainty. The first of these, *single-dippedness*, is the key property identified by [Friedman and Holden \(2008\)](#)—we just re-establish their result in our continuum-district model. The second property, *pack-and-pairedness*, is novel.

3.1. *Single-Dippedness*

We first show that optimal districting plans are *strictly single-dipped*, in that more extreme voters are assigned to stronger districts: formally, any district $P \in \text{supp}(\mathcal{H})$ containing any two voter types $s < s''$ is stronger than any district $P' \in \text{supp}(\mathcal{H})$ containing any intervening voter type $s' \in (s, s'')$, in that $r^*(P') < r^*(P)$.¹⁶ Note that if districting is strictly single-dipped then each district contains at most two distinct voter types. Thus, any district P in the support of a strictly single-dipped districting plan \mathcal{H} is either *segregated* (if $|\text{supp}(P)| = 1$) or *paired* (if $|\text{supp}(P)| = 2$). For example, segregation is strictly single-dipped, but uniform districting and pack-and-crack are not.

LEMMA 1: *Any optimal districting plan is strictly single-dipped.*

Lemma 1 recapitulates Lemma 1 of [Friedman and Holden \(2008\)](#) in our continuum-district model.¹⁷ To see the intuition, suppose a districting plan creates two districts, 1 and 2, with the same threshold aggregate shock r^* , but where District 1 contains moderate voters and District 2 contains a mix of left-wing and right-wing extremists. Since q is log-concave, the vote share is more sensitive to the aggregate shock in District 1 than in District 2, which implies that a marginal voter is more likely to be pivotal in District 2 than in District 1. The designer can then profitably exploit this asymmetry by re-assigning some unfavorable voters to District 1 and re-assigning some favorable voters to District 2, thus weakening the moderate District 1 and strengthening the extreme District 2. Breaking all ties in favor of extreme districts in this manner leads to strictly single-dipped districting.

3.2. *Pack-and-Pairedness*

A strictly single-dipped districting plan can contain a mix of segregated and paired districts of varying strengths. If such a plan has the further property that every segregated district is weaker than every paired district (i.e., there do not exist $P, P' \in \text{supp}(\mathcal{H})$ such that $|\text{supp}(P)| = 1$, $|\text{supp}(P')| = 2$, and $r^*(P) > r^*(P')$), we say that the plan is *pack-and-pair*.

A pack-and-pair plan \mathcal{H} can be described in a simple way. There is a *bifurcation point* $r^b \in [\underline{s}, \bar{s}]$ that divides the segregated and paired districts, so that $r^*(P) \leq r^b$ for all segregated districts $P \in \text{supp}(\mathcal{H})$, and $r^*(P) > r^b$ for all paired districts $P \in \text{supp}(\mathcal{H})$. The assignment

dimensional, the receiver’s utility is supermodular and concave in her action, and the sender’s utility is independent of the state and increasing in the receiver’s action. In the gerrymandering context, the designer’s preferences are state-independent because he only cares about how many districts he wins and not directly about the districts’ composition.

¹⁶We say that a district P “contains” a voter type s if $s \in \text{supp}(P)$.

¹⁷[Kolotilin, Corrao, and Wolitzky \(2024\)](#) give sufficient conditions for single-dippedness in a more general model that allows state-dependent designer preferences.

of voters to paired districts is then described by a decreasing function s_1 and an increasing function s_2 , where the two types in a paired district P are $s_1(r^*(P))$ and $s_2(r^*(P)) > s_1(r^*(P))$. Stronger paired districts thus contain more extreme voters, as single-dippedness requires.¹⁸

Despite this tight characterization, a range of interesting districting plans are pack-and-pair, including the following:

Segregation, where all voters are segregated: $P = \delta_{r^*(P)}$ for all $P \in \text{supp}(\mathcal{H})$, or equivalently $r^b = \bar{s}$.

Pack-Moderates-and-Pair (PMP), where moderate voters are segregated and extreme voters are paired in a negatively assortative manner: \mathcal{H} is pure, $r^b \in (\underline{s}, \bar{s})$, and there exists $\hat{s} \in (\underline{s}, r^b)$ such that a district $P \in \text{supp}(\mathcal{H})$ is segregated iff $r^*(P) \in [\hat{s}, r^b]$.

Pack-Opponents-and-Pair (POP), where unfavorable voters are segregated and more favorable voters are paired in a negatively assortative manner: \mathcal{H} is pure, $r^b \in (\underline{s}, \bar{s})$, and there exists $\hat{s} \in (\underline{s}, r^b)$ such that a district $P \in \text{supp}(\mathcal{H})$ is segregated iff $r^*(P) \in [\underline{s}, \hat{s}]$.

Negative Assortative Districting (NAD), where all voters are paired in a negatively assortative manner: $r^b = \inf_{P \in \text{supp}(\mathcal{H})} r^*(P)$.

These four plans feature prominently in our results (as illustrated in Figures 1–3) and warrant some discussion. First, segregation and NAD are the extreme pack-and-pair plans where all voter types are segregated and where only a single type is segregated. There is a unique segregation plan, but there is a continuum of NAD plans, depending on the weights on the different voter types in each paired district. (Similarly, there is also a continuum of PMP and POP plans.) NAD plans can be viewed as “strictly single-dipped versions” of uniform districting: starting from uniform districting and splitting the pool of voters into pairs in a strictly single-dipped manner yields NAD.

Similarly, POP plans are strictly single-dipped version of [Gul and Pesendorfer’s \(2010\)](#) “ p -segregation” plan, where unfavorable voters are segregated and more favorable voters are pooled: starting from p -segregation and splitting the pool into pairs yields POP. POP can also be obtained from pack-and-crack districting by first splitting the weak districts into segregated ones (yielding p -segregation) and then splitting the strong districts into pairs.

Finally, PMP is the same as [Friedman and Holden’s \(2008\)](#) “matching slices” plan, with the difference that [Friedman and Holden](#) assume a finite number of districts and do not mention the possibility of segregating a non-trivial interval of moderate voter types.

An instructive example of a plan that can be strictly single-dipped but not pack-and-pair is “Pack-Supporters-and-Pair,” where favorable voters are segregated and less favorable voters are paired. This plan can be obtained from pack-and-crack by splitting weak districts into pairs and splitting strong districts into segregated ones.

Our first main result is that pack-and-pair districting is optimal if idiosyncratic uncertainty is larger than aggregate uncertainty.

THEOREM 1: *If idiosyncratic uncertainty is larger than aggregate uncertainty, there is a unique optimal districting plan, which is pack-and-pair.*

Numerically, pack-and-pair districting is also optimal when idiosyncratic uncertainty is smaller than aggregate uncertainty, but we were not able to prove this.¹⁹ However, Theorem 1 covers the empirically relevant case, as we will estimate that γ is much greater than 0.5.

¹⁸Lemma 5 in Appendix A formalizes the description of a pack-and-pair plan by a bifurcation point r^b and functions s_1 and s_2 . In particular, we define the bifurcation point as the infimum of $r^*(P)$ over all paired districts $P \in \text{supp}(\mathcal{H})$.

¹⁹See Figures 1–3, where all optimal plans are pack-and-pair. In these figures, G and Q are normal. Using Lemma 6 in Appendix A, we have checked numerically that pack-and-pair is also always optimal when G and Q are logistic.

The intuition for Theorem 1 is as follows. First, log-concavity of g implies that G , the distribution of the aggregate shock, is first convex and then concave. Second, convexity of G favors segregation (as splitting a district with threshold aggregate shock r^* into districts with threshold shocks $r^* - \varepsilon$ and $r^* + \varepsilon$ increases expected seat share when G is convex and Q is linear), while concavity of G favors pairing. Thus, when G is first convex and then concave, weak districts should be segregated and strong districts should be paired. This is precisely pack-and-pair.²⁰

Theorem 1 is a fundamental result: it is optimal to make paired districts stronger than segregated ones. However, while pack-and-pair plans admit a tight characterization, we have seen that a wide variety of plans are pack-and-pair. The next section characterizes optimal plans as a function of F , Q , and γ .

As an aside, we note that Theorem 1 also contributes to the Bayesian persuasion literature. An important disclosure policy that frequently arises in this literature is *upper censorship*, where states below a cutoff are disclosed and states above the cutoff are pooled. Upper censorship is often optimal in “linear” persuasion problems, where a posterior can be summarized by its mean (Kolotilin, 2018, Kolotilin, Mylovannov, and Zapechelnyuk, 2022). However, in non-linear persuasion problems, a version of strict single-dippedness often holds, so disclosure policies that pool more than two states (like upper censorship) cannot be optimal (Kolotilin, Corrao, and Wolitzky, 2024). This raises the question of when a strictly single-dipped version of upper censorship—such as a pack-and-pair policy—is optimal. Theorem 1 is the first result in the literature to give sufficient conditions for such policies to be optimal.

4. OPTIMAL PARTISAN GERRYMANDERING IN DIFFERENT PARAMETER REGIMES

We now present our results on optimal districting as a function of the designer’s popularity and the ratio of idiosyncratic and aggregate uncertainty. First, NAD is optimal if the designer has strong support from all voter types, and segregation is optimal if the designer has weak support from all voter types and idiosyncratic uncertainty is larger than aggregate uncertainty (Theorem 2). Second, optimal plans approximate NAD or POP with equally strong paired districts if aggregate uncertainty is small (Theorem 3). Since we will estimate that aggregate uncertainty is small empirically, Theorem 3 is our most practically relevant result. However, while exactly optimal plans approximate NAD or POP, uniform districting or pack-and-crack districting are also approximately optimal. Third, optimal plans approximate NAD with a 50-50 voter type split in each district if idiosyncratic uncertainty is small (Theorem 4). Fourth, in the intermediate region where both the designer’s support among voters and the ratio of idiosyncratic and aggregate uncertainty are balanced, mixed versions of POP and PMP can be optimal, and we can numerically trace out the boundaries of the parameter regions where each type of plan emerges (Theorem 5 and the subsequent numerical results). Overall, we give a fairly complete picture of how optimal districting varies with the designer’s support and the ratio of idiosyncratic and aggregate uncertainty, which we illustrate in Figure 3 at the end of this section.

4.1. *Optimal Districting with Imbalanced Voter Support*

We first investigate optimal districting when voter support is highly imbalanced between the parties. This case is relatively simple and is not too unrealistic: we will estimate that in around

The normal and logistic families are the only standard location-scale families we are aware of with symmetric and strictly log-concave densities on \mathbb{R} .

²⁰A complication is that log-concavity of g always favors pairing. In Section 4.1, we explain how $\gamma > 0.5$ ensures that the log-concavity of g “dominates” that of q .

half of US states, voter support is sufficiently imbalanced that NAD (an optimal plans in the high imbalanced case) is optimal for our estimated parameters.²¹

We say that the designer has *uniformly strong support* if $\underline{s} \geq 0$. This means that, when the aggregate shock takes its modal value of 0, *all* voter types vote for the designer with probability at least 50%. Thus, a designer with uniformly strong support can only lose a district when the aggregate shock lands in the right (unfavorable) tail. Conversely, the designer has *uniformly weak support* if $\bar{s} \leq 0$, so he can only win a district when the aggregate shock lands in the left tail. Finally, the designer has *balanced support* if $r^*(F) = 0$, so the overall vote is 50-50 when the aggregate shock takes its modal value.

THEOREM 2: *The following hold:*

1. *If the designer has uniformly strong support, there is a unique optimal districting plan, which is NAD.*
2. *If the designer has uniformly weak support and idiosyncratic uncertainty is larger than aggregate uncertainty, there is a unique optimal districting plan, which is segregation.*
3. *If the designer has balanced support, NAD and segregation are both suboptimal.*

Since NAD plans are strictly single-dipped versions of uniform districting, the optimality of NAD in case 1 is akin to the optimality of uniform districting for a designer with majority support in the absence of aggregate uncertainty. To see why NAD is optimal, recall that any strictly single-dipped plan that never segregates two distinct voter types is NAD. So, since $\underline{s} \geq 0$, it suffices to show that it is sub-optimal for the designer to segregate any two voter types $s < s'$ that lie in a region where G is concave. To see this, suppose the designer pools a few type- s voters in with the type- s' voters. The marginal effect of this change on the designer's expected seat share among type- s voters is

$$G(s') - G(s),$$

which is the increased probability of winning a type- s voter's district when she moves from the weak district δ_s to the strong district $\delta_{s'}$. On the other hand, the marginal effect of this change on the designer's expected seat share among type- s' voters is

$$\frac{Q(s - s') - Q(0)}{q(0)}g(s').$$

This follows because the first term is the marginal effect on the threshold shock to win the strong district, where this comes from using the implicit function theorem to calculate $dr/d\rho$ at $\rho = 0$ from the equation

$$\rho Q(s - r) + (1 - \rho)Q(s' - r) = Q(0),$$

and the second term is the density of the aggregate shock at $r^*(\delta_{s'}) = s'$. Finally, the sum of the two effects is positive, because

$$\frac{G(s') - G(s)}{g(s')} > s' - s > \frac{Q(0) - Q(s - s')}{q(0)},$$

²¹However, the estimated parameters are *not* extreme enough to satisfy the conditions in Theorem 2, which are sufficient but not necessary for NAD to be optimal.

where the first inequality is by strict concavity of G on $[s, s']$, and the second inequality is by strict convexity of Q on $[s - s', 0]$.

The intuition for why segregation is optimal in case 2 is that, for any two voter types s and s' that lie in a region where G is “sufficiently convex” relative to Q , we have

$$\frac{G(s') - G(s)}{g(s')} < \frac{Q(0) - Q(s - s')}{q(0)},$$

which by a similar logic as above implies that it is optimal for the designer to separate any two voter types rather than pooling them. Note that it is not enough for G to be just slightly convex, because now (in contrast to case 1) the convexity of G and the convexity of Q compete in comparing the two effects above: intuitively, log-concavity of Q favors pooling, because a few unfavorable voters are unlikely to be pivotal and thus can be safely added to a stronger district. The proof of Theorem 2 shows that the convexity of G “wins” if $\gamma \geq 0.5$. Intuitively, when $\gamma \geq 0.5$, G is more convex than Q , which suffices for the above inequality.

In case 3, neither NAD nor segregation are optimal, so the optimal plan creates a mix of segregated and paired districts.²² The intuition is that the designer prefers pooling any two positive voter types, so segregation is suboptimal; but at the same time, for any strictly single-dipped NAD plan, there exist nearby voter types that are paired in a district P with $r^*(P) < 0$, and the designer is better-off segregating these types.

We remark that the logic of Theorem 1 is a more intricate variant of Theorem 2’s. Note that a strictly single-dipped plan is *not* pack-and-pair iff there exist $s < r < s' \leq s''$ such that voter types $s < s'$ are paired in a district P with $r^*(P) = r \in (s, s')$ and voter type s'' is segregated. Suppose toward a contradiction that such a plan is optimal. As in case 2 of Theorem 2, if idiosyncratic uncertainty is larger than aggregate uncertainty then pooling type- s voters in district P is worse than segregating them if $r \leq 0$ (the range where G is convex), so we must have $r > 0$. But then, it can be shown that the planner would be better-off removing a few type- s voters from district P and pooling them in with the type- s'' voters, by a similar argument as in Theorem 2.

To turn the above arguments into rigorous proofs of Theorems 1 and 2, we rely on duality and complementary slackness theorems developed in [Kolotilin, Corrao, and Wolitzky \(2024\)](#), which we restate as Lemma 2 in Appendix A. The key implication of Lemma 2 is that there is a well-defined Lagrange multiplier $\lambda(r^*(P))$ on the constraint $\int Q(s - r^*(P))dP(s) = Q(0)$, which is given by the formula

$$\lambda(r^*(P)) = \frac{g(r^*(P))}{\int q(s - r^*(P))dP(s)}$$

for all districts P in the support of an optimal plan \mathcal{H} , and that the designer only assigns type- s voters to districts P that maximize

$$G(r^*(P)) + \lambda(r^*(P))(Q(s - r^*(P)) - Q(0)).$$

Intuitively, $\lambda(r^*(P))$ is the designer’s value of an extra vote in district P , which equals the product of the designer’s marginal utility of increasing $r^*(P)$ (which equals $g(r^*(P))$) and the

²²Proposition 1 of [Friedman and Holden \(2008\)](#) shows that PMP (“matching slices”) is optimal when idiosyncratic uncertainty is sufficiently small, but their discussion focuses on NAD. In contrast, Proposition 2 shows that NAD is never optimal with symmetric parties and a continuum of districts.

derivative of $r^*(P)$ with respect to ε in the equation $\int Q(s - r^*(P))dP(s) = Q(0) - \varepsilon$ (which equals $1/\int q(s - r^*(P))dP(s)$ by the implicit function theorem). The designer’s “total value” of assigning a type- s voter to district P then equals the sum of $G(r^*(P))$ (the probability of winning district P) and $\lambda(r^*(P))(Q(s - r^*(P)) - Q(0))$ (the product of the designer’s value of an extra vote in district P and the number of net votes provided by a type- s voter at the pivotal aggregate shock $r^*(P)$). The proofs of Theorems 1 and 2 combine Lemma 2 and the above arguments.

4.2. Optimal Districting with Small Aggregate or Idiosyncratic Uncertainty

We now consider optimal districting when either aggregate or idiosyncratic uncertainty is small. We will see that the small aggregate uncertainty case is the empirically relevant one. We include the small idiosyncratic uncertainty case for completeness and also to show how the main result of Friedman and Holden (2008) fits in our framework.²³

Aggregate uncertainty is small when F and Q are fixed and $\gamma \rightarrow 1$, so the aggregate shock r is close to 0 with high probability. When aggregate uncertainty is small and $r^*(F) > 0$ (so the designer has majority support at the modal aggregate shock), the designer’s expected seat share is close to 1 under uniform districting. But since uniform districting is not strictly single-dipped, it cannot be exactly optimal for any $\gamma < 1$, by Lemma 1. Instead, we show that optimal districting approximates NAD with equally strong paired districts. Intuitively, when aggregate uncertainty is small and $r^*(F) > 0$, optimal districting starts from uniform districting and then splits pooled districts into equally strong paired districts in a negatively assortative manner.

When aggregate uncertainty is small and $r^*(F) < 0$, the designer’s optimal expected seat share is approximately $1 - F(s^*(0))$, where, for any $r \in (r^*(F), \bar{s})$, $s^*(r)$ is defined so that the designer’s vote share among voter types $s \geq s^*(r)$ at aggregate shock r is 50%.²⁴ This expected seat share can be approximated by a pack-and-crack plan where, for a small $\varepsilon > 0$, voter types $s < s^*(\varepsilon)$ are assigned to identical weak districts that the designer loses with high probability, and voter types $s \geq s^*(\varepsilon)$ are assigned to identical strong districts that the designer wins with a vote share close to 50% with high probability.²⁵ However, since pack-and-crack districting is not strictly single-dipped, it cannot be exactly optimal. Instead, we show that optimal districting approximates POP with equally strong paired districts. Intuitively, when aggregate uncertainty is small and $r^*(F) < 0$, optimal districting starts from pack-and-crack and then splits the packed districts into segregated districts and splits the cracked districts into equally strong paired districts in a negatively assortative manner.

To state our result, let \mathcal{H}^* be the unique districting plan that segregates types below $s^*(0)$ (if $s^*(0) > \underline{s}$, which holds when $r^*(F) < 0$) and pairs types above $s^*(0)$ in equally strong districts in a negatively assortative manner. Formally, letting $r_+^*(F) = \max\{0, r^*(F)\}$, \mathcal{H}^* is the unique plan \mathcal{H} such that, for any $P \in \text{supp}(\mathcal{H})$, either (a) $\text{supp}(P) = \{s(P)\}$ such that $s(P) \in [\underline{s}, s^*(0)] \cup \{r_+^*(F)\}$, or (b) $\text{supp}(P) = \{s_1(P), s_2(P)\}$ such that $r^*(P) = r_+^*(F)$, $s^*(0) \leq s_1(P) < s_2(P) \leq \bar{s}$, and

$$\int_{[s^*(0), s_1(P)] \cup [s_2(P), \bar{s}]} (Q(s - r_+^*(F)) - Q(0))dF(s) = 0. \quad (1)$$

²³The results in this subsection, Theorems 3 and 4, do not require the assumption that G and Q lie in the same location-scale family, although this assumption facilitates the exposition.

²⁴Formally, define $s^*(r)$ as the smallest $\tilde{s} \in [\underline{s}, \bar{s}]$ such that $\int_{\tilde{s}}^{\bar{s}} (Q(s - r) - Q(0))dF(s) \geq 0$. Note that $s^*(r) = \underline{s}$ if $r \leq r^*(F)$, $s^*(r) = \bar{s}$ if $r \geq \bar{s}$, and $s^*(r) \in (\underline{s}, \bar{s})$ if $r \in (r^*(F), \bar{s})$.

