Nonparametric Network Bootstrap

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Abstract

Inference on network data is challenging due to the strong dependence between observations, which renders standard techniques incorrect. To address this, we propose a valid bootstrap procedure for network data based on a nonparametric linking function estimator. We characterise the conditions under which this estimator is uniformly consistent. We prove that the distribution of the bootstrap network is consistent for the distribution of the original network in terms of a Wasserstein distance. We also provide conditions under which distributions of a class of functions related to U-statistics on the bootstrapped networks consistently replicate the distributions of the corresponding statistics on the original network. Monte Carlo simulations show good confidence interval coverage for a wider class of network functions than those accounted for by our theory. We apply our method to the data from Banerjee, Chandrasekhar, Duflo, and Jackson (2013): we replicate their findings, but also show that our method works under weaker assumptions and with a significantly smaller sample size. Finally, we propose an alternative specification of their model which takes advantage of our linking function estimator and may be of interest independently of our bootstrap procedure.

1 Introduction

Many papers in economics studying information diffusion (e.g. Banerjee, Chandrasekhar, Duflo, and Jackson (2013), Alatas et al. (2016)), impact of the most influential individuals (e.g. Banerjee, Chandrasekhar, Duflo, and Jackson (2019), Breza and Chandrasekhar (2019)), inherent features of networks (e.g. Chetty et al. (2022), Banerjee et al. (2024)) and other models on network data run into the issue that it is very difficult to conduct statistical inference on complex, interconnected data structures represented by networks. In this paper, we propose a solution: a bootstrap procedure which does not impose strong assumptions on the form of the network-generating function and can be applied to a wide range of network statistics.

The default approach when analysing the behaviour of statistics is finding an asymptotic approximation to their distribution, a technique easiest to apply to simple models and data with limited dependence. Unfortunately, the models built on networks are often complex and the networks themselves tend to exhibit a deeply interconnected structure. All individuals in a network are closely related: the concept of "six degrees of separation" shows that nearly all users of social media platforms like Facebook or Twitter are at most six connections away from each other, while the average distance is below four. At the same time, the number of connections grows more slowly than the network size. This phenomenon is known as sparsity and can be illustrated by the fact that, during their peak growth periods, social media platforms gained new users at a faster rate than individual users gained new connections. Because of the issues of strong connectedness, sparsity, and complicated functional forms of network statistics, asymptotic theory for network statistics tends to be complicated, specialised to certain classes of estimators, and, in many cases, still underdeveloped.

For similar reasons, standard bootstrap techniques are not valid for network data: there is a need for a specialised bootstrap procedure specifically designed to deal with this kind of dependence. The few existing methods for bootstrapping network data suffer from either limited applicability (they tend to focus on specific classes of network statistics and cannot be easily extended to e.g. regressions controlling for the dependence structure defined by the network) or restrictive parametric functional form for the components of the network-generating process. We address both of those concerns.

We propose a bootstrap procedure which takes a given network, uses it to approximate the data-generating process, and creates new networks with a similar structure to the original one. If we are interested in the distribution of a particular statistic of the original network, we can approximate it by estimating the same statistic on a large number of bootstrapped networks.

We assume a general form of a network-generating process in which the observed network is determined by an unknown distribution over types of individuals and an unknown function determining the the probability of a link between any pair of individuals. Under this assumption, each person is characterised by a set of (possibly unobserved) features that are independent of the features of others. The links are assumed to come from independent draws with probabilities determined by a binary linking function which takes the features of any two individuals as inputs and outputs the probability of a link between them.

If we knew the linking function, we could generate networks similar to the observed one by firstly resampling from the original set of individuals and then adding links based on probabilities determined by the linking function. However, the linking function is unknown, and it depends on possibly unobserved inputs. To address this challenge, we propose a nonparametric method to estimate the linking function which takes advantage of the information provided by the set of observed connections.

