A short tale about determination of functions

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In this seminar, we will revise a series of works concerning the problem of *determination of functions*. It is well-known, as a consequence of the fundamental theorem of calculus, that for two differentiable functions $f, g : \mathbb{R}^n \to \mathbb{R}$, one has that

$$\nabla f(x) = \nabla g(x), \forall x \in \mathbb{R}^n \implies f = g + \text{cst}.$$

We will refer to this type of result as a determination result: a "first order information" about the function determines it completely, up to an additive constant.

In the first part of the seminar, we will revise some literature concerning determination results. The determination result of Rockafellar for convex functions [13], the determination result of Monotone operators of Brézis [3], and the first metric determination result for $\mathcal{C}^2$-convex functions of Boulmezaoud,Cieutat, and Daniilidis [2]: For any two convex functions over a Hilbert space $f, g : \mathcal{H} \to \mathbb{R}$, of class $\mathcal{C}^2$ and bounded from below, one has that:

$$\|\nabla f(x)\| = \|\nabla g(x)\|, \forall x \in \mathcal{H} \implies f = g + \text{cst}.$$

This last result, quite unexpected at the time, can be interpreted as a uniqueness result for the eikonal equation

$$\|Du\| = f(x).$$

By means of metric differentiation, we will discuss some recent advances in existence and uniqueness of the eikonal equation in metric spaces. After quickly visiting the seminal results of Crandall and Lions [4] and Ishii [10], we will discuss three notions of metric viscosity solutions of the eikonal equation: Ambrosio and Feng [1], afterward generalized by Gangbo and Święch [8]; Giga, Hamamuki and Nakayasu [9]; and Liu, Shanmugalingam, Zhou [11]. In a broad setting, it is shown in [11] that the three approaches are in fact equivalent.

In the second part of this presentation, we will visit the recent contributions in the problem of determination of functions: First, we will discuss the convex case studied in [12] and [14], and then its extension for nonconvex inf-compact functions [7]. Both results are based on the metric slope.

Finally, we will discuss about an abstraction of metric slopes, that we call Descent Modulus. We will visit how, following a descent method approach, a

Descent Modulus can determine functions. We will first study the case of inf-compact functions [5] and then, the case of continuous functions in complete metric spaces [6]. We will discuss how to interpret our results in both contexts: determination of functions, and uniqueness of solutions of the eikonal equation.

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