²⁵This is shown formally in Lemma 10 in the appendix.

THEOREM 3: *As aggregate uncertainty vanishes ($\gamma \rightarrow 1$ with F and Q fixed), the optimal expected seat share converges to $1 - F(s^*(0))$, and the optimal districting plan converges to \mathcal{H}^* .²⁶ Thus, when aggregate uncertainty is small, optimal districting approximates NAD with equally strong paired districts if $r^*(F) \geq 0$ and approximates POP with equally strong paired districts if $r^*(F) < 0$.*

The intuition for why paired districts are approximately equally strong is that, when aggregate uncertainty is small, it is approximately optimal for the designer to assign voters among the paired districts so as to make the weakest of these districts as strong as possible. Mathematically, this follows from log-concavity. This simple and intuitive property is the basis for the gerrymandering test we propose in Section 6.4.

To appreciate how optimal districting with small aggregate uncertainty differs from uniform districting (when $r^*(F) \geq 0$) or p -segregation (when $r^*(F) < 0$), consider the difference between pooling an interval of voter types and splitting the pool into equally strong paired districts in a negatively assortative manner. This splitting does not affect the designer's expected seat share. Indeed, it does not affect the joint distribution over voter types s and threshold shocks $r^*(P)$ in districts to which they are assigned, because if $P = \alpha P' + (1 - \alpha)P''$ and $r^*(P') = r^*(P'')$, then $r^*(P) = r^*(P') = r^*(P'')$. Thus, in terms of *outcomes* (joint distributions of s and $r^*(P)$), NAD with equally strong paired districts is equivalent to uniform districting, and POP with equally strong paired districts is equivalent to p -segregation. However, viewed as *districting plans* (distributions of P), NAD with equally strong paired districts is quite different from uniform districting, and POP with equally strong paired districts is quite different from p -segregation. Thus, Theorem 3 implies that optimal districting plans with small aggregate uncertainty are quite different from uniform districting and p -segregation; but also that, at the same time, these differences are not very consequential for the joint distribution of s and $r^*(P)$ or the designer's expected seat share.

We now turn to the case where idiosyncratic uncertainty is small. Here, F and G are fixed and $\gamma \rightarrow 0$, so each idiosyncratic shock t is close to 0 with high probability. In this case, whether the designer wins a district P at aggregate shock r is essentially determined by the median voter type s^P in district P : with a unique median s^P , the designer loses districts where $s^P < r - \varepsilon$ and wins districts where $s^P > r + \varepsilon$. Therefore, any optimal districting plan must approximate the highest feasible distribution of district median voters, which is attained by pairing each voter type s above the population median $s^m = F^{-1}(1/2)$ with below-median types, with 50% weight on the above-median type. Under such a plan with an extra ε weight on the above-median type in each district, the designer's expected seat share is approximately $2 \int_{s^m}^{\bar{s}} G(r) dF(r)$. Moreover, for such a plan to be strictly single-dipped, all voter types must be paired in a negatively assortative manner. The resulting districting plan approximates NAD with a 50-50 voter type split in each district.

Let \mathcal{H}^{**} be NAD with a 50-50 split in each district. Formally, \mathcal{H}^{**} is the unique plan \mathcal{H} such that, for any $P \in \text{supp}(\mathcal{H})$, we have either (a) $\text{supp}(P) = \{s^m\}$, or (b) $\text{supp}(P) = \{s_1(P), s_2(P)\}$ such that $\underline{s} \leq s_1(P) < s^m < s_2(P) \leq \bar{s}$, and $F(s_1(P)) = 1 - F(s_2(P))$.

THEOREM 4: *As idiosyncratic uncertainty vanishes ($\gamma \rightarrow 0$ with F and G fixed), the optimal expected seat share converges to $2 \int_{s^m}^{\bar{s}} G(r) dF(r)$, and the optimal districting plan converges to \mathcal{H}^{**} . Thus, when idiosyncratic uncertainty is small, optimal districting approximates NAD with a 50-50 voter type split in each district.*

²⁶The latter convergence is in the weak* topology.

Theorem 4 is similar to [Friedman and Holden’s \(2008\)](#) main result. With finitely many districts, [Friedman and Holden](#) show that, when idiosyncratic uncertainty is sufficiently small, optimal districting is a discrete version of PMP.²⁷ Theorem 4 adds that, in the limit, only one district is segregated, and the voter type split in all other districts is 50-50.

Comparing Theorems 3 and 4, we see that varying the ratio of aggregate and idiosyncratic uncertainty leads to completely different districting plans. When aggregate uncertainty is small, pack-and-crack is approximately optimal, and the exactly optimal plan is close to NAD or POP with equally strong paired districts. When idiosyncratic uncertainty is small, pack-and-crack is far from optimal, and the optimal plan is close to NAD with a 50-50 voter type split in each district. In particular, while a NAD plan can arise in either case, the plans are extremely different: NAD with equally strong paired districts is outcome-equivalent to uniform districting, while NAD with a 50-50 split (or, away from the limit, a $50 - \varepsilon$ - $50 + \varepsilon$ split in favor of the higher type) in each district is very different from uniform districting with small idiosyncratic uncertainty, as $r^*(P)$ is much higher in $50 - \varepsilon$ - $50 + \varepsilon$ districts with more extreme voter types. (For example, compare Figures 2(d) and 2(f).) The critical role of the ratio of aggregate and idiosyncratic uncertainty motivates estimating this parameter in Section 5.

The distinction between optimal districting under small aggregate uncertainty and small idiosyncratic uncertainty relates to results in the probabilistic voting literature. When aggregate uncertainty is very small, the probability that the designer wins a district is approximately determined by the mean voter type in the district, as in probabilistic voting models with partisan taste shocks (e.g., [Hinich 1977](#), [Lindbeck and Weibull 1993](#)). Optimizing the distribution of district means against a unimodal aggregate shock then requires segregating opposing voters and pooling more favorable voters, as in p -segregation or POP or NAD with equally strong paired districts. In contrast, when aggregate uncertainty is very small, the probability that the designer wins a district is approximately determined by the median voter type in the district, as in probabilistic voting models with an uncertain median bliss point (e.g., [Wittman 1983](#), [Calvert 1985](#)). The distribution of district medians is then optimized by pairing above-population-median and below-population-median voter types, as in NAD with a 50-50 voter type split in each district.²⁸

4.3. The Balanced Case and Regime Transitions

Finally, we analyze optimal districting in the intermediate case where neither the parties’ supporters nor the amounts of aggregate and idiosyncratic uncertainty are highly imbalanced. Here, optimal districting will take the form of either POP, PMP, or a mixed versions of these districting plans that we call “Y-districting.” We say that a pack-and-pair plan \mathcal{H} is *Y-districting* if there exists a positive number $\varepsilon > 0$ such that

1. For all $r \in [r^b - \varepsilon, r^b + \varepsilon]$ (where r^b is the bifurcation point), there exists $P \in \text{supp}(\mathcal{H})$ such that $r^*(P) = r$.
2. The functions s_1 and s_2 describing the voter types in paired districts are twice differentiable and satisfy $\lim_{r \downarrow r^b} s_1(r) = \lim_{r \downarrow r^b} s_2(r)$.²⁹

²⁷[Friedman and Holden’s](#) proof relies on perturbation arguments, while our proofs use duality.

²⁸The distinction between mean and median-dependence applies to several related strands of the literature. In gerrymandering, [Owen and Grofman \(1988\)](#) and [Gul and Pesendorfer \(2010\)](#) study the mean-dependent case, while [Friedman and Holden \(2008\)](#) study an approximately median-dependent case. In persuasion, [Kolotilin \(2015\)](#), [Gentzkow and Kamenica \(2016\)](#), [Kolotilin, Mylovanov, Zapechelnyuk, and Li \(2017\)](#), [Kolotilin \(2018\)](#), [Dworczak and Martini \(2019\)](#), and [Kleiner, Moldovanu, and Strack \(2021\)](#) study the mean-dependent case, while [Kolotilin, Corrao, and Wolitzky \(2024\)](#) study a general case nesting both the mean and quantile (e.g., median)-dependent case, and [Yang and Zentefis \(2024\)](#) and [Kolotilin and Wolitzky \(2024\)](#) study the quantile-dependent case.

²⁹Differentiability is used in the proof of Theorem 5. It may be possible to drop it.

Note that Y-districting encompasses a mixed version of POP, where there exists $\hat{s} \in (\underline{s}, r^b)$ such that voter types in $[\underline{s}, \hat{s})$ are always segregated and types in (\hat{s}, r^b) are sometimes segregated and sometimes paired, as well as a mixed version of PMP, where there exists $\hat{s} \in (\underline{s}, r^b)$ such that types in $[\underline{s}, \hat{s})$ are always paired and types in (\hat{s}, r^b) are sometimes segregated and sometimes paired.³⁰ We will show that, with balanced voter support, POP is optimal when idiosyncratic uncertainty is “substantially” larger than aggregate uncertainty, PMP is optimal when aggregate uncertainty is larger than idiosyncratic uncertainty, and Y-districting (and, in particular, mixed POP or mixed PMP) is optimal in the intermediate range.

To analyze these cases, we let J be the distribution with variance 1 satisfying $Q(t) = J(t/\sqrt{\gamma})$ and $G(r) = J(r/\sqrt{1-\gamma})$, so that the variances of t and r are γ and $1-\gamma$. For example, if Q and G are normal then J is the standard normal distribution. By varying γ while fixing J , we can simultaneously approximate the low-aggregate uncertainty and low-idiosyncratic uncertainty limits analyzed in Theorems 3 and 4, as Q is almost constant as $\gamma \rightarrow 1$ and G is almost constant as $\gamma \rightarrow 0$.

Our analytic result in this section is modest: if Y-districting is optimal, then the ratio of idiosyncratic and aggregate uncertainty must fall in an intermediate range. However, numerically it appears that this result actually fully characterizes optimal districting when voter support is balanced: at least when J is normal and F is uniform, our necessary conditions for optimality of Y-districting are also approximately sufficient, and when the ratio of idiosyncratic uncertainty to aggregate uncertainty is below (resp., above) the range where Y-districting is optimal, then PMP (resp., POP) is optimal.

THEOREM 5: *If Y-districting is optimal, then $r^b = 0$ and $\gamma \in (0.5, \sqrt{3} - 1 \approx 0.732]$.*

The proof of Theorem 5 proceeds by deriving three necessary conditions for optimal Y-districting to involve a bifurcation point at r and showing that these conditions imply that r must equal 0 and γ must lie in an intermediate range. The first condition (equation (18) in Appendix A) says that it is optimal to pair voter types just below and just above r . The second condition (equation (19)) says that it is optimal to segregate types just below r . The third condition (equation (20)) says that the proportions of favorable and unfavorable voters in each district P with $r^*(P) = r'$ just above r actually induce the desired cutoff r' . Intuitively, for it to be optimal to pair nearby voter types around r , G must be weakly concave at r ; and for it to be optimal to segregate voter types just below r , G must be weakly convex at r . Hence, bifurcation can occur only at 0, the inflection point of G . Moreover, if we take parameters where Y-districting is optimal and increase aggregate uncertainty, it eventually becomes optimal to always segregate voter types just below 0 rather than pairing them with higher voter types, at which point optimal districting becomes PMP (with a bifurcation point below 0). On the other hand, if we take parameters where Y-districting is optimal and decrease aggregate uncertainty, it eventually becomes optimal to always pair voter types just below 0 with higher voter types rather than segregating them, at which point optimal districting becomes POP (with a bifurcation point above 0).

If we take for granted that the condition $\gamma \in (0.5, 0.732)$ is sufficient as well as necessary for Y-districting to be optimal, the above intuition suggests that, with balanced voter support, PMP is optimal when $\gamma \leq 0.5$, Y-districting is optimal when $\gamma \in (0.5, 0.732)$, and POP is optimal when $\gamma \geq 0.732$. Figure 1 presents numerical solutions that verify this heuristic. In the figure, J

³⁰In contrast, under POP there exists $\hat{s} \in (\underline{s}, r^b)$ such that types in $[\underline{s}, \hat{s})$ are always segregated and types in (\hat{s}, r^b) are always paired, while under PMP there exists $\hat{s} \in (\underline{s}, r^b)$ such that types in $[\underline{s}, \hat{s})$ are always paired and types in (\hat{s}, r^b) are always segregated.

is standard normal and F is uniform on $[-1, 1]$.³¹ Voter types are on the x -axis, and the threshold shocks to win the districts to which each voter type is assigned are on the y -axis. (Thus, packed districts lie on the 45° line, while paired districts straddle the 45° line.) For mixed districting plans (i.e., Y-districting, the middle row of the figure), the shading intensity indicates the probability that a voter type is assigned to each district. We see that optimal districting takes exactly the conjectured form: PMP is optimal for $\gamma \in \{0.1, 0.3, 0.5\}$, Y-districting is optimal for $\gamma \in \{0.6, 0.65, 0.7\}$, and POP is optimal for $\gamma \in \{0.8, 0.9, 0.95\}$. The highest value of γ in the figure, $\gamma = 0.95$, is the value closest to our empirical estimates. When $\gamma = 0.95$, POP remains optimal but now closely resembles p -segregation. Thus, for what we will see is the empirically relevant parameter range, p -segregation is approximately optimal.

We now explain how optimal districting transitions from PMP to POP as γ increases. First, consider the extreme cases where $\gamma \approx 0$ (small idiosyncratic uncertainty) and $\gamma \approx 1$ (small aggregate uncertainty). When $\gamma \approx 0$, PMP is optimal; moreover, optimal districting approximates NAD with a 50-50 split in each district, which implies that the bifurcation point is below 0 and the range of values of $r^*(P)$ across paired districts P is large.³² When $\gamma \approx 1$, POP is optimal; moreover, paired districts are almost equally strong (the range of $r^*(P)$ across paired districts P is small), which implies that the bifurcation point is above 0.³³ Now, when γ increases from 0 toward 0.5, the range of $r^*(P)$ across paired districts decreases (as the range of probable aggregate shocks decreases), and the proportion of packed districts increases. When γ reaches 0.5, it becomes optimal to pack voters with $s = 0$. Since it cannot be optimal to pack voters with $s > 0$, once γ crosses 0.5 it becomes optimal to pair voters with s just above 0 with a few slightly less favorable voters. At this point, districting takes the form of mixed PMP. As γ increases farther above 0.5, the range of $r^*(P)$ across paired districts continues to decrease, and the left arm of the “Y” gets steeper as the right arm gets flatter.³⁴ At some point, the right arm of the Y becomes flatter than the left arm, so that the most extreme left-wing voters have no right-wing voters to match with, at which point these voters are segregated: this point marks the transition from mixed PMP to mixed POP, which occurs at $\gamma = 2/3$ in the uniform case illustrated in Figure 1.³⁵ The $\gamma = 0.65$ and $\gamma = 0.7$ panels in the figure illustrate points just before and just after this transition. As γ increases further, more and more mixed unfavorable voters are assigned to paired districts, until all such voters are assigned to paired districts, at which point optimal districting becomes POP, and the bifurcation point becomes positive. This occurs when $\gamma \approx 0.732$. Finally, as γ increases beyond 0.732, the range of $r^*(P)$ across paired districts continues to decrease, and the optimal POP plan eventually approximates p -segregation.

Figure 2 illustrates optimal districting for the same parameters as Figure 1, except that now voter types are uniform on $[x - 1, x + 1]$ where x is scaled to give an expected vote share of 40% (top panels) or 60% (bottom panels). The figure shows that a less popular designer segregates more unfavorable voters, while a more popular voter pools more voters. The last panel shows that NAD (approximating uniform districting) is optimal for a designer with a 60% expected vote share and $\gamma = 0.9$.

³¹More precisely, we approximate the designer’s problem by a finite-dimensional linear program and then solve it using Gurobi Optimizer. Our approximation specifies that s is uniformly distributed on $\{-1, -0.99, \dots, 0.99, 1\}$ and that the designer is constrained to create districts P satisfying $r^*(P) \in \{-1, -0.99, \dots, 0.99, 1\}$.

³²Another property of optimal PMP plans is that the left arm of the “Y” is infinitely steep at the bifurcation point, i.e., $\lim_{r \downarrow r^b} s'_1(r) = 0$.

³³Another property of optimal POP plans is that pairing at the bifurcation point is smooth, i.e., $\lim_{r \downarrow r^b} s'_1(r) = -\infty$ and $\lim_{r \downarrow r^b} s'_2(r) = \infty$.

³⁴The proof of Theorem 5 shows that, for all sufficiently small positive r , $|s'_1(r)|$ is decreasing in γ (i.e., the left arm gets steeper) and $s'_2(r)$ is increasing in γ (i.e., the right arm gets flatter).

³⁵The transition point is the unique value of γ at which $\lim_{r \downarrow 0} |s'_1(r)| = \lim_{r \downarrow 0} s'_2(r)$.

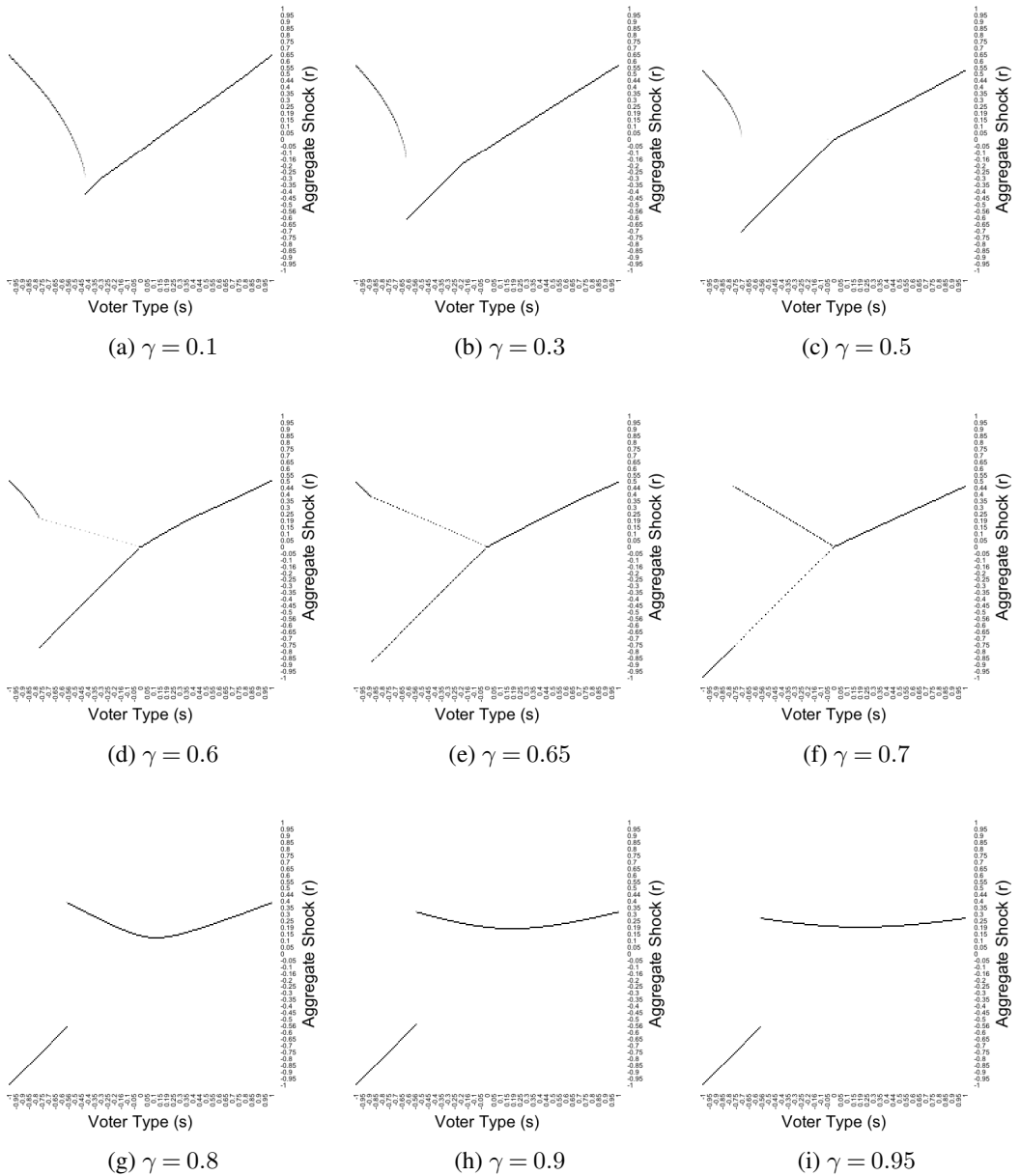


FIGURE 1.—Optimal Districting with Balanced Voter Support as Share of Idiosyncratic Uncertainty Varies
Notes: The optimal districting plan is PMP for $\gamma \in \{0.1, 0.3, 0.5\}$, Y-districting for $\gamma \in \{0.6, 0.65, 0.7\}$ (and, specifically, mixed PMP for $\gamma \in \{0.6, 0.65\}$ and mixed POP for $\gamma = 0.7$), and POP for $\gamma \in \{0.8, 0.9, 0.95\}$. Our empirical estimates of γ in Section 5 are above 0.96 for all US states.