We start by borrowing a distance from Auerbach (2022) (see also Zhang, Levina, and Zhu (2017)), which can be intuitively summarised as: people with similar sets of friends are similar to each other. If we observe two people with similar sets of neighbours, it likely happened because the linking function gave similar probabilities for their links with other individuals. This way we can identify people similar to any person i. We can then determine what proportion of these 'counterfactuals' of i are linked to a person j, providing an estimate of the link probability between i and j. Similarly, by swapping the roles of i and j, we can find what proportion of individuals similar to j are linked to i. Assuming that links are symmetric, we take the average of these two estimates as an estimate of the link probability between i and j. Whenever we sample i and j as nodes of the bootstrap network we draw a link between them with this estimated probability.

Having developed a method to generate bootstrap networks, we characterise the conditions under which our procedure works well. We make a contribution to the literature on the linking function estimation: we propose a nonparametric linking function estimator and provide conditions under which it achieves uniform consistency. We also develop a cross-validation procedure for choosing a bandwidth parameter for our estimator.

Our next contribution is to the network bootstrap literature: we propose a new bootstrap procedure and provide conditions under which it achieves consistency. We show this in two ways: we borrow a notion of Wasserstein distance between network generating distributions from Levin and Levina (2019) and we show that the distance between the bootstrap network generating process and the true network generating process goes to zero in probability as we increase the sample size. Unfortunately, this is not sufficient to ensure that the distribution of any statistic on a bootstrap network replicates the corresponding distribution of that statistic on the original network. We show this directly for a class of statistics which are closely related to U-statistics. The motivation for this is twofold: this is a wide class of functions and includes some estimators we may be directly interested in, for example the density of connections within a network. Additionally, this class includes motif densities, i.e. the densities of different patterns (e.g. triangles, stars, cycles of length m) on subgraphs of the adjacency matrix. These are sometimes referred to as "network moments" because they characterise the network generating distribution: if two networks match on densities of all possible patterns, they come from the same distribution. Hence proving that our bootstrap procedure correctly recovers the distributions of all motif densities implicitly shows that the bootstrap networks share the same asymptotic network generating distribution as the original network.

While we don't currently have explicit asymptotic theory for other classes of network functions, for example measures of centrality, clustering, eigenvalues of the adjacency matrix, or parameters of regressions on networks, our simulations suggest that our method is more widely applicable and can be used to recover distributions of these kinds of statistics.

In our application, we provide an illustration of how our bootstrap method can be extended to an information diffusion model over a network using data from Banerjee, Chandrasekhar, Duflo, and Jackson (2013). Under the setup of the original paper, we are able to provide slightly narrower confidence intervals. Additionally, our method allows us to perform estimation on a significantly smaller sample: while the original paper relies on asymptotics in the number of networks and requires observing many villages, our method is asymptotic in the village size, meaning that we can construct confidence intervals given data on a single village. This has the potential to drastically lower data collection costs. We also propose an alternative model specification which uses our linking function estimator as a proxy for the strength of connection and which could be of interest independently of the bootstrap procedure.

In Section 2 we summarise the related literature. The setup of the model is described in Section 3, where we also provide definitions of our estimators and the bootstrap procedure. Section 4 includes the statements of our main results: the uniform consistency of the linking function estimator in Theorem 1, a bootstrap consistency result for a specific class of estimators related to U-statistics in Theorem 3, and a Wasserstein distance convergence in Theorem 2. Section 5 shows results of Monte Carlo simulations and Section 6 describes an application to the data from Banerjee, Chandrasekhar, Duflo, and Jackson (2013). Section 7 concludes. The appendices start with a list of all notation. Appendix A includes all proofs, Appendix B provides the codes, additional tables, plots for simulations, Appendix C includes extensions.

2 Related literature

2.1 Network Bootstrap

There are a few existing bootstrap procedures for different functions on networks. Most of the literature focuses on bootstrapping a class of network functions closely related to U-statistic, or their subset, motif densities (i.e. the proportions of subgraphs of a given size which take the form of a specific pattern or 'motif,' e.g. the proportion of subgraphs of size three which are fully connected, or the proportion of subgraphs of size four in which there is only one link). Our procedure can be applied to a much wider class of functions.

Green and Shalizi (2022) propose two types of bootstrap: the empirical bootstrap, in which they resample individuals and put a link between them if they were linked in the original graph, and a parametric histogram bootstrap. The empirical bootstrap can be seen as a special case of ours, it is simple and computationally attractive, but it suffers from a few types of bias: whenever an individual gets resampled more than once, these copies are not linked (as there were no self-links in the original graph), and they share the same link patterns with all other individuals (there is correlation between link formation in the bootstrap graph which was not present in the original graph). Green and Shalizi (2022) prove that this dependence and bias is asymptotically negligible for motif densities, but this is not necessarily true for other functions. Our simulations show that their procedure does not perform well e.g. for eigenvalues other than the highest one.

Levin and Levina (2019) assume a specific functional form of the linking function[1]. They propose two methods: one in which they directly estimate a U-statistic and one in which they generate a full network that can be used for estimating more general functions of a network, including eigenvalues and measures of small-world behaviour. This is the only paper we are aware of which provides results for functions of the entire network: they show that the entire bootstrapped network converges to an independent copy of the original network in terms of a new notion of Wasserstein network distance they define. Under our more general nonparametric specification we are able to show convergence in terms of the same distance (see Theorem 2).

Lin, Lunde, and Sarkar (2020) propose a computationally efficient multiplier bootstrap for motif densities, based on approximating the first (for large sparse graphs) or first and second (for smaller denser graphs) order terms of a Hoeffding decomposition of the U-statistic. Their method is specific to this class and, unlike ours, it cannot be extended to other types of network functions. They show higher-order accuracy of their quadratic bootstrap using an Edgeworth expansion. The

1. They assume a random dot product graph with a linking function: $h_{0,n}(\xi_i, \xi_j) = \xi_i' \xi_j$ where ξ_i is a vector of latent positions which can be interpreted as characteristics of individual i.

theory of Edgeworth expansion for motif densities is developed Zhang and Xia (2022) who show higher order correctness of a studentised version of the empirical bootstrap of Green and Shalizi (2022). We believe similar methods could be used to show higher-order accuracy of our method, but we do not pursue this direction in the current work.

Shao and Le (2024) provide a parametric bootstrap in a setting different from all the previously mentioned papers, where the nodes are non-exchangeable. In our notation this corresponds to a situation in which the ξ and the matrix of link probabilities are fixed, $h_{0,n}$ takes a known parametric form, and the randomness comes only from η. Their analysis focuses on quantifying the bias and providing bias-corrected bootstrap procedure for motif densities.

The network setup is a special case of an exchangeable array[2]. Papers which propose bootstrap for exchangeable arrays include Davezies, D'Haultfœuille, and Guyonvarch (2021), whose method in our setting is identical to empirical bootstrap, and Menzel (2021), who proposes a new wild bootstrap procedure based on splitting the statistic of interest into orthogonal components, estimating them by sample analogues, and resampling each component with appropriate scaling. Menzel (2021) also points out that, depending on the dependence structure, the limiting distribution may be nonstandard. Their LLN and CLT results apply to functions of finite k-dimensional subgraphs which take a form of U- or V-statistics (including degenerate cases), their smooth functionals and Z-estimators. Their methods are local, they can be applied to functions of finite subgraphs and cannot account for dependence over the whole adjacency matrix, as in the case of eigenvalues or some centrality measures covered by our method.

In terms of the allowed level of sparsity, we impose a stronger requirement for acyclic motifs than both models in Green and Shalizi (2022) as well as Lin, Lunde, and Sarkar (2020), but our requirement is the same as for cycles in Lin, Lunde, and Sarkar (2020) and is weaker than that for general motifs for Green and Shalizi (2022) empirical graphon. In comparison with Green and Shalizi (2022) histogram graphon, our sparsity condition for general motifs becomes weaker only when $m > 4$, and we also impose weaker conditions than L-Lipschitz on the linking function. Levin and Levina (2019) only include sparsity considerations in one result, for acyclic motifs and cycles. Their assumption is weaker than ours, which is not surprising given their model is parametric.