Figure 3 illustrates the form of optimal districting as a function of the designer’s expected vote share and γ . The figure continues to assume that J is standard normal and F is uniform on $[x - 1, x + 1]$, where x is scaled so that the designer’s expected vote share ranges from 0 to 1. The figure shows that segregation is optimal for an unpopular designer (unless aggregate uncertainty dominates), NAD is optimal for a popular one, and optimal districting ranges from

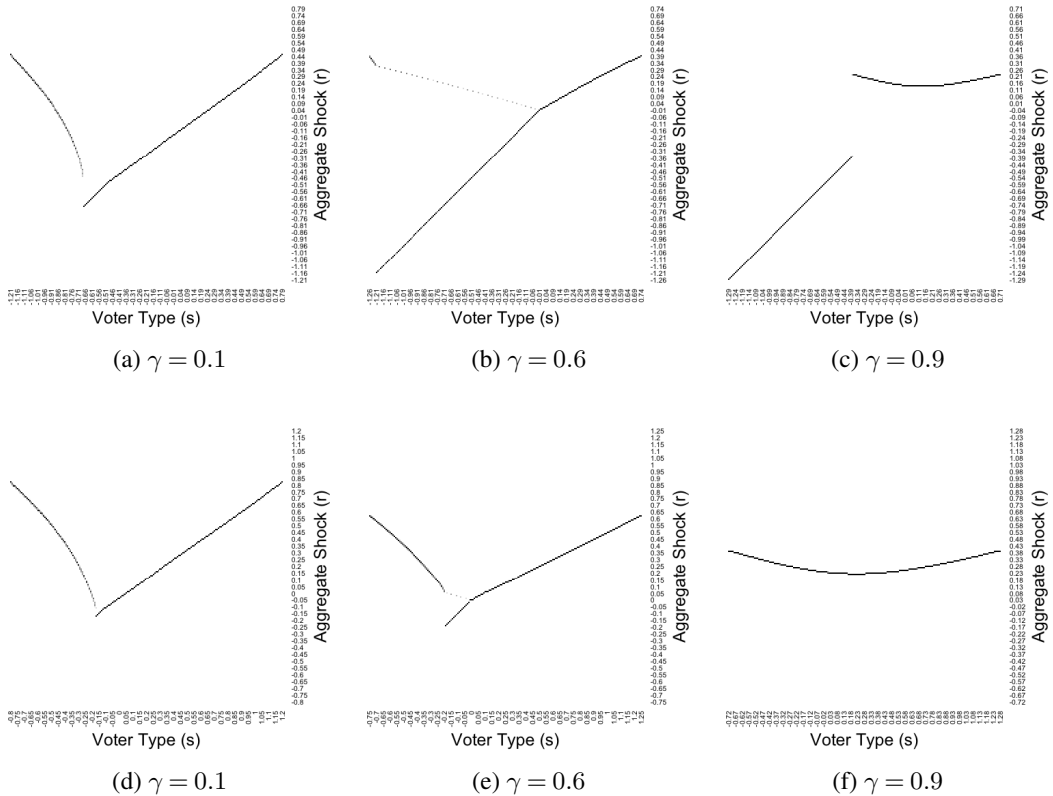


FIGURE 2.—Optimal Districting with Imbalanced Voter Support as Share of Idiosyncratic Uncertainty Varies
Notes: In the top and bottom panels, the designer’s expected vote share is 40% and 60%.

PMP to Y-districting to POP as γ ranges from 0 to 1 with balanced voter support. These results match Theorems 2–5.³⁶

Figure 3 also plots our point estimates of the (Republican) designer’s expected vote share and γ for every US state. (The data and estimation procedure is described in the next section.) The most important observation is that γ is close to 1 in every state: the mean estimate for γ (weighted by the number of districts in each state) is 0.986, and the lowest estimate (in North Carolina) is 0.962. These estimates are all far above the cutoff of 0.732 above which POP is optimal with balanced voter support. Thus, NAD (approximating uniform districting) is optimal for a Republican designer in Republican states like Alabama, Wyoming, and Louisiana, while POP (approximating pack-and-crack) is optimal for a Republican designer in swing states like Michigan and North Carolina, as well as in Democratic states like New York and California (in the fanciful event that the Republicans found themselves controlling districting in such a state).³⁷

³⁶Due to numerical error, it is difficult to confidently classify optimal plans within one or two grid points of the boundaries between the regions where different plans are optimal in Figure 3. (By continuity, plans of different forms are both approximately optimal near the boundary.) The boundaries should thus be viewed as approximations.

³⁷A caveat is that Figure 3 is a 2-dimensional plot and thus neglects heterogeneity in the variance of s across states, which we also estimate. It turns out that assuming that the variance of s is $1/\sqrt{3} \approx 0.577$ in all states—which is implicitly what Figure 3 does—yields the correct classification of optimal districting for every state except Hawaii, where Republican-optimal districting is actually segregation.

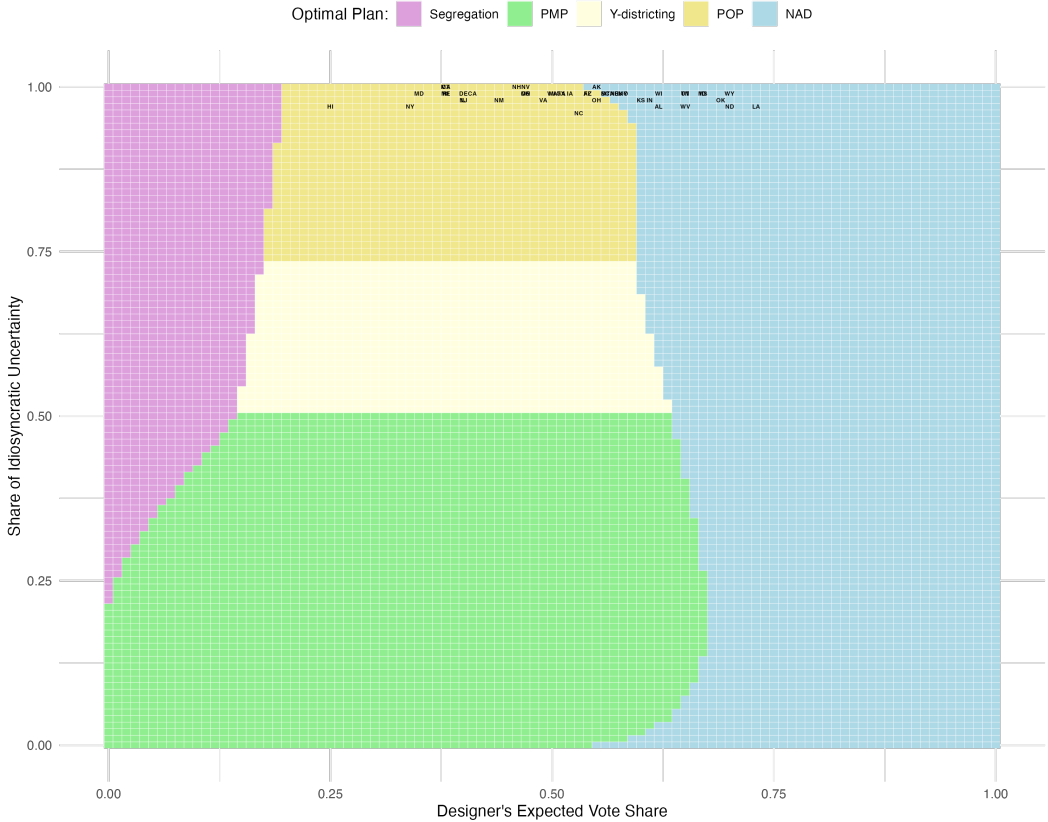


FIGURE 3.—Optimal Districting as Designer’s Expected Vote Share and Share of Idiosyncratic Uncertainty Vary
Notes: Each US state is located at its point estimate in Table I in Section 5.

REMARK 1—Approximate Optimality of Pack-and-Crack: Lemma 10 in the appendix shows that uniform districting (for a designer with majority support) or pack-and-crack districting (for a designer with minority support) is approximately optimal with small aggregate uncertainty. The intuition is simple: with small aggregate uncertainty, the designer with minority support can obtain an expected seat share of approximately $1 - F(s^*(0))$ by creating slightly fewer than $1 - F(s^*(0))$ identical districts each with an expected vote share slightly greater than $1/2$, and $1 - F(s^*(0))$ is the optimal expected seat share in the limit. Moreover, pack-and-crack districting is also approximately optimal for realistic parameters. For the same parameters as in Figure 1 (which, recall, has balanced voter support), Figure 4 plots the expected seat share under the optimal pack-and-crack plan and under the unconstrained optimal plan. The figure shows that the unconstrained optimal expected seat share never exceeds the pack-and-crack optimal expected seat share by more than 0.1% for any value of γ above 0.95. (Recall that our lowest point estimate of γ for any US state is 0.962.) We also estimate that the maximum loss from pack-and-crack relative to optimal districting in any US state (accounting for unbalanced voter support) is 0.56% (see Table I in Section 5).³⁸

³⁸The maximum loss is attained by Rhode Island, a Democratic state where Republicans are very unlikely to ever control districting. If we exclude the Democratic states of Rhode Island, Massachusetts, and Maine, the maximum estimated loss from pack-and-crack relative to optimal districting in any US state is 0.09%.

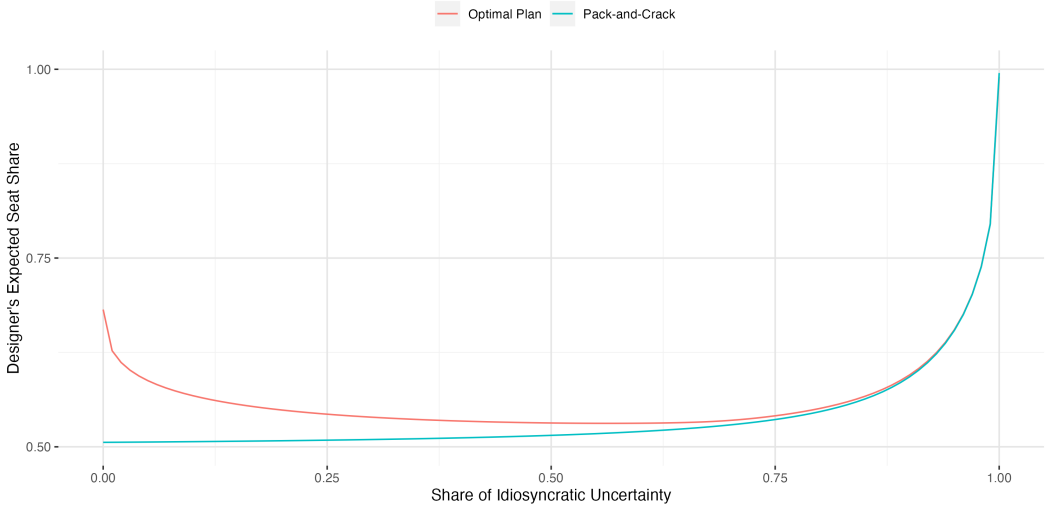


FIGURE 4.—Expected Seat Share under Optimal Districting and Optimal Pack-and-Crack Districting

5. ESTIMATION

We have seen that the form of optimal districting depends on a comparison of the amount of aggregate and idiosyncratic uncertainty facing the designer, and in particular on the parameter γ (the share of idiosyncratic uncertainty), as well as the designer’s expected vote share. We now estimate these parameters using precinct-level returns from recent US House elections. We first describe our data and empirical model, then present some simple summary statistics and plots, and finally estimate γ .

5.1. Data and Empirical Model

Our data are the precinct-level returns from the US House elections in 2016, 2018, and 2020, which were recently standardized and made freely available by Baltz et al. (2022). For each precinct n and election year $y \in \{2016, 2018, 2020\}$, we observe the total two-party vote k_{ny} and the share of the two-party vote for the Republican candidate v_{ny} .³⁹ The data are a repeated cross-section rather than a panel, because there is no general way to match precincts across elections (for example, because precinct boundaries change frequently; Baltz et al. 2022, p. 6). We drop all districts with an uncontested House race in any of 2016, 2018, or 2020 (which drops 25% of all districts).⁴⁰ This results in dropping South Dakota and Vermont from the analysis, as

³⁹ A “precinct” is the smallest election-reporting unit in a state, which typically corresponds to a geographic area where all voters vote at the same polling place. Maine and New Jersey report election returns only at the township level, so for these states n indexes townships rather than precincts. Also, for some elections where a nominally third-party candidate runs in place of an official Democratic or Republican candidate, we manually re-label this candidate as a Democrat or Republican. For example, in New York, we re-assign Working Families Party candidates as Democrats and re-assign Conservative Party candidates as Republicans. Throughout, we focus on the two-party vote k_{ny} and the Republican share of the two-party vote v_{ny} , ignoring third parties.

⁴⁰ Keeping these districts would bias our estimate of γ , because the relevant vote shares are for contested elections, and if these districts were contested their vote shares would be different from 0 or 1. Keeping a district with one or two uncontested elections only for the elections where it is contested would also bias our estimate of γ , by distorting the estimated swing across elections. Dropping uncontested districts does likely bias our estimate of the voter type

in these states the single at-large district was uncontested in 2020 and 2016, respectively; we also drop Pennsylvania, as it was redistricted between 2016 and 2018. Moreover, for each election, we drop precincts with fewer than 50 total votes (which drops 0.14% of all votes) or where the Republican vote share is 0 or 1 (which drops an additional 0.015% of votes).

To take the model to these data, we assume that s indexes precincts, so that $Q(s - r)$ is the designer's vote share in a type- s precinct at aggregate shock r .⁴¹ Formally, this is equivalent to assuming that all voters in a precinct have the same type. (As we clarify below, this does not mean that all voters in a precinct vote the same way.) We also assume that precincts are relatively large (in the data, the mean precinct vote count is 794 with standard deviation 1,434, after dropping precincts with fewer than 50 total votes or a 0 or 1 vote share), and idiosyncratic voter taste shocks are normally distributed.⁴² By the law of large numbers, this implies that the designer's vote share in a precinct n with type s_n in district d and election y is given by

$$Q(s_n - r_{dy}) = \Phi\left(\frac{s_n - r_{dy}}{\sqrt{\gamma}}\right), \quad (2)$$

where Φ is the standard normal distribution. To see this, recall that each voter i in precinct n votes for the designer's party iff $s_n - r_{dy} - t_{iy} \geq 0$, where t_{iy} is the voter's normally distributed idiosyncratic taste shock, and hence votes for the designer's party with probability $\Phi((s_n - r_{dy})/\sqrt{\gamma})$.⁴³ Applying the law of large numbers at the precinct level gives (2).

We emphasize that this empirical model does not allow precinct-level aggregate shocks: the vote share $Q(s_n - r_{dy})$ in precinct n in district d and election y is given by (2), which is a deterministic function of the persistent precinct type s_n and the district-level aggregate shock in election y , r_{dy} .

To interpret the assumption that all voters in a precinct have the same type, note that a voter's type and taste shock enter only through their difference $s_n - t_{iy}$. For this discussion, let us call this difference the voter's "preference." Our assumption is that voter preferences in precinct s_n are normally distributed with mean s_n and variance γ . Also, while voter preferences must be independent across voters in each district to justify (2), the correlation of each voter's preference across elections is arbitrary. Thus, voters in a precinct can differ in their persistent tastes for the parties as well as in their election-specific tastes.

5.2. Descriptive Figures and Summary Statistics

We first present a histogram (Figure 5(a)) showing the number of voters in the United States who live in a precinct with Republican vote share v , with bin breaks $\{0, 0.05, \dots, 0.95, 1\}$,

distribution F , as uncontested districts are presumably more extreme; however, this bias is irrelevant for our main goal of estimating γ .

⁴¹In practice, the smallest "districtable unit" is usually not a precinct but a *census block*, which is the smallest geographic unit for which the US Census tabulates complete data. However, the number of voters in a precinct or a census block are roughly similar (typically around 1,000, albeit with fairly wide variation), and [Bouton, Genicot, Castanheira, and Stashko \(2023\)](#) report that less than 2% of precincts are split across proposed congressional districts in their sample, so we believe there is little loss in proceeding as if designers assign precincts rather than census blocks or some smaller subdivision. However, precincts sometimes are split, and if technological or political constraints evolve to make this practice more common, our analysis would have to be redone at the level of the resulting smallest districtable unit.

⁴²Our estimates are not sensitive to assuming normality: because we will find that γ is very large, the taste shock distribution is approximately uniform over the relevant range, so specifying any smooth taste shock distribution leaves our estimates almost unchanged. For example, our point estimate of γ for the US as a whole is 0.986 with normal taste shocks, 0.987 with logistic taste shocks, 0.989 with Laplace taste shocks, and 0.981 with uniform taste shocks.

⁴³In this section, as in Section 4.3, we assume that $Q(t) = \Phi(t/\sqrt{\gamma})$ and $G(r) = \Phi(r/\sqrt{1-\gamma})$.

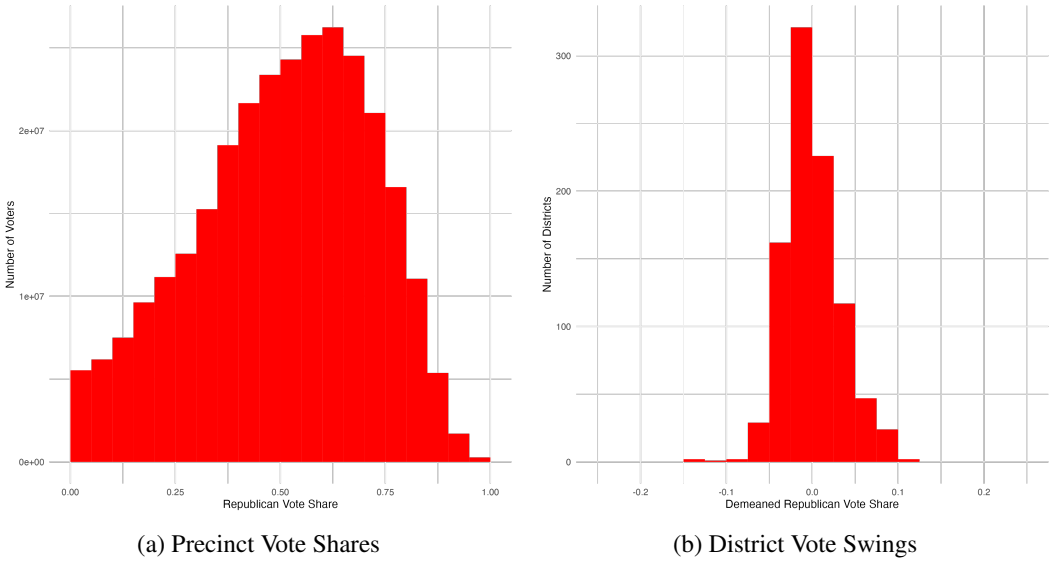


FIGURE 5.—Distributions of Precinct Vote Shares and District Vote Swings

averaging over elections $y \in \{2016, 2018, 2020\}$. The histogram shows that the distribution of v_{ny} is unimodal, with a large majority (74%) of the mass on $v \in [0.25, 0.75]$. This pattern has two simple, but important, implications for our model. First, the distribution of voter/precinct types is far from bimodal: there is a continuum of types, with most mass “toward the middle.” A designer choosing how to assign precincts to districts thus faces a continuum of types, as in our model. Second, idiosyncratic uncertainty appears large relative to aggregate uncertainty. To see this, note that as if idiosyncratic uncertainty dominates ($\gamma \rightarrow 1$), Figure 5(a) would show a normal density centered at $v = 1/2$, while if aggregate uncertainty dominates ($\gamma \rightarrow 0$), it would show a bimodal distribution with all mass at 0 and 1. The former case is a much better approximation, as the distribution in Figure 5(a) is unimodal, with 74% of the mass on $v \in [0.25, 0.75]$. While we quantitatively estimate γ in the next subsection, this observation already suggests what we will find, which is that γ is much greater than 0.5.

Next we present another histogram (Figure 5(b)), which shows the number of (*district, election*) pairs where the district-wide Republican vote share deviated from its mean over the three elections we consider by x , with bin breaks $\{-0.25, -0.225, \dots, 0.225, 0.25\}$.⁴⁴ This histogram gives another way of showing that aggregate shocks are small: the distribution is centrally unimodal, and most of the mass (59%) is on $x \in [-0.025, 0.025]$. In contrast, if aggregate shocks were very large, we would again have a bimodal distribution with all mass far from 0.

5.3. Estimates for γ

We now estimate the key parameter γ under the assumption that aggregate and idiosyncratic shocks are both normally distributed. Since districting plans in the US are drawn at the state level, we estimate γ separately for each US state.⁴⁵ We assume that aggregate shocks are jointly

⁴⁴This histogram is compiled at the district level because precincts are not matched across elections.

⁴⁵While our model assumes a large number of districts, we estimate γ for each state (including states with only one congressional district) to give as complete estimates of this parameter as possible.

normally distributed across districts and independent across elections, so that the variance of r_{dy} is $1 - \gamma$; the correlation between r_{dy} and $r_{d'y}$ is ρ for each $d \neq d'$ and y ; and the correlation between r_{dy} and $r_{d'y'}$ is 0 for each d, d' , and $y \neq y'$. Recall that the results in Section 4.3 show that, with balanced voter support, PMP is optimal if $\gamma \leq 0.5$, Y-districting is optimal if $\gamma \in (0.5, 0.732)$, and POP is optimal if $\gamma \geq 0.732$. Thus, a key question of interest is which of these three regions contains our estimate of γ .

We estimate γ for each state by method of moments. Recall that v_{ny} is the Republican share of the two-party vote in precinct n and election y . Let $w_{ny} = \Phi^{-1}(v_{ny})$ be the corresponding quantile of the standard normal distribution. Let $T = 3$ denote the number of elections, D the number of districts in the state, and \mathcal{N}_{dy} the set of precincts in district d and election y . Next, define

$$w_{dy} = \frac{\sum_{n \in \mathcal{N}_{dy}} k_{ny} w_{ny}}{\sum_{n \in \mathcal{N}_{dy}} k_{ny}} \quad \text{and} \quad w_{d\bullet} = \frac{\sum_y w_{dy}}{T}.$$

That is, w_{dy} is the average value of w_{ny} over precincts in district d , weighted by the number of votes in each precinct; and $w_{d\bullet}$ is the average value of w_{dy} over elections y . It can then be shown that a consistent estimator of γ is given by

$$\hat{\gamma} = 1 / \left(1 + \frac{1}{D(T-1)} \sum_{d,y} (w_{dy} - w_{d\bullet})^2 \right).$$

In the Online Appendix, we also construct a confidence interval for γ , as well as a point estimator of the correlation among the district-level aggregate shocks, and point estimators of the mean and standard deviation of the distribution of precinct types.

Table I displays the resulting estimates for each US state, as well as for the state average weighed by the number of districts included in the analysis (row WS) and the US as a whole (row US). The states are ordered by column v , the designers expected vote share in the districts included in the estimation. Columns D_T and D_A are the total number of districts and the number of districts included in the analysis.

Columns γ and $\underline{\gamma}$ are our point estimate and the lower bound of a 95% one-sided confidence interval for γ . The confidence interval is wide because we only have data from three elections: $T = 3$. However, it is clear that γ is far above the critical value of 0.732. The lowest point estimate for γ for any state is 0.962 in North Carolina, and the weighted mean estimate for γ and the estimate for γ for the US as a whole are both 0.986. Moreover, even with $T = 3$, the lower bound of a 95% one-sided confidence interval is above 0.732 for all available states except North Dakota, where the lower endpoint is 0.619. If we expand our dataset to include the returns from the 2012 and 2014 elections (thus covering all five congressional elections held under the 2010 districting plans), the lower endpoints of the 95% confidence interval exceeds 0.732 for all states, including North Dakota.⁴⁶ Together with the results in Section 4.3 (including Figure 2, which accounts for imbalances in voter support), this provides strong evidence that optimal gerrymandering is given by POP (for a designer with minority support) or NAD (for a designer with majority support) for realistic parameters. Moreover, our estimates for γ are high enough that the optimal POP plan approximates p -segregation and the optimal

⁴⁶Precinct-level returns for 2012 and 2014 have been compiled by [Ansolabehere, Palmer, and Lee \(2014\)](#) but are less complete and less standardized than the [Baltz et al. \(2022\)](#) data we use, which only cover 2016, 2018, and 2020. We have checked that all of our empirical results are robust to including the 2012 and 2014 data.

NAD plan approximates uniform districting (recall Figures 1–4), and that pack-and-crack (with minority support) or uniform districting (with majority support) is approximately optimal.

Columns v and σ_s are the designer’s expected vote share and the standard deviation of s . The latter estimates are similar to those in Figure 1. However, our estimates of v and σ_s may be biased by dropping uncontested elections (unlike our estimates of γ , which remain unbiased after dropping any set of districts).⁴⁷ Column σ_s^c is the standard deviation of s across counties rather than precincts. We discuss county-level estimates in Section 6.2.

Columns V and \underline{V} are the designer’s expected seat share under optimal unconstrained districting and optimal pack-and-crack districting, respectively. As illustrated in Figure 4, the shares are very similar. Column V^c is the expected seat share under optimal districting where the designer assigns counties rather than precincts: see Section 6.2.

Finally, Column \mathcal{H} is the form of the optimal districting plan at the estimated parameters. We estimate that if Republicans somehow found themselves in charge of districting Hawaii, they would segregate the state. Otherwise, POP is optimal (and pack-and-crack is approximately optimal) in states where the expected Republican vote share is less than 55%, and NAD is optimal (and uniform districting is approximately optimal) in states where the expected Republican vote share is greater than 55%. This reflects the fact that, for our estimated value of γ , the optimal pack-and-crack plan creates cracked districts where the designer’s expected seat share is around 55%.

6. DISCUSSION: WHY DOES THE FORM OF GERRYMANDERING MATTER?

Gerrymandering has long been a major concern in American politics and has been tied to several important political and legal issues. In this section, we briefly discuss potential implications of our results on the form of optimal partisan gerrymandering for some of these broader issues. We consider three areas: implications for how regulations and restrictions on districting affect partisan representation; implications for how gerrymandering affects political competition and polarization; and implications for detecting and measuring gerrymandering.

6.1. *Effects of Districting Reforms on Seat Shares I: Majority-Minority Districts*

US state and federal election laws have long recognized potential harms associated with gerrymandering and have therefore restricted gerrymandering in various ways. At the federal level, the key laws are the Equal Protection Clause of the Fourteenth Amendment and the Voting Rights Act of 1965. These laws have been interpreted as not only prohibiting adverse racial gerrymandering, but also as affirmatively requiring states to create electoral districts where racial or ethnic minority voters form either a majority (a so-called “majority-minority district”) or a large enough minority so as to have a strong opportunity to elect their candidate of choice, perhaps in coalition with some majority voters (often called a “minority opportunity district”; e.g., Canon 2022). The creation of such districts played a significant role in increasing Black representation in state legislatures and the US Congress from the 1970’s onward, especially in the South (Grofman and Handley 1991, Cox and Holden 2011). However, the overall partisan impact of majority-minority and minority opportunity districts has long been hotly contested, with some observers arguing that these districts effectively pack strong Democratic supporters and thus resemble a component of a Republican-optimal districting plan. This issue came to

⁴⁷We also estimate the correlation ρ among the district-level aggregate shocks to be 0.317 (at the country level). Since this estimate is not close to either 0 or 1, estimating a simpler empirical model where district-level shocks are either uncorrelated or perfectly correlated would yield biased estimates of γ .

US	D_T	D_A	γ	$\underline{\gamma}$	v	σ_s	σ_s^c	V	\underline{V}	V^c	\mathcal{H}
HI	2	2	0.972	0.839	0.250	0.181	0.076	0.001	0.001	0.000	Seg
NY	27	19	0.966	0.937	0.342	0.826	0.659	0.416	0.415	0.356	POP
MD	8	8	0.990	0.978	0.346	0.728	0.624	0.456	0.456	0.410	POP
RI	2	1	0.990	0.833	0.375	0.302	0.259	0.199	0.194	0.145	POP
CT	5	5	0.995	0.987	0.379	0.377	0.327	0.328	0.328	0.263	POP
ME	2	1	0.992	0.866	0.385	0.311	0.304	0.246	0.244	0.236	POP
MA	9	1	0.998	0.956	0.385	0.233	0.211	0.142	0.139	0.098	POP
DE	1	1	0.990	0.836	0.397	0.488	0.268	0.456	0.456	0.228	POP
IL	18	13	0.984	0.962	0.399	0.737	0.545	0.560	0.560	0.463	POP
NJ	12	12	0.981	0.962	0.402	0.590	0.445	0.492	0.492	0.411	POP
CA	53	35	0.992	0.987	0.412	0.483	0.337	0.508	0.508	0.382	POP
NM	3	3	0.979	0.925	0.436	0.543	0.412	0.548	0.548	0.476	POP
NH	2	2	0.997	0.960	0.463	0.263	0.256	0.575	0.575	0.568	POP
NV	4	4	0.998	0.992	0.467	0.449	0.310	0.741	0.741	0.662	POP
MN	8	8	0.987	0.973	0.470	0.436	0.342	0.626	0.626	0.569	POP
CO	7	7	0.989	0.953	0.470	0.527	0.429	0.691	0.691	0.649	POP
OR	5	4	0.987	0.961	0.471	0.498	0.377	0.662	0.662	0.603	POP
VA	11	8	0.985	0.934	0.491	0.548	0.428	0.726	0.726	0.684	POP
WA	10	5	0.987	0.960	0.496	0.375	0.264	0.689	0.689	0.624	POP
MI	14	13	0.990	0.980	0.501	0.596	0.503	0.803	0.802	0.772	POP
GA	14	7	0.985	0.959	0.509	0.718	0.525	0.821	0.820	0.781	POP
TX	36	23	0.989	0.978	0.514	0.645	0.472	0.841	0.841	0.811	POP
IA	4	4	0.986	0.949	0.519	0.372	0.270	0.763	0.763	0.726	POP
NC	13	11	0.962	0.933	0.526	0.560	0.373	0.740	0.739	0.699	POP
AZ	9	6	0.990	0.974	0.537	0.402	0.286	0.868	0.868	0.852	POP
FL	27	20	0.994	0.987	0.545	0.444	0.291	0.949	0.948	0.948	POP
OH	16	16	0.984	0.967	0.552	0.635	0.469	0.908	0.908	0.896	NAD
AK	1	1	0.996	0.922	0.554	0.396	0.298	0.987	0.987	0.983	NAD
MT	1	1	0.993	0.884	0.556	0.490	0.325	0.973	0.973	0.973	NAD
SC	7	7	0.994	0.988	0.559	0.622	0.402	0.990	0.990	0.988	NAD
AR	4	1	0.985	0.773	0.566	0.629	0.391	0.950	0.949	0.945	NAD
NE	3	2	0.990	0.945	0.575	0.446	0.297	0.981	0.981	0.979	NAD
KY	6	4	0.991	0.968	0.584	0.548	0.408	0.994	0.994	0.992	NAD
MO	8	8	0.995	0.981	0.584	0.702	0.579	1.000	1.000	1.000	NAD
KS	4	4	0.978	0.905	0.598	0.463	0.355	0.968	0.968	0.971	NAD
IN	9	8	0.983	0.963	0.608	0.524	0.351	0.991	0.991	0.990	NAD
WI	8	5	0.989	0.970	0.617	0.301	0.228	0.998	0.998	0.999	NAD
AL	7	2	0.971	0.846	0.624	0.674	0.409	0.988	0.988	0.992	NAD
WV	3	3	0.971	0.883	0.646	0.340	0.252	0.989	0.989	0.989	NAD
UT	4	4	0.989	0.947	0.647	0.585	0.472	1.000	1.000	1.000	NAD
TN	9	8	0.991	0.976	0.650	0.691	0.526	1.000	1.000	1.000	NAD
MS	4	2	0.993	0.934	0.672	0.671	0.340	1.000	1.000	1.000	NAD
ID	2	2	0.987	0.930	0.673	0.462	0.365	1.000	1.000	1.000	NAD
OK	5	4	0.983	0.922	0.685	0.454	0.318	1.000	1.000	1.000	NAD
ND	1	1	0.969	0.619	0.696	0.426	0.335	0.999	0.999	0.999	NAD
WY	1	1	0.990	0.835	0.701	0.478	0.375	1.000	1.000	1.000	NAD
LA	6	4	0.974	0.898	0.725	0.595	0.288	1.000	1.000	1.000	NAD
WS	18	13	0.986	0.959	0.497	0.561	0.415	0.755	0.754	0.709	POP
US	417	311	0.986	0.979	0.497	0.643	0.508	0.777	0.776	0.745	POP

TABLE I: Estimates

a head following the 1994 Republican takeover of the US House, which many journalists and political scientists blamed in part on the creation of majority-minority districts in the 1990 redistricting cycle; however, other observers have disputed this narrative (see, e.g., [Cox and Holden 2011](#) and references therein, [Washington 2012](#)).

Previous studies have observed that the impact of a requirement to create majority-minority or minority opportunity districts on overall partisan representation can hinge on the form of optimal gerrymandering. The conventional view in the 1990's (what [Cox and Holden 2011](#) call the “pack-and-crack consensus”) was that optimal gerrymandering packs opponents, and hence that a requirement to create majority-minority districts that pack strong Democratic supporters is likely to increase overall Republican representation.⁴⁸ [Shotts \(2001\)](#) adds an important caveat by noting that, since uniform districting is optimal for a designer with majority voter support in the population (without aggregate uncertainty), majority-minority mandates hurt Republican designers in strongly Republican states. More dramatically, based on the analysis of [Friedman and Holden \(2008\)](#), [Cox and Holden \(2011\)](#) challenge the pack-and-crack consensus by arguing that optimal districting is given by NAD, and thus packs moderates rather than opponents. Since NAD does not create districts packed with strong Democratic supporters, [Cox and Holden](#) argue that a requirement to create such districts precludes NAD and is therefore likely to reduce overall Republican representation.

Our results contribute to this debate as follows. [Cox and Holden](#)'s argument that NAD is optimal in practice rests on an implicit assumption that the low-idiosyncratic-uncertainty case studied by [Friedman and Holden \(2008\)](#) is representative. For example, [Cox and Holden](#) write, “In a world with diverse voter types, however, there is no plausible distribution of African American voters that would make it optimal for Republican redistricting authorities to create districts in which African Americans make up a supermajority of voters. Within the model, packing one's opponents is never the optimal strategy,” (p. 574). We instead show that, empirically, idiosyncratic uncertainty is much larger than aggregate uncertainty, and that this implies that packing opponents is optimal for a designer with majority voter support, while NAD is optimal for a designer with minority support. Thus, majority-minority mandates can increase Republican representation in closely divided states where POP is optimal and pack-and-crack is approximately optimal (as in the pack-and-crack consensus), but are likely to decrease Republican representation in strongly Republican states where NAD is optimal and uniform districting is approximately optimal (as argued by [Shotts \(2001\)](#) in a model without aggregate uncertainty). Overall, by analyzing a general model that does not restrict the relative amounts of idiosyncratic and aggregate uncertainty, we reach a conclusion similar to that of [Shotts \(2001\)](#) and quite different from that of [Cox and Holden \(2011\)](#).

6.2. *Effects of Districting Reforms on Seat Shares II: Respecting Political Subdivisions*

Among the practical restrictions on districting beyond the equipopulation requirement (see the Introduction), one that is amendable to our analysis is a requirement not to split counties or other political subdivisions. Preserving counties or other subdivisions is one of the six traditional redistricting criteria according to the National Conference of State Legislators and is currently required in 29 of the 50 US states.⁴⁹ From the perspective of partisan gerrymandering, a requirement to preserve counties constrains the designer to choose among a coarser set

⁴⁸Minority opportunity districts may or may not raise similar issues, depending on the share of strong Democratic supporters in these districts ([Lublin, Handley, Brunell, and Grofman, 2020](#)).

⁴⁹The other criteria are compactness, contiguity, preservation of communities of interest, preservation of the “cores” of previous districts, and avoiding incumbent pairing (<https://www.ncsl.org/elections-and-campaigns/2020-redistricting-criteria>).

of districting plans, where counties rather than census blocks or precincts become the object of districting.

We can assess the impact of a requirement to preserve counties by re-running our estimation of γ and F , taking the unit of districting as counties rather than precincts. Our estimates of γ are similar in both cases but are slightly higher with counties, because precinct vote shares swing slightly more from election to election than county vote shares: our mean (weighted by the number of districts in each state) precinct-based estimate of γ is 0.986, while our weighted mean county-based estimate is 0.987. More importantly, our estimate of the standard deviation of F is considerably smaller with counties: the weighted mean precinct-based estimate is 0.561, while the weighted mean county-based estimate is 0.415. This gap is the key consequence of constraining the designer to assigning coarser units. Finally, this constraint significantly affects the designer’s optimal expected seat share in closely divided states where POP is optimal, as now fewer highly unfavorable units can be packed; however, it has only a small effect on the optimal seat share in states where the designer has strong support and NAD is optimal, as uniform districting (which is unaffected by a requirement to preserve counties) is approximately optimal in these states. In particular, our estimate of the reduction in a Republican designer’s seat share from requiring him to preserve counties ranges from essentially 0 in strongly Republican states to 23% in Delaware, with a weighted average across states of 4.5%.⁵⁰

6.3. *Effects of Gerrymandering on Political Competition and Polarization*

Another major debate concerns the impact of gerrymandering on the intensity of electoral competition (e.g., the fraction of “competitive” districts or the extent of incumbency advantage) and political polarization. Popular discourse often blames gerrymandering for reducing competition and increasing polarization. While the scholarly literature is generally skeptical of the claim that gerrymandering plays a large role in explaining overall secular trends in competition and polarization (e.g., [Gelman and King 1994a](#), [Abramowitz, Alexander, and Gunning 2006](#), [McCarty, Poole, and Rosenthal 2009](#), [Friedman and Holden 2009](#)), some work does find such effects (e.g., [Cottrell 2019](#), [Kenny, McCartan, Simko, Kuriwaki, and Imai 2022](#)).

Regardless of the size of the overall effects of gerrymandering on competition and polarization, the nature of these effects depends on the form that gerrymandering takes. In particular, under POP, intra-district polarization is relatively low while inter-district polarization is relatively high; while under NAD or PMP, intra-district polarization is high and inter-district polarization is low. To see this, note that, with a right-wing designer, POP or pack-and-crack creates a few strongly left-leaning districts and many slightly right-leaning districts, with a gap between the left-leaning and right-leaning districts: formally, under POP, there is a gap between the highest value of $r^*(P)$ among segregated districts and the lowest value of $r^*(P)$ among paired districts (see the last three panels in [Figure 1](#)). POP also involves relatively low intra-district polarization within each district, since the lowest voter types in paired districts are “moderates” rather than extreme left-wingers. In contrast, NAD or PMP creates a continuum of districts ranging from left-leaning to right-leaning—formally, the set $\{r : r = r^*(P) \text{ for some } P \in \text{supp}(\mathcal{H})\}$ is an interval (see the first three panels in [Figure 1](#))—with less extreme left-leaning districts than under POP. NAD or PMP also involves greater intra-district polarization than POP, in that the maximum range of voter types that are pooled

⁵⁰Our weighted mean estimate of a Republican designer’s optimal seat share is 70.9% under county-based districting and 75.4% under precinct-based districting. A limitation of this comparison is that our assumption that the designer assigns a continuum of units is more accurate when units are precincts rather than counties. The omitted integer constraint would bind more harshly for county-based districting, which biases our estimates of the designer’s loss from being restricted to assigning counties downward.

together under PMP is greater than under POP (since this range is as large as possible under PMP, but is strictly smaller under POP).

Our model does not encompass endogenous political responses to districting, such as effects of districting on which politicians run for office and on what platforms. With this caveat in mind, we can draw some tentative implications of the above features of optimal districting for political competition and polarization. First, since the distribution of threshold shocks $r^*(P)$ has a gap under POP or pack-and-crack but not under NAD or PMP, POP or pack-and-crack may lead to more polarized legislatures, where the packed districts elect left-wing representatives and the cracked districts elect right-leaning ones. Indeed, the possibility that packing opponents can increase polarization in this manner is a long-standing concern (e.g., [Cox and Holden 2011](#), p. 595); [Coate and Knight \(2007\)](#), [Besley and Preston \(2007\)](#), and [Bracco \(2013\)](#) develop models with this feature. In contrast, NAD or PMP may lead to less polarized legislatures. Second, POP or pack-and-crack may produce more “uncompetitive,” far-left districts. Creating uncompetitive districts is usually viewed as a socially undesirable feature of a districting plan, but see [Buchler \(2005\)](#) and [Brunell \(2008\)](#) for opposing views. Finally, lower intra-district polarization under POP or pack-and-crack may be socially desirable if voters benefit from being ideologically close to their representative, as in [Besley and Preston \(2007\)](#) and [Gomberg, Pance, and Sharma \(2023\)](#). These and other implications of optimal districting for political processes and outcomes can be studied more fully in models that endogenize additional aspects of political competition. This is a promising direction for future research.