Apart from bootstrap, other ways of estimating distributions of network statistics include the asymptotic theory for motif densities provided byBickel, Chen, and Levina (2011). Subsampling methods have been proposed by Bhattacharyya and Bickel (2015), who give results for motif

2. Using notation from Davezies, D'Haultfœuille, and Guyonvarch (2021), our model is a special case of an exchangeable and dissociated array with $k = 2$ and U_{ij} corresponding to the randomness due to Bernoulli trials $\tau(u_i, u_j, u_{ij}) = \mathbb{1}(h(u_i, u_j) \leq u_{ij})$. The kernels of U-statistics on networks can be represented as higher-dimensional $(2 \leq k < \infty)$ exchangeable arrays.

densities, and Lunde and Sarkar (2022), who provide consistency results for general functions and specify them to two classes: motif densities and eigenvalues of graphons of finite rank. Their methods require minimal assumptions and allow for sparser graphs than ours.

2.2 Linking function estimation

The first step in our bootstrap procedure involves the estimation of a linking function, for which we propose a nonparametric estimator and provide conditions for its uniform consistency. Our idea for the nonparametric linking function estimator was inspired by Auerbach (2022), who provides a way of controlling for a network-dependent latent covariate in a partially linear regression setting. Our estimator has been previously proposed by Zeleneev (2020) (whose focus is different than ours and who does not analyse the theoretical properties of the estimator). Our contribution to the literature on linking function estimation most closely resembles Zhang, Levina, and Zhu (2017), who propose a similar procedure to ours, with the difference that their distance is defined in terms of the maximum norm and their estimator is a nearest-neighbour estimator instead of a kernel estimator.

One key contribution we make is providing a numerical procedure for selecting an optimal value for a tuning parameter. In Zhang, Levina, and Zhu (2017) the equivalent parameter is motivated theoretically and derived from the rate of convergence. We find that our procedure with the numerically chosen optimal parameter values outperforms Zhang, Levina, and Zhu (2017) with the theoretically best optimal values when used as a first step in our bootstrap procedure. We believe this is largely due to the bandwidth choice.

2.3 Other

In the proofs that our bootstrap procedure is reliable we use a framework inspired by Politis et al. 1999. Our results can be seen as an extension of Bickel and Freedman (1981), the classic paper providing conditions for consistency of bootstrap for U-statistics. We extend their analysis to the case where the objective function becomes a U-statistic only after taking expectation conditional on a vector of unobserved characteristics and after substituting the true linking function for its estimator as the input to the kernel function. We show that our linking function estimator converges to the true linking function in a sense which is sufficient for the bootstrap equivalent to converge weakly in probability to the same limiting distribution as the object of interest in the original sample. Because of the additional levels of approximation, we achieve a weaker notion of convergence than convergence weakly almost surely (see Definition 3) in Bickel and Freedman (1981), but our result is still sufficient to provide asymptotically correct bootstrap confidence intervals.

For our empirical application, we use the data and some of the codes from Banerjee, Chandrasekhar, Duflo, and Jackson (2013). We confirm their results using our method and we repeat a part of their analysis under weaker assumptions: where the original paper performs the estimation aggregating over many villages, we are able to provide estimates and confidence intervals on individual village level. We are also able to run a related model based on the strength of connections between household rather than the less informative binary information on presence or lack of connection. We find that removing this one level of approximation has a significant effect on our conclusions.

3 Model: setup, definitions and the bootstrap procedure

3.1 Setup

We follow the standard setup in the literature known as the latent space model.