6.4. *Detecting and Measuring Gerrymandering*

A large literature proposes various metrics that attempt to detect and measure gerrymandering. Most of these metrics rely on comparing a party’s seat share and its vote share, with a high seat share being taken as an indicator of gerrymandering.⁵¹ However, a limitation of this approach is that one can debate what range of seat shares is “reasonable” for a given vote share. Indeed, the Supreme Court has objected that this class of measures bakes in some form of proportionality between seat and vote shares: as Justice Roberts wrote for the Court in *Rucho v. Common Cause*, “Partisan gerrymandering claims rest on an instinct that groups with a certain level of political support should enjoy a commensurate level of political power and influence. Such claims invariably sound in a desire for proportional representation, but the Constitution does not require proportional representation.”

Our analysis suggest an alternative test for gerrymandering that compares vote shares across districts, rather than comparing seat and vote shares. A novel and robust prediction of our analysis is that, in the realistic case of small aggregate uncertainty, optimal plans make favorable districts equally strong: a designer with majority support creates equally strong districts under NAD or uniform districting, while a designer with minority support creates some packed districts that are lost with high probability and creates equally strong districts that are won with high probability under POP or pack-and-crack. In contrast, there is no reason to expect favorable districts to be equally strong under a non-gerrymandered districting plan. Thus, a proposed test for gerrymandering is whether a districting plan displays an unusually low variance in votes share among districts won by the designer’s party. Such a test can be operationalized in future work.

⁵¹Such measures include the partisan bias ([King and Browning, 1987](#)), efficiency gap ([Stephanopoulos and McGhee, 2015](#)), mean-median gap ([Wang, 2016](#)), and declination ([Warrington, 2018](#)). An alternative approach relies on statistical analysis of an ensemble of simulated maps ([Deford, Duchin, and Solomon, 2021](#)).

7. CONCLUSION

This paper has developed a simple and general model of optimal partisan gerrymandering. Our main message has four parts. First, optimal districting is “pack-and-pair”: weak districts are segregated; strong districts are paired. Second, the optimal form of pack-and-pair depends on the gerrymandering party’s popularity and—more subtly—on the relative amounts of aggregate and idiosyncratic uncertainty facing the gerrymanderer. Packing opposing voters is optimal when idiosyncratic uncertainty dominates, while packing moderate voters is optimal when aggregate uncertainty dominates. Third, empirically, idiosyncratic uncertainty dominates, implying that pack-opponents-and-pair (POP) districting is optimal for a designer with minority support, while negative assortative districting (NAD) is optimal for a designer with majority support. This finding also establishes that the relevant parameter range for future research on gerrymandering (and electoral competition more generally) is that where idiosyncratic uncertainty is much larger than aggregate uncertainty. Fourth, estimated aggregate uncertainty is so small that a simple pack-and-crack plan is approximately optimal for a designer with minority support, while uniform districting is approximately optimal for a designer with majority support. This last observation can help rationalize the use of simple districting plans in practice.

Methodologically, we develop and exploit a tight connection between gerrymandering and information design. We show that a general model of partisan gerrymandering is equivalent to a general Bayesian persuasion problem where the state of the world and the receiver’s action are both one-dimensional and the sender’s preferences are state-independent. This common framework nests the important prior contributions of [Owen and Grofman \(1988\)](#), [Friedman and Holden \(2008\)](#), and [Gul and Pesendorfer \(2010\)](#), and facilitates a more general and realistic analysis that allows diverse voter types and non-linear vote swings without restricting the relative amounts of aggregate and idiosyncratic uncertainty.

We hope our model can inform future research on various aspects of redistricting. We mention a few directions for future research.

First, we have assumed that the designer maximizes his party’s expected seat share. It may be more realistic to assume that the designer’s utility is non-linear in seat shares, for example because of a premium on winning a majority of seats. We examined this case in an earlier version of this paper ([Kolotilin and Wolitzky, 2020](#)). While non-linear designer utility introduces new complications, the extreme case where the designer simply maximizes the probability of winning a majority is straightforward: here, optimal districting maximizes seats conditional on the threshold aggregate shock at which the designer is barely able to attain a majority, and hence reduces to optimal districting without aggregate uncertainty.

Second, we have assumed that all voters always vote, or at least always vote at the same rate (as is equivalent). It would be interesting to incorporate heterogeneous turnout in the analysis. Recently, [Bouton, Genicot, Castanheira, and Stashko \(2023\)](#) consider voters with a binary partisan type (as in [Owen and Grofman 1988](#)), uniform aggregate shocks, and a continuous “turnout type,” which captures fixed turnout heterogeneity across voters. It is promising to explore mutual generalizations of our models that allow more general forms of aggregate uncertainty as well as heterogeneous turnout. An alternative model, which captures variable turnout heterogeneity, would retain one-dimensional voter types but assume that voters abstain when they are close to indifferent between the parties. It is interesting to compare these models, as in practice turnout heterogeneity has both exogenous sources (e.g., education, race) and endogenous ones (e.g., almost-indifferent voters turn out less).

Finally, further questions include: What does the model imply for political competition and the resulting policy choices? What are the model’s comparative statics—for example, what

factors determine the proportion of packed and cracked districts?⁵² And, what does the model imply about how gerrymandering should be measured and regulated? A better understanding of the form of optimal partisan gerrymandering can contribute to the study of these questions and related ones.

REFERENCES

- ABRAMOWITZ, A. I., B. ALEXANDER, AND M. GUNNING (2006): “Incumbency, Redistricting, and the Decline of Competition in US House Elections,” *Journal of Politics*, 68, 75–88. [27]
- ANSOLABEHERE, S., M. PALMER, AND A. LEE (2014): “Precinct-Level Election Data,” *Harvard Dataverse*. [23]
- BAGNOLI, M., AND T. BERGSTROM (2005): “Log-Concave Probability and Its Applications,” *Economic Theory*, 26(2), 445–469. [5]
- BALTZ, S., ET AL. (2022): “American Election Results at the Precinct Level,” *Scientific Data*, 9(651). [20, 23]
- BESLEY, T., AND I. PRESTON (2007): “Electoral Bias and Policy Choice: Theory and Evidence,” *Quarterly Journal of Economics*, 122, 1473–1510. [4, 28]
- BICKERSTAFF, S. (2020): *Election Systems and Gerrymandering Worldwide*. Springer. [1]
- BOUTON, L., G. GENICOT, M. CASTANHEIRA, AND A. STASHKO (2023): “Gerrymandering when Turnout Rates Differ,” Georgetown University. [4, 21, 29]
- BRACCO, E. (2013): “Optimal Districting with Endogenous Party Platforms,” *Journal of Public Economics*, 104, 1–13. [4, 28]
- BRUNELL, T. (2008): *Redistricting and Representation: Why Competitive Elections Are Bad for America*. Routledge. [28]
- BUCHLER, J. (2005): “Competition, Representation, and Redistricting: The Case against Competitive Congressional Districts,” *Journal of Theoretical Politics*, 17, 431–463. [28]
- CALVERT, R. L. (1985): “Robustness of the Multidimensional Voting Model: Candidate Motivations, Uncertainty, and Convergence,” *American Journal of Political Science*, 29(1), 69–95. [14]
- CANON, D. (2022): “Race and Redistricting,” *Annual Review of Political Science*, 25, 509–528. [25]
- CHAMBERS, C. P., AND A. D. MILLER (2010): “A Measure of Bizarreness,” *Quarterly Journal of Political Science*, 5, 27–44. [4]
- COATE, S., AND B. KNIGHT (2007): “Socially Optimal Districting: A Theoretical and Empirical Exploration,” *Quarterly Journal of Economics*, 122, 1409–1471. [4, 28]
- COTTRELL, D. (2019): “Using Computer Simulations to Measure the Effect of Gerrymandering on Electoral Competition in the U.S. Congress,” *Legislative Studies Quarterly*, 44, 487–514. [27]
- COX, A. B., AND R. R. HOLDEN (2011): “Reconsidering Racial and Partisan Gerrymandering,” *University of Chicago Law Review*, 78, 553–604. [25, 26, 28]
- DEFORD, D., M. DUCHIN, AND J. SOLOMON (2021): “Recombination: A Family of Markov Chains for Redistricting,” *Harvard Data Science Review*, 3. [4, 28]
- DWORCZAK, P., AND G. MARTINI (2019): “The Simple Economics of Optimal Persuasion,” *Journal of Political Economy*, 127(5), 1993–2048. [14]
- ELY, J. (2022): “A Cake-Cutting Solution to Gerrymandering,” Northwestern University. [4]
- FRIEDMAN, J. N., AND R. HOLDEN (2009): “The Rising Incumbent Reelection Rate: What’s Gerrymandering Got To Do with It?,” *Journal of Politics*, pp. 593–611. [27]
- (2020): “Optimal Gerrymandering in a Competitive Environment,” *Economic Theory Bulletin*, pp. 1–21. [4]
- FRIEDMAN, J. N., AND R. T. HOLDEN (2008): “Optimal Gerrymandering: Sometimes Pack, but Never Crack,” *American Economic Review*, 98(1), 113–44. [2, 3, 4, 5, 6, 7, 8, 11, 12, 14, 26, 29]
- FRYER, R. G., AND R. HOLDEN (2011): “Measuring the Compactness of Political Districting Plans,” *Journal of Law and Economics*, 54, 493–535. [4]
- GELMAN, A., AND G. KING (1994a): “Enhancing Democracy through Legislative Redistricting,” *American Political Science Review*, 88, 541–559. [27]
- (1994b): “A Unified Method of Evaluating Electoral Systems and Redistricting Plans,” *American Journal of Political Science*, 38, 514–554. [4]
- GENTZKOW, M., AND E. KAMENICA (2016): “A Rothschild-Stiglitz Approach to Bayesian Persuasion,” *American Economic Review, Papers & Proceedings*, 106, 597–601. [14]
- GILLIGAN, T. W., AND J. G. MATSUSAKA (2006): “Public Choice Principles of Redistricting,” *Public Choice*, 129, 381–398. [4]

⁵²Kolotilin and Wolitzky (2020) analyze comparative statics with binary voter types.

- GOMBERG, A., R. PANCS, AND T. SHARMA (2023): “Electoral Maldistricting,” *International Economic Review*, Forthcoming. [4, 28]
- (2024): “Padding and Pruning: Gerrymandering under Turnout Heterogeneity,” *Social Choice and Welfare*. [4]
- GROFMAN, B., AND L. HANDLEY (1991): “The Impact of the Voting Rights Act on Black Representation in Southern State Legislatures,” *Legislative Studies Quarterly*, 16, 111–128. [25]
- GROLL, T., AND S. O’HALLORAN (2024): “Redistricting and Representation: The Paradox of Minority Power,” Working Paper, Columbia University. [4]
- GUL, F., AND W. PESENDORFER (2010): “Strategic Redistricting,” *American Economic Review*, 100(4), 1616–1414. [4, 5, 6, 8, 14, 29]
- HAYES, D., AND S. C. MCKEE (2009): “The Participatory Effects of Redistricting,” *American Journal of Political Science*, 53, 1006–1023. [4]
- HINICH, M. J. (1977): “Equilibrium in Spatial Voting: The Median Voter Result is an Artifact,” *Journal of Economic Theory*, 16(2), 208–219. [14]
- JEONG, D., AND A. SHENOY (2022): “The Targeting and Impact of Partisan Gerrymandering: Evidence from a Legislative Discontinuity,” *Review of Economics and Statistics*, Forthcoming. [4]
- KAMENICA, E., AND M. GENTZKOW (2011): “Bayesian Persuasion,” *American Economic Review*, 101, 2590–2615. [6]
- KENNY, C., C. MCCARTAN, T. SIMKO, S. KURIWAKI, AND K. IMAI (2022): “Widespread Partisan Gerrymandering Mostly Cancels Nationally, but Reduces Electoral Competition,” Harvard University. [27]
- KING, G., AND R. X. BROWNING (1987): “Democratic Representation and Partisan Bias in Congressional Elections,” *American Political Science Review*, 81(4), 1251–1273. [4, 28]
- KLEINER, A., B. MOLDOVANU, AND P. STRACK (2021): “Extreme Points and Majorization: Economic Applications,” *Econometrica*, 89(4), 1557–1593. [14]
- KOLOTLIN, A. (2015): “Experimental Design to Persuade,” *Games and Economic Behavior*, 90, 215–226. [14]
- (2018): “Optimal Information Disclosure: A Linear Programming Approach,” *Theoretical Economics*, 13, 607–636. [9, 14]
- KOLOTLIN, A., R. CORRAO, AND A. WOLITZKY (2024): “Persuasion and Matching: Optimal Productive Transport,” *Journal of Political Economy*, forthcoming. [4, 6, 7, 9, 11, 14, 33, 36, 37]
- KOLOTLIN, A., T. MYLOVANOV, AND A. ZAPECHELNYUK (2022): “Censorship as Optimal Persuasion,” *Theoretical Economics*, 17(2), 561–585. [9]
- KOLOTLIN, A., T. MYLOVANOV, A. ZAPECHELNYUK, AND M. LI (2017): “Persuasion of a Privately Informed Receiver,” *Econometrica*, 85, 1949–1964. [14]
- KOLOTLIN, A., AND A. WOLITZKY (2020): “The Economics of Partisan Gerrymandering,” MIT. [5, 6, 29]
- (2024): “Distributions of Posterior Quantiles via Matching,” *Theoretical Economics*. [14]
- LAGARDE, A., AND T. TOMALA (2021): “Optimality and Fairness of Partisan Gerrymandering,” *Mathematical Programming*, pp. 1–37. [4]
- LINDBECK, A., AND J. WEIBULL (1993): “A Model of Political Equilibrium in a Representative Democracy,” *Journal of Public Economics*, 51, 195–209. [14]
- LUBLIN, D., L. HANDLEY, T. BRUNELL, AND B. GROFMAN (2020): “Minority Success in Non-Majority Minority Districts: Finding the “Sweet Spot”,” *Journal of Race, Ethnicity, and Politics*, 5, 275–298. [26]
- MCCARTY, N., K. T. POOLE, AND H. ROSENTHAL (2009): “Does Gerrymandering Cause Polarization?,” *American Journal of Political Science*, 53, 666–680. [4, 27]
- MCGANN, A. J., C. A. SMITH, M. LATNER, AND A. KEENA (2016): *Gerrymandering in America: The House of Representatives, the Supreme Court, and the Future of Popular Sovereignty*. Cambridge University Press. [1]
- MCGHEE, E. (2014): “Measuring Partisan Bias in Single-Member District Electoral Systems,” *Legislative Studies Quarterly*, 39(1), 55–85. [4]
- (2020): “Partisan Gerrymandering and Political Science,” *Annual Review of Political Science*, 23, 171–185. [1]
- NEWKIRK, V. R. (2017): “How Redistricting Became a Technological Arms Race,” *The Atlantic*, 28 October. [1]
- OWEN, G., AND B. GROFMAN (1988): “Optimal Partisan Gerrymandering,” *Political Geography Quarterly*, 7(1), 5–22. [3, 4, 6, 14, 29]
- PUPPE, C., AND A. TASNÁDI (2009): “Optimal Redistricting under Geographical Constraints: Why “Pack and Crack” Does Not Work,” *Economics Letters*, 105, 93–96. [4]
- RAKICH, N., AND E. MEJIA (2022): “Did Redistricting Cost Democrats the House?,” <https://fivethirtyeight.com/features/redistricting-house-2022/>. [2]
- RAKICH, N., AND N. SILVER (2018): “Election Update: The Most (And Least) Elastic States And Districts,” <https://fivethirtyeight.com/features/election-update-the-house-districts-that-swing-the-most-and-least-with-the-national-mood/>. [5]

- SABET, N., AND N. YUCHTMAN (2024): “Identifying Partisan Gerrymandering and Its Consequences: Evidence from the 1990 US Census Redistricting,” *Journal of Political Economy: Microeconomics*, forthcoming. [4]
- SANTAMBROGIO, F. (2015): *Optimal Transport for Applied Mathematicians*, vol. 55. Springer. [41]
- SHERSTYUK, K. (1998): “How to Gerrymander: A Formal Analysis,” *Public Choice*, 95, 27–49. [4]
- SHOTTS, K. W. (2001): “The Effect of Majority-Minority Mandates on Partisan Gerrymandering,” *American Journal of Political Science*, pp. 120–135. [4, 26]
- (2002): “Gerrymandering, Legislative Composition, and National Policy Outcomes,” *American Journal of Political Science*, pp. 398–414. [4]
- STEPHANOPOULOS, N. O., AND E. M. MCGHEE (2015): “Partisan Gerrymandering and the Efficiency Gap,” *University of Chicago Law Review*, 82, 831–900. [4, 28]
- WANG, S. (2016): “Three Tests for Practical Evaluation of Partisan Gerrymandering,” *Stanford Law Review*, 69, 1263–1321. [28]
- WARRINGTON, G. (2018): “Quantifying Gerrymandering Using the Vote Distribution,” *Election Law Journal*, 17, 39–57. [28]
- WASHINGTON, E. (2012): “Do Majority-Black Districts Limit Blacks’ Representation? The Case of the 1990 Redistricting,” *Journal of Law and Economics*, 55, 251–274. [25]
- WITTMAN, D. (1983): “Candidate Motivation: A Synthesis of Alternative Theories,” *American Political Science Review*, 77(1), 142–157. [14]
- YANG, K. H., AND A. ZENTEFIS (2024): “Monotone Function Intervals: Theory and Applications,” *American Economic Review*, 114(8), 2239–2270. [14]

APPENDIX A: PROOFS

A.1. Duality Lemma

We start with a duality result restating Theorem 1 in [Kolotilin, Corrao, and Wolitzky \(2024\)](#).

LEMMA 2: *There exists a bounded, measurable function $\lambda : [\underline{s}, \bar{s}] \rightarrow \mathbb{R}$ such that, for any optimal districting plan \mathcal{H} , the following hold:*

1. *For all $P \in \text{supp}(\mathcal{H})$, all $s \in \text{supp}(P)$, and all $r \in [\underline{s}, \bar{s}]$, we have*

$$G(r^*(P)) + \lambda(r^*(P))(Q(s - r^*(P)) - Q(0)) \geq G(r) + \lambda(r)(Q(s - r) - Q(0)).$$

2. *For all $P \in \text{supp}(\mathcal{H})$, we have*

$$\lambda(r^*(P)) = \frac{g(r^*(P))}{\int q(s - r^*(P))dP(s)}.$$

3. *For all non-degenerate $P \in \text{supp}(\mathcal{H})$, λ has derivative $\lambda'(r^*(P))$ at $r^*(P)$ satisfying, for all $s \in \text{supp}(P)$,*

$$g(r^*(P)) - \lambda(r^*(P))q(s - r^*(P)) + \lambda'(r^*(P))(Q(s - r^*(P)) - Q(0)) = 0.$$

As explained in the text, $\lambda(r^*(P))$ is the multiplier on the constraint $\int Q(s - r^*(P))dP(s) = Q(0)$. Part 2 of the lemma gives the formula for $\lambda(r^*(P))$ from the implicit function theorem. Part 1 says that the designer assigns a type- s voter to a district P to maximize $G(r^*(P)) + \lambda(r^*(P))(Q(s - r^*(P)) - Q(0))$. Part 3 says that the first-order condition of this maximization problem with respect to r holds for all non-degenerate $P \in \text{supp}(\mathcal{H})$ and all $s \in \text{supp}(P)$.

A.2. Proof of Lemma 1

Lemma 1 follows from Theorem 4 in [Kolotilin, Corrao, and Wolitzky \(2024\)](#) for the translation-invariant subcase of the state-independent sender case. For completeness, we prove Lemmas 3 and 4, which immediately yield Lemma 1.