We observe an adjacency matrix A which corresponds to an undirected, unweighted graph on n nodes (also referred to as individuals) indexed by $i \in \{1, 2, \ldots, n\}$. The matrix is symmetric, has zeros on the main diagonal and ones in positions corresponding to edges in the graph ($A_{ij} = 1$ if and only if there is an edge between nodes i and j). For a vector of index numbers $\iota = (\iota_1, \ldots, \iota_m)'$ with $\iota_i \in \{1, 2, \ldots, n\}$ we let $A(\iota)$ denote the corresponding submatrix defined on nodes in ι (i.e. A from which we remove rows and columns not in ι). Each node i is characterised by a vector of unobserved features[3] ξ_i, drawn independently from their common distribution F_0 with support $Supp(\xi_i)$. We denote the vector of all $\{\xi_i\}_{i=1}^n$ by ξ and we let $\xi(\iota) = (\xi_{\iota_1}, \ldots, \xi_{\iota_m})$. We assume that the distribution has no point mass, i.e. for $\xi_i, \xi_j \sim F_0$ we have $P_{F_0}(\xi_i = \xi_j) = 0$[4]. We impose more assumptions on F_0 in Assumption 1.2[5].

Let $h_{0,n} : Supp(\xi_i) \times Supp(\xi_i) \to [0, 1]$ be a symmetric, measurable linking function[6] which can be decomposed as:

$$h_{0,n}(u, v) = \rho_n w_0(u, v) \tag{1}$$

where $\int w_0(u, v) dF_0(u) dF_0(v) = 1$.

3. This corresponds to the vector of latent positions X_i in Levin and Levina (2019).

4. This is without loss of generality: if we had a distribution with a point mass we could define a new support of ξ and a new F_0 in which the point mass would be replaced by a region of ξ of total measure equal to the probability at the original point.

5. The assumptions are implicit and would be implied by F_0 bounded above and separated away from zero with $h_{0,n}$ piecewise Lipschitz.

6. The linking function has been referred to as the coupling function $g(.,.)$ in Zeleneev (2020) and the graphon function in Green and Shalizi (2022).

For each pair of nodes i, j, $h_{0,n}(\xi_i, \xi_j)$ maps their unobserved characteristics ξ_i, ξ_j into the probability of a link (edge) between them, i.e. the probability with which $A_{ij} = 1$. For concise notation, we use $h_{0,n}(\xi(\iota))$ to mean the collection of pairwise linking probabilities between the elements of $\xi(\iota)$. We treat the linking function as unknown, making minimal assumptions on its properties in Assumption 1.2: we require that for each input there is a neighbourhood of sufficiently large measure in which the behaviour of the function remains similar. Importantly, we do not require a specific form (e.g. random dot product structure: $h_{0,n}(\xi_i, \xi_j) = \xi_i' \xi_j$ like in Levin and Levina (2019)), we do not impose any shape constraints (e.g. that the function is strictly increasing in its inputs).

The decomposition into ρ_n and w_0 is without loss of generality and can be seen as a normalisation which allows us to interpret ρ_n as the expected edge density (the marginal probability of an edge between two nodes). We assume $\rho_n \to 0$ as $n \to \infty$, which captures the common feature of real economic networks known as sparsity. Intuitively, it says that the number of expected friends grows at a slower rate than the size of the network: no matter how large the potential pool of friends is, people tend to have a fairly small friendship group. This causes issues for estimation because, even as the size of the network grows at a rate n, the amount of information about the links of a specific node i grows at a slower rate of $\rho_n n$. In the extreme case of $\rho_n n$ being bounded we can't hope to get consistency of our estimates. In our results we specify bounds on the rate at which ρ_n approaches zero which still allow us to reliably estimate parameters and their distributions. For the linking function estimator we require that the density decreases at a slower rate than $\sqrt{\frac{\log(n)}{n}}$ (see Assumption 1.1), while in other sections we may strengthen this requirement, e.g. in Theorem 3 the allowed level of sparsity depends on how complicated the statistic we are estimating is.

w_0 is the underlying linking/graphon function after accounting for sparsity. While w_0 cannot be interpreted directly as a probability, it has similar properties, e.g. it is bounded[7]This is the function which determines the data generating process and the function the statistics of which we want to analyse. Although in a sample of size n we encounter its rescaled version $h_{0,n}$, for any asymptotic results we need to remove the effect of sparsity and we look at normalisations which are function of $\frac{h_{0,n}}{\rho_n}$.