LEMMA 3: *For any optimal \mathcal{H} and any $P, P' \in \text{supp}(\mathcal{H})$ such that P contains types $s < s''$ and P' contains a type $s' \in (s, s'')$, we have $r^*(P) \geq r^*(P')$.*

PROOF OF LEMMA 3: Suppose by contradiction that there exist districts P and P' such that P contains $s < s''$, P' contains $s' \in (s, s'')$, and $r = r^*(P) < r^*(P') = r'$. Then, by part 1 of Lemma 2, we have

$$G(r) + \lambda(r)(Q(s - r) - Q(0)) \geq G(r') + \lambda(r')(Q(s - r') - Q(0)), \quad (3)$$

$$G(r') + \lambda(r')(Q(s' - r') - Q(0)) \geq G(r) + \lambda(r)(Q(s' - r) - Q(0)), \quad \text{and} \quad (4)$$

$$G(r) + \lambda(r)(Q(s'' - r) - Q(0)) \geq G(r') + \lambda(r')(Q(s'' - r') - Q(0)). \quad (5)$$

This yields a contradiction because

$$\begin{aligned} & 0 \geq (Q(s'' - r') - Q(s' - r'))(Q(s' - r) - Q(s - r)) \\ & \quad - (Q(s'' - r) - Q(s' - r))(Q(s' - r') - Q(s - r')) \\ & = \int_{s'}^{s''} \int_s^{s'} q(\tilde{s}' - r')q(\tilde{s} - r)d\tilde{s}d\tilde{s}' - \int_{s'}^{s''} \int_s^{s'} q(\tilde{s}' - r)q(\tilde{s} - r')d\tilde{s}d\tilde{s}' \\ & = \int_{s'}^{s''} \int_s^{s'} (q(\tilde{s}' - r')q(\tilde{s} - r) - q(\tilde{s}' - r)q(\tilde{s} - r')) d\tilde{s}d\tilde{s}' > 0, \end{aligned}$$

where the first inequality holds by summing (3) multiplied by $Q(s'' - r) - Q(s' - r)$, (4) multiplied by $Q(s'' - r) - Q(s - r)$, and (5) multiplied by $Q(s' - r) - Q(s - r)$, and then dividing by $\lambda(r')$, which is strictly positive by part 2 of Lemma 2; and the second inequality holds because the integrand is strictly positive for $r < r'$ and $\tilde{s} < \tilde{s}'$ by strict log-concavity of q . *Q.E.D.*

LEMMA 4: For any optimal \mathcal{H} and any $P \in \text{supp}(\mathcal{H})$, we have $|\text{supp}(P)| \leq 2$.

PROOF OF LEMMA 4: Suppose by contradiction that some district $P \in \text{supp}(\mathcal{H})$ contains three types $s < s' < s''$ and $r^*(P) = r$. Then, by part 3 of Lemma 2, we have

$$g(r) - \lambda(r)q(s - r) + \lambda'(r)(Q(s - r) - Q(0)) = 0, \quad (6)$$

$$g(r) - \lambda(r)q(s' - r) + \lambda'(r)(Q(s' - r) - Q(0)) = 0, \quad \text{and} \quad (7)$$

$$g(r) - \lambda(r)q(s'' - r) + \lambda'(r)(Q(s'' - r) - Q(0)) = 0. \quad (8)$$

This yields a contradiction because

$$\begin{aligned} 0 & = \det \begin{pmatrix} g(r) & q(s - r) & Q(s - r) - Q(0) \\ g(r) & q(s' - r) & Q(s' - r) - Q(0) \\ g(r) & q(s'' - r) & Q(s'' - r) - Q(0) \end{pmatrix} \\ & = g(r)(q(s' - r) - q(s - r))(Q(s'' - r) - Q(s' - r)) \\ & \quad - g(r)(q(s'' - r) - q(s' - r))(Q(s' - r) - Q(s - r)) \\ & = g(r) \left[\int_s^{s'} q'(\tilde{s} - r)d\tilde{s} \int_{s'}^{s''} q(\tilde{s}' - r)d\tilde{s}' - \int_{s'}^{s''} q'(\tilde{s}' - r)d\tilde{s}' \int_s^{s'} q(\tilde{s} - r)d\tilde{s} \right] \end{aligned}$$

$$> \frac{g(r)q'(s' - r)}{q(s' - r)} \left[\int_s^{s'} q(\tilde{s} - r) d\tilde{s} \int_{s'}^{s''} q(\tilde{s}' - r) d\tilde{s}' - \int_{s'}^{s''} q(\tilde{s}' - r) d\tilde{s}' \int_s^{s'} q(\tilde{s} - r) d\tilde{s} \right] = 0,$$

where the first equality is by (6)–(8), and the inequality is by strict log-concavity of q , which implies that the derivative of $\ln q$ is strictly decreasing, yielding

$$\frac{q'(\tilde{s} - r)}{q(\tilde{s} - r)} > \frac{q'(s' - r)}{q(s' - r)} > \frac{q'(\tilde{s}' - r)}{q(\tilde{s}' - r)}, \quad \text{for } \tilde{s} < s' < \tilde{s}'. \quad Q.E.D.$$

A.3. Characterization of Pack-and-Pair Districting

LEMMA 5: For any pack-and-pair districting plan \mathcal{H} , there exists a bifurcation point $r^b \in [\underline{s}, \bar{s}]$, a decreasing function $s_1 : (r^b, \bar{s}] \rightarrow [\underline{s}, r^b)$, and an increasing function $s_2 : (r^b, \bar{s}] \rightarrow (r^b, \bar{s}]$ satisfying $s_1(r) < r < s_2(r)$, such that for each $P \in \text{supp}(\mathcal{H})$, we have $P = \delta_{r^*(P)}$ if $r^*(P) \leq r^b$ and $\text{supp}(P) = \{s_1(r^*(P)), s_2(r^*(P))\}$ if $r^*(P) > r^b$.

PROOF OF LEMMA 5: Let \mathcal{H} be a pack-and-pair districting plan. Since \mathcal{H} is strictly single-dipped, the support of each $P \in \text{supp}(\mathcal{H})$ has at most two elements and thus can be represented as $\{s_1(r^*(P)), s_2(r^*(P))\}$ with $s_1(r^*(P)) \leq r^*(P) \leq s_2(r^*(P))$. Moreover, for each $P, P' \in \text{supp}(\mathcal{H})$ with $r^*(P) < r^*(P')$, we have $s_2(r^*(P)) \leq s_2(r^*(P'))$, as otherwise we would have $s_2(r^*(P')) \in (s_1(r^*(P)), s_2(r^*(P)))$, contradicting strict single-dippedness of \mathcal{H} .

Assume that there exists $P \in \text{supp}(\mathcal{H})$ such that $s_1(r^*(P)) < s_2(r^*(P))$, as otherwise the lemma obviously holds with $r^b = \bar{s}$. Define $r^b = \inf\{r^*(\tilde{P}) : \tilde{P} \in \text{supp}(\mathcal{H}), s_1(r^*(\tilde{P})) < s_2(r^*(\tilde{P}))\}$, so that, for each $P \in \text{supp}(\mathcal{H})$ with $r^*(P) < r^b$, we have $\text{supp}(P) = \{r^*(P)\}$. Since $\text{supp}(\mathcal{H})$ is compact, there exists $P^b \in \text{supp}(\mathcal{H})$ with $r^*(P^b) = r^b$. It follows that $\text{supp}(P^b) = \{r^b\}$, as otherwise (i.e., if $s_1(r^*(P^b)) < r^b < s_2(r^*(P^b))$) voter types in $(r^b, s_2(r^*(P^b)))$ (which have strictly positive mass since f is strictly positive on $[\underline{s}, \bar{s}]$) cannot be segregated, as this would contradict strict single-dippedness of \mathcal{H} , and also cannot be paired with other types, as this would contradict either strict single-dippedness of \mathcal{H} or the definition of r^b .

Finally, we show that, for each $P, P' \in \text{supp}(\mathcal{H})$ with $r^b < r^*(P) < r^*(P')$, we have $s_1(r^*(P)) \geq s_1(r^*(P'))$. Suppose by contradiction that $s_1(r^*(P)) < s_1(r^*(P'))$. Since \mathcal{H} is a strictly single-dipped pack-and-pair districting plan, by the definition of r^b , we have $s_1(r^*(P)) < r^*(P) < s_2(r^*(P)) \leq s_1(r^*(P')) < r^*(P') < s_2(r^*(P'))$. Define $r^\dagger = \inf\{r^*(\tilde{P}) : \tilde{P} \in \text{supp}(\mathcal{H}), s_1(r^*(P')) \leq s_1(r^*(\tilde{P})) < s_2(r^*(\tilde{P})) \leq s_2(r^*(P'))\} \geq s_1(r^*(P'))$. By the same argument as in the previous paragraph, we have $\delta_{r^\dagger} \in \text{supp}(\mathcal{H})$, contradicting that \mathcal{H} is pack-and-pair. Q.E.D.

A.4. Auxiliary Lemmas

Lemmas 6–8 are used to prove Theorems 1 and 2.

LEMMA 6: *If for all $s < r < s'$ such that*

$$G(r) + \lambda(r)(Q(s-r) - Q(0)) \geq G(s), \quad (9)$$

where

$$\lambda(r) = \frac{g(r)(Q(s'-r) - Q(s-r))}{(Q(s'-r) - Q(0))q(s-r) - (Q(s-r) - Q(0))q(s'-r)}, \quad (10)$$

we have, for all $s'' \geq s'$,

$$G(r) + \lambda(r)(Q(s-r) - Q(0)) < G(s'') + \frac{g(s'')}{q(0)}(Q(s-s'') - Q(0)), \quad (11)$$

then there is a unique optimal districting plan, which is pack-and-pair.

PROOF OF LEMMA 6: Suppose by contradiction that there exists an optimal non-pack-and-pair plan \mathcal{H} . By Lemma 1, \mathcal{H} is strictly single-dipped. Consequently, since \mathcal{H} is not pack-and-pair, there exist $s < r < s' \leq s''$ and $P, P' \in \text{supp}(\mathcal{H})$ such that $r^*(P) = r$, $\text{supp}(P) = \{s, s'\}$, and $\text{supp}(P') = \{s''\}$. By Lemma 2, condition (9) holds and condition (11) fails, yielding a contradiction.⁵³ Finally, for uniqueness, by Theorem 7 in Kolotilin, Corrao, and Wolitzky (2024), it suffices to show that \mathcal{H} is *regular*, in that for each $P \in \text{supp}(\mathcal{H})$, there exists $\varepsilon > 0$ such that either (i) $|\text{supp}(\tilde{P})| = 1$ for all $\tilde{P} \in \text{supp}(\mathcal{H})$ satisfying $r^*(\tilde{P}) \in (r^*(P) - \varepsilon, r^*(P))$, or (ii) $|\text{supp}(\tilde{P})| = 2$ for all $\tilde{P} \in \text{supp}(\mathcal{H})$ satisfying $r^*(\tilde{P}) \in (r^*(P) - \varepsilon, r^*(P))$. But each pack-and-pair plan \mathcal{H} is clearly regular, with any $\varepsilon > 0$ for $r^*(P) \leq r^b$ and with any $\varepsilon \in (0, r^*(P) - r^b)$ for $r^*(P) > r^b$. Q.E.D.

LEMMA 7: *If $\eta \geq 1$ and $s < r < s'$ satisfy (9), with $\lambda(r)$ given by (10), then $r > 0$.*

PROOF OF LEMMA 7: If $r \leq 0$, then (9) fails, because

$$\begin{aligned} G(r) - G(s) &= \int_s^r g(x)dx \leq \frac{g(r)}{g(0)} \int_s^r g(x-r)dx = \frac{g(r)}{g(0)}(G(0) - G(s-r)) \\ &= \frac{g(r)}{\eta q(0)}(Q(\eta 0) - Q(\eta(s-r))) \leq \frac{g(r)}{q(0)}(Q(0) - Q(s-r)) < \lambda(r)(Q(0) - Q(s-r)), \end{aligned}$$

where the first inequality is by strict log-concavity of g on $[s, 0]$, the second inequality is by $\eta \geq 1$ and strict convexity of Q on $[s-r, 0]$, and the last inequality is by $\lambda(r) > g(r)/q(0)$, which holds because q is uniquely maximized at 0. Q.E.D.

LEMMA 8: *If \mathcal{H} is optimal and $\delta_s, \delta_{s'} \in \text{supp}(\mathcal{H})$ with $s < s'$, then $s < 0$.*

⁵³Intuitively, (9) says that the designer prefers not to move a few type- s voters from district P to district δ_s , and (11) says that the designer strictly prefers to move a few type- s voters from district P to district $\delta_{s''}$.

PROOF: Suppose by contradiction that $s \geq 0$. By Lemma 2, for each $r \in (s, s')$ there exists $\lambda(r)$ such that

$$G(s) \geq G(r) + \lambda(r)(Q(s-r) - Q(0)) \quad \text{and} \quad (12)$$

$$G(s') \geq G(r) + \lambda(r)(Q(s'-r) - Q(0)). \quad (13)$$

Summing (12) multiplied by $Q(s'-r) - Q(0)$ and (13) multiplied by $Q(0) - Q(s-r)$ yields

$$(G(s') - G(r))(Q(0) - Q(s-r)) \geq (G(r) - G(s))(Q(s'-r) - Q(0)).$$

But this inequality cannot hold for r sufficiently close to s' , because

$$g(s')(Q(0) - Q(s-s')) < (G(s') - G(s))q(0), \quad (14)$$

since G is strictly concave on $[s, s']$ and Q is strictly convex on $[s-s', 0]$. *Q.E.D.*

A.5. Proof of Theorem 1

If $s < r < s'$ satisfy (9), with $\lambda(r)$ given by (10), then $r > 0$, by Lemma 7. Theorem 1 then follows from Lemma 6, as (11) holds for all $s'' \geq s'$, because

$$\begin{aligned} & G(s'') + \frac{g(s'')}{q(0)}(Q(s-s'') - Q(0)) - G(r) - \lambda(r)(Q(s-r) - Q(0)) \\ & > \frac{g(r)}{q(0)}(Q(0) - Q(s-r)) + G(s'') - G(r) - \frac{g(s'')}{q(0)}(Q(0) - Q(s-s'')) \\ & > \frac{g(s'')}{q(0)}[Q(0) - Q(s-r) + q(0)(s''-r) - (Q(0) - Q(s-s''))] > 0, \end{aligned}$$

where the first inequality is by $\lambda(r) > g(r)/q(0)$, the second inequality is by strict concavity of G on $[r, s'']$, and the third inequality is by strict convexity of Q on $[s-s'', 0]$.

A.6. Proof of Theorem 2

Part 1. Let \mathcal{H} be an optimal strictly single-dipped plan. By Lemma 8, there do not exist $s < s'$ in $[\underline{s}, \bar{s}]$ such that $\delta_s, \delta_{s'} \in \text{supp}(\mathcal{H})$. Then, by Theorem 6 in Kolotilin, Corrao, and Wolitzky (2024), \mathcal{H} is NAD.

Part 2. Suppose by contradiction that there exist an optimal strictly single-dipped plan \mathcal{H} and $P \in \text{supp}(\mathcal{H})$ such that $r^*(P) = r$ and $\text{supp}(P) = \{s, s'\}$ with $s < r < s'$. By Lemma 2, (9) holds with $\lambda(r)$ given by (10). So, by Lemma 7, $r^*(P) > 0$, contradicting that $r < s' \leq \bar{s} \leq 0$.

Part 3. Since f is strictly positive on $[\underline{s}, \bar{s}]$ and $\underline{s} < \bar{s}$, we have $\underline{s} < r^*(F) = 0 < \bar{s}$, so segregation is suboptimal by Lemma 8.

Suppose by contradiction that there exists an optimal NAD plan \mathcal{H} . By Lemma 5, for each $P \in \text{supp}(\mathcal{H})$ except for $\delta_{r,b}$, we have $s_1(r^*(P)) < r^*(P) < s_2(r^*(P))$, where s_1 is decreasing

and s_2 is increasing. Note that $r^b < r^*(F) = 0$, because

$$\begin{aligned} \int Q(s - r^*(F))dF(s) &= Q(0) = \iint Q(s - r^*(P))dP(s)d\mathcal{H}(P) \\ &< \iint Q(s - r^b)dP(s)d\mathcal{H}(P) = \int Q(s - r^b)dF(s), \end{aligned}$$

where the first two equalities hold by the definition of $r^*(F)$ and $r^*(P)$, the inequality holds by $r^*(P) > r^b$ for all $P \in \text{supp}(\mathcal{H})$ except for $P = \delta_{r^b}$, and the last equality holds by $\int P d\mathcal{H}(P) = F$. Since f is strictly positive on $[\underline{s}, \bar{s}]$, we get $\lim_{r \downarrow r^b} s_1(r) = \lim_{r \downarrow r^b} s_2(r) = r^b$, as otherwise voter types in $(\lim_{r \downarrow r^b} s_1(r), \lim_{r \downarrow r^b} s_2(r))$ are not assigned to any district. Thus, there exists $\varepsilon \in (0, -r^b)$ and $P \in \text{supp}(\mathcal{H})$ such that $r^b - \varepsilon < s_1(r^*(P)) < s_2(r^*(P)) < r^b + \varepsilon < 0$. By Lemma 2, $s = s_1(r^*(P))$, $r = r^*(P)$, and $s' = s_2(r^*(P))$ satisfy (9), with $\lambda(r)$ given by (10). Then, by Lemma 7, we have $r^*(P) > 0$, contradicting that $r^*(P) < s_2(r^*(P)) < 0$.

A.7. Proof of Theorem 3

Theorem 3 follows from Lemmas 9–13.

For each r , let $\mathcal{R}(r)$ be the set of all plans \mathcal{H} that maximize the designer's seat share when the aggregate shock is r . Lemma 9 characterizes $\mathcal{R}(r)$. If $r^*(F) \geq r$, then $\mathcal{H} \in \mathcal{R}(r)$ assigns all voters to districts that the designer wins. If $r^*(F) < r$, then $\mathcal{H} \in \mathcal{R}(r)$ assigns all voter types above $s^*(r)$ to cracked districts that the designer wins with exactly 50% of the vote and packs the remaining voters arbitrarily.

LEMMA 9: *The following hold.*

1. *Let $r^*(F) \geq r$. Then $\mathcal{H} \in \mathcal{R}(r)$ iff, for each $P \in \text{supp}(\mathcal{H})$, we have $r^*(P) \geq r$.*
2. *Let $r^*(F) < r$. Then $\mathcal{H} \in \mathcal{R}(r)$ iff, for each $P \in \text{supp}(\mathcal{H})$, we have either $r^*(P) \leq s^*(r)$ and $\text{supp}(P) \subset [\underline{s}, s^*(r)]$ or $r^*(P) = r$ and $\text{supp}(P) \subset [s^*(r), \bar{s}]$.*

PROOF OF LEMMA 9: *Part 1.* Since $r^*(F) \geq r$, $s^*(r) = \underline{s}$ and $\delta_F \in \mathcal{R}(r)$ is optimal. Hence, $\mathcal{H} \in \mathcal{R}(r)$ iff $\int \mathbf{1}\{r \leq r^*(P)\}d\mathcal{H}(P) = 1$. Moreover, this is equivalent to $r^*(P) \geq r$ for all $P \in \text{supp}(\mathcal{H})$, because the set $\{P \in \Delta[\underline{s}, \bar{s}] : r^*(P) \geq r\}$ is closed by the continuity of r^* , which follows from the continuity and strict monotonicity of Q .

Part 2. For each plan \mathcal{H} , we have

$$\begin{aligned}
\int \mathbf{1}\{r \leq r^*(P)\} d\mathcal{H}(P) &= \int \mathbf{1}\{\mathbb{E}_P[Q(s-r) - Q(0)] \geq 0\} d\mathcal{H}(P) \\
&\leq \int \max\left\{0, \frac{\mathbb{E}_P[Q(s-r)] - Q(s^*(r) - r)}{Q(0) - Q(s^*(r) - r)}\right\} d\mathcal{H}(P) \\
&\leq \iint \max\left\{0, \frac{Q(s-r) - Q(s^*(r) - r)}{Q(0) - Q(s^*(r) - r)}\right\} dP(s) d\mathcal{H}(P) \\
&= \int \max\left\{0, \frac{Q(s-r) - Q(s^*(r) - r)}{Q(0) - Q(s^*(r) - r)}\right\} dF(s) \\
&= \int_{s^*(r)}^{\bar{s}} \frac{Q(s-r) - Q(s^*(r) - r)}{Q(0) - Q(s^*(r) - r)} dF(s) = 1 - F(s^*(r)),
\end{aligned} \tag{15}$$

where the first equality is by the definition of $r^*(P)$, the first inequality is by pointwise dominance of the integrands, the second inequality is by Jensen's inequality, the second equality is by $\int P d\mathcal{H}(P) = F$, and the last equality is by the definition of $s^*(r)$. Hence, $\mathcal{H} \in \mathcal{R}(r)$ iff, for a measure-1 set of districts P under \mathcal{H} , we have (a) $r^*(P) \in [\underline{s}, s^*(r)] \cup \{0\}$ (as otherwise the first inequality in (15) is strict) and (b) $\text{supp}(P) \subset [s^*(r), \bar{s}]$ if $r^*(P) = 0$ and $\text{supp}(P) \subset [\underline{s}, s^*(r)]$ if $r^*(P) \leq s^*(r)$ (as otherwise the second inequality in (15) is strict). Finally, as above, continuity implies that that properties (a) and (b) hold for all $P \in \text{supp}(\mathcal{H})$, rather than just for a measure-1 set. *Q.E.D.*

Lemma 10 shows that pack-and-crack districting is approximately optimal. An upper bound on the designer's optimal expected seat share V_η can be obtained by allowing the designer to choose $\mathcal{H}_r \in \mathcal{R}(r)$ after observing each realization r ,

$$\bar{V}_\eta = \int (1 - F(s^*(r))) dG_\eta(r).$$

A lower bound on V_η can be obtained by restricting attention to $\mathcal{H}_{\tilde{r}} \in \mathcal{R}(\tilde{r})$ for some \tilde{r} ,

$$\underline{V}_\eta(\tilde{r}) = \int (1 - F(s^*(\tilde{r}))) \mathbf{1}\{r \leq \tilde{r}\} dG_\eta(r).$$

LEMMA 10: For all η and all \tilde{r} , we have $\underline{V}_\eta(\tilde{r}) < V_\eta < \bar{V}_\eta$. Moreover, if $\eta \rightarrow \infty$, then $\bar{V}_\eta \rightarrow 1 - F(s^*(0))$, $\underline{V}_\eta(\tilde{r}) \rightarrow 1 - F(s^*(\tilde{r}))$ for all $\tilde{r} > 0$, and $V_\eta \rightarrow 1 - F(s^*(0))$.