To capture the way in which the linking function $h_{0,n}$ is translated into the observed links in A we introduce a random noise parameter: for $1 \leq i \leq j \leq n$ let $\eta_{ij} \overset{ind}{\sim} \mathcal{U}[0, 1]$ be independent of

7. This is a common assumption in the literature, though it is sometimes relaxed to allow $w_0(u, v) \in \mathbb{R}_+$ and let $h_{0,n}(u, v) = \min\{w_0(u, v), 1\}$. This affects the interpretation of ρ_n as the density and makes it more difficult to infer $h_{0,m}$ from $h_{0,n}$. Our results could be generalised to allow for unbounded w_0 at the expense of more complicated proofs and additional assumptions on bounded moments of w_0 or its functions.

ξ. We denote the vector of η_{ij} by η. We assume[8]:

$$A_{ij} = A_{ji} = \mathbb{1}\left(h_{0,n}(\xi_i, \xi_j) \geq \eta_{ij}\right) \tag{2}$$

$$A_{ii} = 0. \tag{3}$$

Note that $E(A_{ij}|\xi_i, \xi_j) = P(A_{ij} = 1|\xi_i, \xi_j) = h_{0,n}(\xi_i, \xi_j) = \rho_n w_0(\xi_i, \xi_j)$. To distinguish between adjacency matrices based on the true and estimated/simulated inputs we sometimes explicitly write A as a function: $A(h_{0,n}(\xi), \eta)$.

3.2 The object of interest

We are interested in some property of the network and we have an estimate of this property which is a function of the adjacency matrix. In order to learn about the property of interest we want to approximate the distribution of its estimator. More precisely, let $f_n(A(h_{0,n}(\xi), \eta), \rho_n, F_0)$, or $f_n(A)$ in short, be a function of the observed adjacency matrix A based on Bernoulli trials with probabilities determined by a linking function $h_{0,n}$ of i.i.d observations ξ from the distribution F_0, on the sparsity ρ_n, and on the distribution F_0 itself. The distribution we would like to approximate is:

$$J_n\left(t, h_{0,n}, F_0\right) = P\left(f_n\left(A\left(h_{0,n}\left(\xi\right), \eta\right), \rho_n, F_0\right) \leq t\right). \tag{4}$$

Example. *To fix ideas, suppose we want to learn about the density $\rho_n = E(A_{ij})$. We may want to test if it takes a specific value predicted by our theory, or we may wish to test if two networks (or perhaps the same network at two points in time) have the same density level. We can estimate the density using the density estimator from the observed adjacency matrix A:*

$$\hat{\rho}_n = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} A_{ij}. \tag{5}$$

We could use $f_n(A) = \hat{\rho}_n$ directly, or we could recentre and normalise the above expression:

$$f_n^{\rho_n}(A(h_{0,n}(\xi), \eta), \rho_n, F_0) = \frac{\sqrt{n}}{\binom{n}{2}\rho_n} \sum_{1 \leq i < j \leq n} A_{ij} - E_{F_0}(h_{0,n}(\xi_i, \xi_j)) \tag{6}$$

to get a function which has a well-defined asymptotic distribution. The results in Theorem 3 imply that $\hat{\rho}_n$ is consistent for ρ_n and $f_n^{\rho_n}(A(h_{0,n}(\xi), \eta), \rho_n, F_0)$ is asymptotically normal. The finite-

8. This is one specific way of achieving:

$$A_{ij}|\xi = A_{ji}|\xi \overset{ind}{\sim} \text{Bernoulli}\left(h_{0,n}(\xi_i, \xi_j)\right)$$
$$A_{ii} = 0$$

sample distribution is non-trivial and depends on F_0.

Our goal is to find a good approximation to this finite-sample distribution, e.g. in order to form confidence intervals for ρ_n. We do it by defining estimators $\hat{h}_n$ of the linking function $h_{0,n}$; $\hat{F}_n$ of the distribution of ξ; and $\hat{\rho}_n$ of the density parameter. We use these estimates to form B bootstrap adjacency matrices $A\left(\hat{h}_n\left(\xi_b^*\right), \eta_b^*\right)$, where the bth bootstrap adjacency matrix is evaluated using $\hat{h}_n$ based on ξ_b^* from $\hat{F}_n$, the bootstrap equivalent of ξ, and η_b^* is the bootstrap equivalent of η.

We evaluate $f_n\left(A\left(\hat{h}_n\left(\xi^*\right), \eta^*\right), \hat{\rho}_n, \hat{F}_n\right)$ for B bootstrap samples to get the simulated distribution:

$$\hat{J}_{n,B}\left(t, \hat{h}_n, \hat{F}_n\right) = \frac{1}{B}\sum_{b=1}^{B} \mathbb{1}\left(f_n\left(A\left(\hat{h}_n\left(\xi_b^*\right), \eta_b^*\right), \hat{\rho}_n, \hat{F}_n\right) \leq t\right). \tag{7}$$

For B large enough this provides an arbitrarily good approximation to $J_n\left(t, \hat{h}_n, \hat{F}_n\right)$[9] and can be used to approximate $J_n\left(t, h_{0,n}, F_0\right)$.

In the reminder of this section we define all the estimators and the bootstrap procedure.

3.3 Distance: definition and estimator

Based on the observed matrix A, we want to estimate the linking probability for any pair of nodes. We start by defining a distance between individuals i and j, taking the measure from Auerbach (2022)[10]. Intuitively, if two people have similar friendship groups, they should be similar to each other: they likely ended up with similar friendship groups because their linking functions were similar. We let $\varphi(\xi_i, \xi_t) = E\left(w_0\left(\xi_i, \xi_s\right) w_0\left(\xi_t, \xi_s\right) \middle| \xi_i, \xi_t\right) = E\left(\frac{A_{is}}{\rho_n}\frac{A_{ts}}{\rho_n}\middle| \xi_i, \xi_t\right)$ be a function measuring the probability of a common friend between i and t, normalised to remove the effect of sparsity. Similarly, $\varphi(\xi_j, \xi_t)$ gives a normalised measure of the probability of common friends between j and t. To measure the similarity in friendship groups between i and j we look at the expected difference $\varphi(\xi_i, \xi_t) - \varphi(\xi_j, \xi_t)$ for any individual t. This motivates the definition of distance

9.
$$P_{F_0}\left(\sup_t \left|\hat{J}_{n,B}(t, \hat{h}_n, \hat{F}_n) - J_n\left(t, \hat{h}_n, \hat{F}_n\right)\right| > \varepsilon\right) \leq 4\sqrt{2}e^{-2B\varepsilon^2},$$
see references on p.5 of Politis et al. (1999) for more details.

10. Auerbach (2022) refers to $\varphi_{\xi_i}(\tau) = \varphi(\xi_i, \tau)$ as the codegree function of agent i. In his model there is no sparsity: $\rho_n = 1$, which is why he does not need the normalisation by $\frac{1}{\rho_n}$.

between i and j:

$$d_{ij} = \sqrt{E\left(\left(\varphi(\xi_i, \xi_t) - \varphi(\xi_j, \xi_t)\right)^2 \middle| \xi_i, \xi_j \right)} \tag{8}$$

$$= \sqrt{E\left(\left(E\left(w_0\left(\xi_t, \xi_s\right)\left(w_0\left(\xi_i, \xi_s\right) - w_0\left(\xi_j, \xi_s\right)\right) \middle| \xi_i, \xi_j, \xi_t\right)\right)^2 \middle| \xi_i, \xi_j \right)} \tag{9}$$

$$= \sqrt{E\left(E\left(\frac{A_{ts}}{\rho_n}\left(\frac{A_{is}}{\rho_n} - \frac{A_{js}}{\rho_n}\right) \middle| \xi_i, \xi_j, \xi_t \right)^2 \middle| \xi_i, \xi_j \right)}. \tag{10}$$