PROOF OF LEMMA 10: Let \mathcal{H}_η be the optimal plan and let \mathcal{H}_r be any districting plan in $\mathcal{R}(r)$. On the one hand, we have

$$V_\eta = \iint \mathbf{1}\{r \leq r^*(P)\} d\mathcal{H}_\eta(P) dG_\eta(r)$$

$$< \iint \mathbf{1}\{r \leq r^*(P)\} d\mathcal{H}_r(P) dG_\eta(r) = \int (1 - F(s^*(r))) dG_\eta(r) = \bar{V}_\eta,$$

where the inequality holds weakly because $\int \mathbf{1}\{r \leq r^*(P)\} d\mathcal{H}_\eta(P) \leq \int \mathbf{1}\{r \leq r^*(P)\} d\mathcal{H}_r(P)$ for each r by the definition of \mathcal{H}_r , and it holds strictly because $g(r) > 0$ and $\mathcal{R}(r) \cap \mathcal{R}(r')$ is empty for $r' > r \geq r^*(F)$, as follows from Lemma 9.

On the other hand, for any \tilde{r} , we have

$$\begin{aligned} V_\eta &= \iint \mathbf{1}\{r \leq r^*(P)\} d\mathcal{H}_\eta(P) dG_\eta(r) \\ &\geq \iint \mathbf{1}\{r \leq r^*(P)\} d\mathcal{H}_{\tilde{r}}(P) dG_\eta(r) > \int (1 - F(s^*(\tilde{r}))) \mathbf{1}\{r \leq \tilde{r}\} dG_\eta(r) = \underline{V}_\eta(\tilde{r}), \end{aligned}$$

where the first inequality holds because \mathcal{H}_η is optimal and $\mathcal{H}_{\tilde{r}}$ is feasible, and the second inequality holds strictly because the designer wins even districts with voter types $s < s^*(\tilde{r})$ when $r \leq \underline{s}$.

Suppose now that $\eta \rightarrow \infty$, which implies that $G_\eta \rightarrow \delta_0$. By the implicit function theorem, $F(s^*(r))$ is continuous in r , so $\bar{V}_\eta \rightarrow 1 - F(s^*(0))$. For $\tilde{r} > 0$, $\underline{V}_\eta(\tilde{r}) \rightarrow 1 - F(s^*(\tilde{r}))$, which converges to $1 - F(s^*(0))$ as $\tilde{r} \downarrow 0$, implying that $V_\eta \rightarrow 1 - F(s^*(0))$. *Q.E.D.*

Lemma 11 shows that limit points of optimal plans $H_n = H_{\eta_n}$, for $\eta_n \rightarrow \infty$, belong to $\mathcal{P}(0)$.

LEMMA 11: *Let $\mathcal{H}_n \rightarrow \mathcal{H}$ as $\eta_n \rightarrow \infty$. Then $\mathcal{H} \in \mathcal{R}(0)$.*

PROOF OF LEMMA 11: Suppose by contradiction that there exists a sequence $\eta_n \rightarrow \infty$ such that an optimal plan \mathcal{H}_n converges weakly to $\mathcal{H} \notin \mathcal{R}(0)$. Then we have

$$1 - F(s^*(0)) = \lim_{n \rightarrow \infty} \int Q(\eta_n r^*(P)) d\mathcal{H}_n(P) \leq \int \mathbf{1}\{r^*(P) \geq 0\} d\mathcal{H}(P) < 1 - F(s^*(0)),$$

where the equality is by Lemma 10, the first inequality is by the Portmanteau theorem, and the second inequality is by Lemma 9. *Q.E.D.*

Lemma 12 shows that, in the limit, all districts are equally strong when $r^*(F) \geq 0$.

LEMMA 12: *Let $r^*(F) \geq 0$ and $\mathcal{H}_n \rightarrow \mathcal{H}$ as $\eta_n \rightarrow \infty$. Then, for each $P \in \text{supp}(\mathcal{H})$, we have $r^*(P) = r^*(F)$.*

PROOF OF LEMMA 12: If $r^*(F) = 0$, then, for each $P \in \text{supp}(\mathcal{H})$, we have $r^*(P) \geq 0$ by Lemma 9, so $r^*(P) = 0$ by $\int P d\mathcal{H}(P) = F$. So suppose that $r^*(F) > 0$. Moreover, suppose by contradiction that there exists $\varepsilon \in (0, r^*(F))$, $\delta \in (0, 1)$, and a sequence $\eta_n \rightarrow \infty$ such that $\int \mathbf{1}\{r^*(P) \leq r^*(F) - \varepsilon\} d\mathcal{H}_n(P) \geq \delta$ for all n . We obtain a contradiction for sufficiently large

n , because

$$\int Q(\eta_n r^*(P)) d\mathcal{H}_n(P) \leq \delta Q(\eta_n(r^*(F) - \varepsilon)) + (1 - \delta) < Q(\eta_n r^*(F)),$$

where the first inequality is by the supposition and the second inequality is by

$$\frac{1 - Q(\eta r^*(F))}{1 - Q(\eta(r^*(F) - \varepsilon))} \rightarrow 0, \quad \text{as } \eta \rightarrow \infty, \quad (16)$$

which we prove below. Denote $c = q'(r^*(F) - \varepsilon)/q(r^*(F) - \varepsilon)$. Since $q'(0) = 0$ and q is strictly log-concave, for all $\eta > 1$, we have

$$0 = \frac{q'(0)}{q(0)} > c = \frac{q'(r^*(F) - \varepsilon)}{q(r^*(F) - \varepsilon)} > \frac{q'(\eta(r^*(F) - \varepsilon))}{q(\eta(r^*(F) - \varepsilon))} > \frac{q'(x)}{q(x)}, \quad \text{for all } x > \eta(r^*(F) - \varepsilon).$$

Hence Gronwall's inequality gives $\lim_{\eta \rightarrow \infty} q(\eta r^*(F))/q(\eta(r^*(F) - \varepsilon)) \leq \lim_{\eta \rightarrow \infty} e^{c\varepsilon\eta} = 0$, so, by L'Hopital's rule, we have

$$\lim_{\eta \rightarrow \infty} \frac{1 - Q(\eta r^*(F))}{1 - Q(\eta(r^*(F) - \varepsilon))} = \lim_{\eta \rightarrow \infty} \frac{q(\eta r^*(F))r^*(F)}{q(\eta(r^*(F) - \varepsilon))(r^*(F) - \varepsilon)} = 0. \quad Q.E.D.$$

Lemma 13 shows that, in the limit, types below $s^*(0)$ are segregated and types above $s^*(0)$ are paired in a negatively assortative manner.

LEMMA 13: Let $\mathcal{H}_n \rightarrow \mathcal{H}$ as $\eta_n \rightarrow \infty$.

1. For any $P \in \text{supp}(\mathcal{H})$ with $r^*(P) \leq s^*(0)$, we have $|\text{supp}(P)| = 1$.
2. For any $P, P' \in \text{supp}(\mathcal{H})$ with $r^*(P) = r^*(P') \geq 0$, we have $\text{supp}(P) = \{s_1(P), s_2(P)\}$ and $\text{supp}(P') = \{s_1(P'), s_2(P')\}$ with $s_1(P) \leq s_2(P)$, $s_1(P') \leq s_2(P')$, and $(s_2(P') - s_2(P))(s_1(P) - s_1(P')) \geq 0$.

PROOF OF LEMMA 13: Denote $\Lambda_n = \text{supp}(\mathcal{H}_n)$. Since the set of compact subsets of a compact set is compact (in the Hausdorff topology), taking a subsequence if necessary, Λ_n converges to some compact set Λ . By Box 1.13 in Santambrogio (2015), we have $\text{supp}(\mathcal{H}) \subset \Lambda$. Since \mathcal{H}_n is strictly single-dipped by Lemma 1, we have $|\text{supp}(P_n)| \leq 2$ for all $P_n \in \Lambda_n$, and thus $|\text{supp}(P)| \leq 2$ for all $P \in \Lambda$.

Suppose part 2 fails. Then, by Lemmas 9, 11, and 12, there exist $P, P' \in \text{supp}(\mathcal{H})$ such that $s_1(P') < s_1(P) < r_+^*(F) < s_2(P') < s_2(P)$. But then since $\Lambda_n \rightarrow \Lambda$, there exist n and $P_n, P'_n \in \Lambda_n$ such that $\text{supp}(P_n) = \{s_1(P_n), s_2(P_n)\}$, $\text{supp}(P'_n) = \{s_1(P'_n), s_2(P'_n)\}$, and $s_1(P'_n) < s_1(P_n) < r_+^*(F) < s_2(P'_n) < s_2(P_n)$, contradicting that \mathcal{H}_n is strictly single-dipped.

Suppose part 1 fails. Then, by Lemmas 9 and 11, there exists $P \in \text{supp}(\mathcal{H})$ such that $\text{supp}(P) = \{s, s'\}$ with $\underline{s} \leq s < s' \leq s^*(0)$. Moreover, by Lemmas 9 and 11 and part 2, there exists $P' \in \text{supp}(\mathcal{H})$ such that $\text{supp}(P') = \{s^*(0), \bar{s}\}$. But then since $\Lambda_n \rightarrow \Lambda$, there exist

n and $P_n, P'_n \in \Lambda_n$ with $(s_1(P_n), s_2(P_n), s_1(P'_n), s_2(P'_n))$ close to $(s, s', s^*(0), \bar{s})$. Then, by Lemma 5, \mathcal{H}_n cannot be pack-and-pair, contradicting Theorem 1. Q.E.D.

To complete the proof of Theorem 3, note that Lemmas 9 (for $r = 0$), 11, 12, and 13 show that if a sequence of optimal plans \mathcal{H}_η converge to \mathcal{H} , then \mathcal{H} must segregate types below $s^*(0)$ and pair types above $s^*(0)$ in a negatively assortative manner in equally strong districts. By (1), the unique such plan is $\mathcal{H} = \mathcal{H}^*$. Finally, since every convergent sequence \mathcal{H}_n converges to \mathcal{H}^* , compactness of $\Delta\Delta[\underline{s}, \bar{s}]$ implies that \mathcal{H}_η also converges to \mathcal{H}^* .

A.8. Proof of Theorem 4

Theorem 4 follows from Lemmas 14–17.

Let \mathcal{T} be the set of all plans \mathcal{H} that maximize the designer's seat share when each voter's idiosyncratic shock is 0. Lemma 14 shows that $\mathcal{H} \in \mathcal{T}$ iff each district $P \in \text{supp}(\mathcal{H})$ contains 50% voters with some type $s^P \geq s^m$ and 50% voters with types $s \leq s^m$ (so the designer wins district P iff $r \leq s^P$).

For $P \in \Delta[\underline{s}, \bar{s}]$, define $\bar{P}(r) = \int \mathbf{1}\{s \geq r\} dP(s) = 1 - P(r_-)$ for all r . The designer wins district P iff the aggregate shock r satisfies $r \leq r_0^*(P) = \{\max r : \bar{P}(r) \geq 1/2\}$. For $\mathcal{H} \in \Delta\Delta[\underline{s}, \bar{s}]$, define $\bar{H}(r) = \int \mathbf{1}\{r_0^*(P) \geq r\} d\mathcal{H}(P)$ for all r .

LEMMA 14: $\mathcal{H} \in \mathcal{T}$ iff, for each $P \in \text{supp}(\mathcal{H})$, there exists $s^P \geq s^m$ such that $P(s) = 1$ for all $s \geq s^P$, $P(s) = 1/2$ for all $s \in [s^m, s^P)$, and $P(s) \leq 1/2$ for all $s < s^m$.

PROOF OF LEMMA 14: For each $r \geq s^m$, we have

$$\begin{aligned} \bar{F}(r) &= \int \bar{P}(r) d\mathcal{H}(P) = \int \mathbf{1}\{\bar{P}(r) \geq \frac{1}{2}\} \bar{P}(r) d\mathcal{H}(P) + \int \mathbf{1}\{\bar{P}(r) < \frac{1}{2}\} \bar{P}(r) d\mathcal{H}(P) \\ &\geq \int \mathbf{1}\{\bar{P}(r) \geq \frac{1}{2}\} \frac{1}{2} d\mathcal{H}(P) = \int \mathbf{1}\{r_0^*(P) \geq r\} \frac{1}{2} d\mathcal{H}(P) = \frac{1}{2} \bar{H}(r). \end{aligned} \tag{17}$$

So, any feasible \mathcal{H} satisfies $\bar{H}(r) \leq \bar{H}^*(r)$ for all r , where

$$\bar{H}^*(r) = \begin{cases} 1, & \text{if } r \leq s^m, \\ 2\bar{F}(r), & \text{if } r > s^m. \end{cases}$$

Thus, the designer's expected seat share for any feasible plan is $\int \bar{H}(r) dG(r) \leq \int \bar{H}^*(r) dG(r)$, with strict inequality if $\bar{H}(r) < \bar{H}^*(r)$ for some r (and thus on some interval (r', r) with $r' < r$, by continuity of \bar{H}^* and monotonicity and left-continuity of \bar{H}), because $G(r)$ is strictly increasing in r . Hence, a districting plan \mathcal{H} is optimal iff it induces $\bar{H} = \bar{H}^*$. In turn, $\bar{H} = \bar{H}^*$ iff, for each $r \geq s^m$, the inequality in (17) holds with equality, or equivalently, $\int \mathbf{1}\{\bar{P}(r) = 1/2\} d\mathcal{H}(P) = 2\bar{F}(r)$ and $\int \mathbf{1}\{\bar{P}(r) = 0\} d\mathcal{H}(P) = 1 - 2\bar{F}(r)$. Finally, this holds for all $r \geq s^m$ iff, for each $P \in \text{supp}(\mathcal{H})$, there exists $s^P \geq s^m$ such that $\bar{P}(s) = 0$ for all $s > s^P$, $\bar{P}(s) = 1/2$ for all $s \in (s^m, s^P)$, and $\bar{P}(s) \geq 1/2$ for all $s \leq s^m$. Q.E.D.

LEMMA 15: If $\eta \rightarrow 0$, then $V_\eta \rightarrow 2 \int_{s^m}^{\bar{s}} G(r) dF(r)$.

PROOF OF LEMMA 15: Let \mathcal{H}_η be the optimal plan and let \mathcal{H}_r be any districting plan in $\mathcal{R}_\eta(r)$. Then

$$\begin{aligned} V_\eta &= \iint \mathbf{1}\{r \leq r_\eta^*(P)\} d\mathcal{H}_\eta(P) dG(r) \\ &\leq \iint \mathbf{1}\{r \leq r_\eta^*(P)\} d\mathcal{H}_r(P) dG(r) = \int (1 - F(s_\eta^*(r))) dG(r) = \bar{V}_\eta, \end{aligned}$$

where the inequality holds because $\int \mathbf{1}\{r \leq r_\eta^*(P)\} d\mathcal{H}_\eta(P) \leq \int \mathbf{1}\{r \leq r_\eta^*(P)\} d\mathcal{H}_r(P)$ for each r by the definition of \mathcal{H}_r .

Let \mathcal{H}_q^* , with $q \in (0, 1/2)$, be NAD with a $q-1-q$ split in each district. Formally, \mathcal{H}_q^* is the unique plan \mathcal{H} such that, for any $P \in \text{supp}(\mathcal{H})$, we have either (a) $\text{supp}(P) = \{s^q\}$ with $s^q = F^{-1}(q)$ or (b) $\text{supp}(P) = \{s_1(P), s_2(P)\}$ such that $\underline{s} \leq s_1(P) < s^q < s_2(P) \leq \bar{s}$, and $(1-q)F(s_1(P)) = q(1 - F(s_2(P)))$. We have

$$V_\eta = \int G(r_\eta^*(P)) d\mathcal{H}_\eta(P) \geq \int G(r_\eta^*(P)) d\mathcal{H}_q^*(P) = \underline{V}_\eta(q),$$

where the inequality holds because \mathcal{H}_η is optimal and \mathcal{H}_q^* is feasible.

Suppose now that $\eta \rightarrow 0$, which implies that $Q_\eta \rightarrow \delta_0$. For each r , $1 - F(s_\eta^*(r)) \rightarrow \bar{H}^*(r)$, so, by the dominated convergence theorem and integration by parts, $\bar{V}_\eta \rightarrow \int \bar{H}^*(r) dG(r) = 2 \int_{s^m}^{\bar{s}} G(r) dF(r)$. For $q < 1/2$ and $s < s'$, $r_\eta^*(q\delta_s + (1-q)\delta_{s'}) \rightarrow s'$, so, by the dominated convergence theorem, $\underline{V}_\eta(q) \rightarrow \int_{s^q}^{\bar{s}} G(r) dF(r)/(1-q)$, which converges to $2 \int_{s^m}^{\bar{s}} G(r) dF(r)$ as $q \uparrow 1/2$. Q.E.D.

Lemma 16 shows that limit points of optimal plans $H_n = H_{\eta_n}$, for $\eta_n \rightarrow 0$, belong to \mathcal{T} .

LEMMA 16: Let $\mathcal{H}_n \rightarrow \mathcal{H}$ as $\eta_n \rightarrow \infty$. Then $\mathcal{H} \in \mathcal{T}$.

PROOF OF LEMMA 16: Suppose by contradiction that there exists a sequence $\eta_n \rightarrow 0$ such that \mathcal{H}_n converges weakly to $\mathcal{H} \notin \mathcal{T}$. Then we have

$$2 \int_{s^m}^{\bar{s}} G(r) dF(r) = \lim_{n \rightarrow \infty} \int G(r_{\eta_n}^*(P)) d\mathcal{H}_n(P) \leq \int \bar{H}(r) dG(r) < 2 \int_{s^m}^{\bar{s}} G(r) dF(r),$$

where the equality is by Lemma 15, the first inequality is by the Portmanteau theorem, and the second inequality is by Lemma 14. Q.E.D.

Lemma 17 shows that, in the limit, all types are paired in a negatively assortative manner.

LEMMA 17: Let $\mathcal{H}_n \rightarrow \mathcal{H}$ as $\eta_n \rightarrow 0$. For any $P, P' \in \text{supp}(\mathcal{H})$, we have $\text{supp}(P) = \{s_1(P), s_2(P)\}$ and $\text{supp}(P') = \{s_1(P'), s_2(P')\}$ with $s_1(P) \leq s_2(P)$, $s_1(P') \leq s_2(P')$, and $(s_2(P') - s_2(P))(s_1(P) - s_1(P')) \geq 0$.

PROOF OF LEMMA 17: Denoting $\Lambda_n = \text{supp}(\mathcal{H}_n)$, the same argument as in the proof of Lemma 13 implies that there exists Λ such that, up to a subsequence, $\Lambda_n \rightarrow \Lambda$, $\text{supp}(\mathcal{H}) \subset \Lambda$, and $|\text{supp}(P)| \leq 2$ for all $P \in \Lambda$.

By Lemmas 14 and 16, if the conclusion of the lemma fails, there must exist $P, P' \in \text{supp}(\mathcal{H})$ such that $\text{supp}(P) = \{s_1(P), s_2(P)\}$ and $\text{supp}(P') = \{s_1(P'), s_2(P')\}$ with $s_1(P') < s_1(P) < s^m < s_2(P') < s_2(P)$. Then, since $\Lambda_n \rightarrow \Lambda$, there exist n and $P_n, P'_n \in \Lambda_n$ such that $\text{supp}(P_n) = \{s_1(P_n), s_2(P_n)\}$, $\text{supp}(P'_n) = \{s_1(P'_n), s_2(P'_n)\}$, and $s_1(P'_n) < s_1(P_n) < s^m < s_2(P'_n) < s_2(P_n)$, contradicting that \mathcal{H}_n is strictly single-dipped. *Q.E.D.*

To complete the proof of Theorem 4, note that Lemmas 14, 16, and 17 show that if a sequence of optimal plans \mathcal{H}_η converge to \mathcal{H} , then \mathcal{H} must pair all types in a negatively assortative manner, with 50% mass on the higher type. Clearly, the unique such plan is $\mathcal{H} = \mathcal{H}^{**}$. Since every convergent sequence \mathcal{H}_n converges to \mathcal{H}^{**} , compactness of $\Delta\Delta[\underline{s}, \bar{s}]$ implies that \mathcal{H}_η also converges to \mathcal{H}^{**} .

A.9. Proof of Theorem 5

By Lemma 2, $\lambda(r)$ has a derivative $\lambda'(r)$ at each $r \in (r^b, r^b + \varepsilon]$ satisfying

$$\begin{aligned} g(r) - \lambda(r)q(s_2(r) - r) + \lambda'(r)(Q(s_2(r) - r) - Q(0)) &= 0, \\ g(r) - \lambda(r)q(s_1(r) - r) + \lambda'(r)(Q(s_1(r) - r) - Q(0)) &= 0. \end{aligned}$$

Solving for $\lambda(r)$ and $\lambda'(r)$ yields, for all $r \in (r^b, r^b + \varepsilon]$,

$$\begin{aligned} \lambda(r) &= \frac{g(r)[Q(s_2(r) - r) - Q(s_1(r) - r)]}{(Q(s_2(r) - r) - Q(0))q(s_1(r) - r) - (Q(s_1(r) - r) - Q(0))q(s_2(r) - r)}, \\ \lambda'(r) &= \frac{g(r)[q(s_2(r) - r) - q(s_1(r) - r)]}{(Q(s_2(r) - r) - Q(0))q(s_1(r) - r) - (Q(s_1(r) - r) - Q(0))q(s_2(r) - r)}. \end{aligned}$$

Since λ' is the derivative of λ , we have $d\lambda(r)/dr = \lambda'(r)$ for all $r \in (r^b, r^b + \varepsilon]$. Taking into account that s_1 and s_2 are twice differentiable and satisfy $\lim_{r \downarrow r^b} s_1(r) = \lim_{r \downarrow r^b} s_2(r) = r^b$, we can apply L'Hopital's rule to evaluate $d\lambda(r)/dr = \lambda'(r)$ in the limit $r \downarrow r^b$ to obtain

$$\frac{g'(r^b)q(0)}{(q(0))^2} = \frac{g(r^b)q'(0)}{(q(0))^2},$$

which implies that $r^b = 0$, because $G(r) = Q(\eta r)$ for all r and $q'(r) = 0$ iff $r = 0$. Denote $\lim_{r \downarrow r^b} s_1'(r) = 1 - \beta_1$ and $\lim_{r \downarrow r^b} s_2'(r) = 1 + \beta_2$, where $\beta_1 \geq 1$ (because s_1 is decreasing) and $\beta_2 \geq 0$ (because $s_2(r) > r$). Differentiating $d\lambda(r)/dr = \lambda'(r)$ with respect to r and taking

the limit $r \downarrow 0$, we get

$$\frac{\eta q''(0)(\eta^2 - \beta_2\beta_1)}{q(0)} = \frac{\eta q''(0)(\beta_2 - \beta_1)}{2q(0)},$$

and hence

$$2\eta^2 = 2\beta_2\beta_1 + \beta_2 - \beta_1. \quad (18)$$

Since, for small enough $r > 0$, type $s_1(r)$ is assigned to both district $\delta_{s_1(r)}$ and district P with $r^*(P) = r$ and $\text{supp}(P) = \{s_1(r), s_2(r)\}$, we must have, by Lemma 2,

$$Q(\eta s_1(r)) = Q(\eta r) + \lambda(r)(Q(s_1(r) - r) - Q(0)).$$

In the limit $r \downarrow 0$, the values and the derivatives up to order 2 of both sides always coincide, while the third derivatives coincide iff

$$q''(0)\eta^3(-\beta_1 + 1)^3 = q''(0)\eta^3 - 3q''(0)\eta^3\beta_1 + 3q''(0)\eta\beta_2\beta_1^2 - q''(0)\eta\beta_1^3,$$

which simplifies to

$$-\eta^2\beta_1 + 3\eta^2 = 3\beta_2 - \beta_1. \quad (19)$$

Since, for small enough $r > 0$, type $s_1(r)$ is assigned to both district $\delta_{s_1(r)}$ and district P with $r^*(P) = r$, while type $s_2(r)$ is assigned only to district P , we have

$$f(s_1(r))s'_1(r)(Q(s_1(r) - r) - Q(0)) \geq f(s_2(r))s'_2(r)(Q(s_2(r) - r) - Q(0)).$$

In the limit $r \downarrow 0$, both sides are equal, and hence their derivatives must satisfy

$$-f(0)q(0)\beta_1(1 - \beta_1) \geq f(0)q(0)\beta_2(\beta_2 + 1),$$

which, given that $\beta_1 + \beta_2 > 0$, simplifies to

$$\beta_1 \geq \beta_2 + 1. \quad (20)$$

Recalling that $\gamma = \eta^2/(1 + \eta^2)$, equations (18) and (19) have two solutions

$$(\beta_1, \beta_2) = \left(\frac{3\eta^2}{2(\eta^2-1)}, \frac{\eta^2}{2}\right) = \left(\frac{3\gamma}{2(2\gamma-1)}, \frac{\gamma}{2(1-\gamma)}\right) \quad \text{and} \quad (\beta'_1, \beta'_2) = \left(1, \frac{(2\eta^2+1)}{3}\right) = \left(1, \frac{\gamma+1}{3(1-\gamma)}\right),$$

unless $\gamma = 1/2$, in which case (18) and (19) have only one solution $(\beta_1, \beta_2) = (1, 1)$. The solution (β'_1, β'_2) never satisfies (20) and thus is discarded. Moreover, for the solution (β_1, β_2) , condition $\beta_1 \geq 1$ yields $\gamma > 1/2$, and condition (20) yields $\gamma \leq \sqrt{3} - 1$. Thus, for Y-districting to be optimal, we must have $\gamma \in (1/2, \sqrt{3} - 1]$. Finally, the statement in Footnote 34 holds because

$$\lim_{r \downarrow 0} s'_1(r) = 1 - \beta_1 = -\frac{2 - \gamma}{2(2\gamma - 1)} < 0 \quad \text{and} \quad \lim_{r \downarrow 0} s'_2(r) = 1 + \beta_2 = \frac{2 - \gamma}{2(1 - \gamma)} > 0$$

are both strictly increasing in γ .

APPENDIX B: ESTIMATORS

In this section, we formally define our estimators and show that they satisfy standard statistical properties. Fix a US state. We assume throughout that there is a large number of voters, so that the vote share in a precinct n with type s_n in district d and election y with aggregate shock r_{dy} is given by $v_{ny} = \Phi((s_n - r_{dy})/\sqrt{\gamma})$. Let μ_s and σ_s^2 be the mean and variance of the distribution of precinct types, defined by $\mu_s = \mathbb{E}_F[s]$ and $\sigma_s^2 = \text{Var}_F[s]$. For convenience, we repeat some definitions from the main text. Let $w_{ny} = \Phi^{-1}(v_{ny})$, T denote the number of elections, D the number of districts, and \mathcal{N}_{dy} the set of precincts in district d and election y . Define

$$w_{dy} = \frac{\sum_{n \in \mathcal{N}_{dy}} k_{ny} w_{ny}}{\sum_{n \in \mathcal{N}_{dy}} k_{ny}}, \quad w_{d\bullet} = \frac{\sum_y w_{dy}}{T}, \quad w_{\bullet y} = \frac{\sum_d w_{dy}}{D}, \quad w_{\bullet\bullet} = \frac{\sum_d w_{d\bullet}}{D},$$

$$e_n^2 = \frac{1}{DT} \sum_{d,y} \frac{\sum_{n \in \mathcal{N}_{dy}} k_{ny} (w_{ny} - w_{\bullet y})^2}{\sum_{n \in \mathcal{N}_{dy}} k_{ny}},$$

$$e_d^2 = \frac{\sum_{d,y} (w_{dy} - w_{d\bullet})^2}{D(T-1)}, \quad e^2 = \frac{\sum_y (w_{\bullet y} - w_{\bullet\bullet})^2}{T-1},$$

$$\text{cov} = \frac{\sum_y \sum_d \sum_{d' > d} (w_{dy} - w_{d\bullet})(w_{d'y} - w_{d'\bullet})}{\frac{D(D-1)}{2}(T-1)} = \frac{De^2 - e_d^2}{D-1},$$

where the last equality follows from

$$e^2 = \frac{\sum_y \left(\sum_d \frac{1}{D} (w_{\bullet y} - w_{\bullet\bullet}) \right)^2}{D(T-1)}$$

$$= \frac{1}{D} \frac{\sum_{d,y} (w_{dy} - w_{d\bullet})^2}{D(T-1)} + \frac{D-1}{D} \frac{\sum_y \sum_d \sum_{d' > d} (w_{dy} - w_{d\bullet})(w_{d'y} - w_{d'\bullet})}{\frac{D(D-1)}{2}(T-1)}$$

$$= \frac{1}{D} e_d^2 + \frac{D-1}{D} \text{cov}.$$

To construct our estimators, we use the following proposition.

PROPOSITION 1: *In our empirical model,*

$$\mathbb{E}e_d^2 = \frac{1-\gamma}{\gamma}, \quad \mathbb{E}cov = \rho \frac{1-\gamma}{\gamma}, \quad \mathbb{E}w_{\bullet\bullet} = \frac{\mu_s}{\sqrt{\gamma}}, \quad \text{and} \quad \mathbb{E}e_n^2 = \frac{\sigma_s^2}{\gamma} + (1-\rho) \frac{D-1}{D} \frac{1-\gamma}{\gamma},$$

and

$$e_d^2 \stackrel{d}{=} \frac{1-\gamma}{D(T-1)\gamma} \left[(1-\rho)\chi_{(D-1)(T-1)}^2 + (1+(D-1)\rho)\chi_{T-1}^2 \right],$$

where $\stackrel{d}{=}$ denotes equality in distribution, and $\chi_{(D-1)(T-1)}^2$ and χ_{T-1}^2 denote independent χ^2 random variables with $(D-1)(T-1)$ and $T-1$ degrees of freedom, respectively.

Consider the following point estimators of γ , ρ , μ_s , and σ_s :

$$\hat{\gamma} = \frac{1}{1+e_d^2}, \quad \hat{\rho} = \frac{cov}{e_d^2}, \quad \hat{\mu}_s = \frac{w_{\bullet\bullet}}{\sqrt{1+e_d^2}}, \quad \text{and} \quad \hat{\sigma}_s = \sqrt{\frac{e_n^2 - \frac{D-1}{D}(e_d^2 - cov)}{1+e_d^2}}.$$

By Proposition 1, $1/\hat{\gamma}$, $\hat{\rho}/\hat{\gamma} - \hat{\rho}$, $\hat{\mu}_s/\sqrt{\hat{\gamma}}$, and $\hat{\sigma}_s^2/\hat{\gamma}$ are unbiased estimators of $1/\gamma$, $\rho/\gamma - \rho$, $\mu_s/\sqrt{\gamma}$, and σ_s^2/γ . Moreover, by the law of large numbers for $D(T-1) \rightarrow \infty$, we have that $\hat{\gamma}$, $\hat{\rho}$, $\hat{\mu}_s$, and $\hat{\sigma}_s$ are consistent estimators of γ , ρ , μ_s , and σ_s .

Proposition 1 also gives us a confidence interval for γ . Specifically, for any $\alpha \in (0, 1)$, let q_α be the α -quantile for $(1-\hat{\rho})\chi_{(D-1)(T-1)}^2 + (1+(D-1)\hat{\rho})\chi_{T-1}^2$. Then, a one-sided $1-\alpha$ confidence interval for γ is $(\hat{\gamma}_\alpha, 1)$ where

$$\hat{\gamma}_\alpha = \frac{1}{1 + \frac{D(T-1)}{q(\alpha)} e_d^2}.$$

PROOF OF PROPOSITION 1: Denote

$$r_{d\bullet} = \frac{\sum_y r_{dy}}{T}, \quad r_{\bullet y} = \frac{\sum_d r_{dy}}{D}, \quad s_{dy} = \frac{\sum_{n \in \mathcal{N}_{dy}} k_{ny} s_n}{\sum_{n \in \mathcal{N}_{dy}} k_{ny}}, \quad s_{\bullet y} = \frac{\sum_d s_{dy}}{D}.$$

First, we have

$$\mathbb{E}w_{\bullet\bullet} = \mathbb{E} \frac{1}{DT} \sum_{d,y} \frac{\sum_{n \in \mathcal{N}_{dt}} k_{nt} \frac{s_n - r_{dt}}{\sqrt{\gamma}}}{\sum_{n \in \mathcal{N}_{dt}} k_{nt}} = \mathbb{E} \frac{1}{DT} \sum_{d,t} \frac{\sum_{n \in \mathcal{N}_{dt}} k_{nt} \frac{s_n}{\sqrt{\gamma}}}{\sum_{n \in \mathcal{N}_{dt}} k_{nt}} = \mathbb{E} \frac{\sum_n k_{nt} \frac{s_n}{\sqrt{\gamma}}}{\sum_n k_{nt}} = \frac{\mu_s}{\sqrt{\gamma}},$$

where the first equality is by $v_{ny} = \Phi((s_n - r_{dy})/\sqrt{\gamma})$ and the definition of v_{ny} and $w_{\bullet\bullet}$, the second is by $\mathbb{E}[r_{dy}] = 0$, the third is by district equipopulation, and the fourth is by the definition

of μ_s . Second, we have

$$\mathbb{E}e_d^2 = \mathbb{E} \frac{\sum_{d,y} \left(\frac{T-1}{T} \frac{r_{dy}}{\sqrt{\gamma}} - \frac{1}{T} \sum_{y' \neq y} \frac{r_{dy'}}{\sqrt{\gamma}} \right)^2}{D(T-1)} = \frac{DT \left[\left(\frac{T-1}{T} \right)^2 + \frac{T-1}{T^2} \right] \frac{1-\gamma}{\gamma}}{D(T-1)} = \frac{1-\gamma}{\gamma},$$

where the first equality is by $v_{ny} = \Phi((s_n - r_{dy})/\sqrt{\gamma})$, the definition of w_{dy} and $w_{d\bullet}$, and rearrangement, the second is by $Var[r_{dy}] = 1 - \gamma$ and $Cov[r_{dy}, r_{dy'}] = 0$ for $y \neq y'$, and the third is by rearrangement. Third, we have

$$\mathbb{E}cov = \mathbb{E} \frac{\sum_y \sum_d \sum_{d' > d} \frac{r_{dy} - r_{d\bullet}}{\sqrt{\gamma}} \frac{r_{d'y} - r_{d'\bullet}}{\sqrt{\gamma}}}{\frac{D(D-1)}{2}(T-1)} = \rho \frac{1-\gamma}{\gamma},$$

where the first equality is again by $v_{ny} = \Phi((s_n - r_{dy})/\sqrt{\gamma})$ and the definition of w_{dy} and $w_{d\bullet}$, and the second is by $Cov[r_{dy}, r_{d'y}] = \rho(1 - \gamma)$ for $d \neq d'$, $Cov[r_{dy}, r_{d'y'}] = 0$ for $y \neq y'$, and rearrangement. Fourth, we have

$$\begin{aligned} \mathbb{E}e_n^2 &= \mathbb{E} \frac{1}{DT} \sum_{d,y} \frac{\sum_{n \in \mathcal{N}_{dy}} k_{ny} \frac{(s_n - s_{\bullet y} + r_{dy} - r_{\bullet y})^2}{\gamma}}{\sum_{n \in \mathcal{N}_{dy}} k_{ny}} = \mathbb{E} \frac{\sum_n k_{ny} \frac{(s_n - s_{\bullet y})^2}{\gamma}}{\sum_n k_{ny}} \\ &\quad + \mathbb{E} \frac{\sum_d \frac{(r_{dy} - r_{\bullet y})^2}{\gamma}}{D} = \frac{\sigma_s^2}{\gamma} + \mathbb{E} \frac{\sum_d \left(\frac{D-1}{D} r_{dy} - \frac{1}{D} \sum_{d' \neq d} r_{d'y} \right)^2}{\gamma D} \\ &= \frac{\sigma_s^2}{\gamma} + \mathbb{E} \frac{\sum_d \left[\left(\frac{D-1}{D} \right)^2 r_{dy}^2 + \frac{1}{D^2} \sum_{d' \neq d} r_{d'y}^2 - \frac{2(D-1)}{D^2} r_{dy} r_{d'y} + \frac{2}{D^2} \sum_{d' \neq d, d'' > d'} r_{d'y} r_{d''y} \right]}{\gamma D} \\ &= \frac{\sigma_s^2}{\gamma} + \left[\left(\frac{D-1}{D} \right)^2 + \frac{D-1}{D^2} - \rho \frac{2(D-1)}{D^2} + \rho \frac{(D-1)(D-2)}{D^2} \right] \frac{1-\gamma}{\gamma} \\ &= \sigma_s^2 + (1-\rho) \frac{D-1}{D} \frac{1-\gamma}{\gamma}, \end{aligned}$$

where the first equality is by $v_{ny} = \Phi((s_n - r_{dy})/\sqrt{\gamma})$, the definition of w_{ny} and $w_{\bullet y}$, and rearrangement, the second is by independence across elections and district equipopulation, the third is by the large number of voters and rearrangement of the second term, the fourth is by

quadratic expansion, the fifth is by $\mathbb{E}[r_{dy}^2] = 1 - \gamma$ and $\mathbb{E}[r_{dy}r_{d'y}] = \rho(1 - \gamma)$ for $d' \neq d$, and the sixth is by rearrangement.

Finally, let $r = (r_{11}, \dots, r_{1T}, \dots, r_{D1}, \dots, r_{DT})'$. Then we can write

$$\sum_{d,y} (r_{dy} - r_{d\bullet})^2 = r'Ar$$

where

$$A = \begin{pmatrix} \frac{T-1}{T} & \dots & -\frac{1}{T} & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\frac{1}{T} & \dots & \frac{T-1}{T} & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & \frac{T-1}{T} & \dots & -\frac{1}{T} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & -\frac{1}{T} & \dots & \frac{T-1}{T} \end{pmatrix}.$$

Note that

$$\frac{\mathbb{E}[rr']}{1 - \gamma} = \Sigma = \begin{pmatrix} 1 & \dots & 0 & \dots & \rho & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & \dots & 0 & \dots & \rho \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \rho & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \rho & \dots & 0 & \dots & 1 \end{pmatrix}.$$

By the spectral theorem, there is an orthogonal matrix P (so that $P'P = P'P = I$) and a diagonal matrix Λ with positive diagonal elements $\lambda_1, \dots, \lambda_{DT}$ such that $\Sigma^{1/2}A\Sigma^{1/2} = P'\Lambda P$. Define $u = \sqrt{1 - \gamma}P\Sigma^{-1/2}r$ (so that $r = \Sigma^{1/2}P'u/\sqrt{1 - \gamma}$). Then

$$\frac{r'Ar}{1 - \gamma} = u'P\Sigma^{1/2}A\Sigma^{1/2}P'u = u'PP'\Lambda PP'u = u'\Lambda u = \sum_{i=1}^{DT} \lambda_i u_i^2$$

where $u \sim N(0, I)$, and $\lambda_1, \dots, \lambda_{DT}$ are the roots of the characteristic equation

$$|\Sigma^{1/2}A\Sigma^{1/2} - \lambda I| = 0 \iff |A\Sigma - \lambda I| = 0.$$

Note that

$$A\Sigma = \begin{pmatrix} \frac{T-1}{T} & \dots & -\frac{1}{T} & \dots & \rho\frac{T-1}{T} & \dots & -\rho\frac{1}{T} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\frac{1}{T} & \dots & \frac{T-1}{T} & \dots & -\rho\frac{1}{T} & \dots & \rho\frac{T-1}{T} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \rho\frac{T-1}{T} & \dots & -\rho\frac{1}{T} & \dots & \frac{T-1}{T} & \dots & -\frac{1}{T} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\rho\frac{1}{T} & \dots & \rho\frac{T-1}{T} & \dots & -\frac{1}{T} & \dots & \frac{T-1}{T} \end{pmatrix}.$$

After some algebra, we obtain

$$|A\Sigma - \lambda I| = (-1)^{DT} \lambda^D (\lambda - 1 + \rho)^{(D-1)(T-1)} (\lambda - 1 - (D-1)\rho)^{T-1},$$

showing that $r'Ar/(1-\gamma) \stackrel{d}{=} (1-\rho)\chi_{(D-1)(T-1)}^2 + (1+(D-1)\rho)\chi_{T-1}^2$, and hence

$$e_d^2 = \frac{r'Ar}{D(T-1)\gamma} \stackrel{d}{=} \frac{1-\gamma}{D(T-1)\gamma} [(1-\rho)\chi_{(D-1)(T-1)}^2 + (1+(D-1)\rho)\chi_{T-1}^2]. \quad Q.E.D.$$