After an appropriate normalisation by the sparsity level ρ_n, we get an expression in terms of the linking function $h_{0,n} = \rho_n w_0$ at sample size n:

$$\rho_n^2 d_{ij} = \sqrt{E\left(\left(E\left(h_{0,n}\left(\xi_t, \xi_s\right)\left(h_{0,n}\left(\xi_i, \xi_s\right) - h_{0,n}\left(\xi_j, \xi_s\right)\right) \middle| \xi_i, \xi_j, \xi_t\right)\right)^2 \middle| \xi_i, \xi_j \right)} \tag{11}$$

$$= \sqrt{E\left(E\left(A_{ts}\left(A_{is} - A_{js}\right) \middle| \xi_i, \xi_j, \xi_t\right)^2 \middle| \xi_i, \xi_j \right)}. \tag{12}$$

Eq. (11) highlights the close relation between the normalised distance and the similarity between the linking functions of i and j at sample size n: a low value of $\rho_n^2 d_{ij}$ means i and j are similar to each other in the sense that their $h_{0,n}(\xi_i, \cdot)$ and $h_{0,n}(\xi_j, \cdot)$ are close. We exploit this when defining an estimator for $h_{0,n}$. The normalised expression is also attractive because the sample equivalent of its representation in Eq. (12) provides us with an estimate of $\rho_n^2 d_{ij}$:

$$\rho_n^2 \hat{d}_{ij} = \sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(\frac{1}{n}\sum_{s=1}^{n} A_{ts}\left(A_{is} - A_{js}\right)\right)^2}. \tag{13}$$

Remark. *If we needed to estimate $\hat{d}_{ij}$ without the normalisation we could substitute the estimated density $\hat{\rho}_n$ (Eq. (5)) for the unknown ρ_n. However, in practice the way we use the distance is with a normalisation by a bandwidth parameter a_n (chosen by the researcher), we look at functions of: $\frac{\rho_n^4 \hat{d}_{ij}^2}{a_n} \equiv \frac{\hat{d}_{ij}^2}{b_n}$, and we can think of the ρ_n as being absorbed into the renormalised bandwidth b_n.*

Remark. *We could also consider estimators of related distances from Zeleneev (2020), who took it from Zhang, Levina, and Zhu (2017): $\rho_n \hat{d}_{ij}^{(\infty)} = \left(\max_{t \neq i,j}\left|\frac{1}{n}\sum_{s \neq i,j,t} A_{ts}(A_{is} - A_{js})\right|\right)^{\frac{1}{2}}$ and from Lovász (2012) (sections 13.4, 15.4): $\rho_n^2 \hat{d}_{ij}^{(1)} = \frac{1}{n}\sum_{t=1}^{n}\left|\frac{1}{n}\sum_{s \neq i,j,t} A_{ts}(A_{is} - A_{js})\right|$. All of these distances are based on the same idea but average $\varphi(\xi_i, \xi_t) - \varphi(\xi_j, \xi_t)$ using L_2, L_∞ and L_1 distances, respectively.*

3.4 Linking function estimator

We are now ready to define an estimator $\hat{h}_n$ for the linking function $h_{0,n}$. We rely on a kernel approximation: let $K(\cdot)$ be a kernel function (for properties see Assumption 1.3), let a_n be a bandwidth parameter (for its rates of convergence see Assumption 1.4, see Section 3.5 for a method of choosing an optimal bandwidth). We can estimate $h_{0,n}(\xi_i, \xi_j)$ as:

$$\hat{h}_n(\xi_i, \xi_j) = \frac{\tilde{h}_n(\xi_i, \xi_j) + \tilde{h}_n(\xi_j, \xi_i)}{2} \tag{14}$$

where

$$\tilde{h}_n(\xi_i, \xi_j) = \frac{\sum_{\substack{t=1 \\ t\neq j}}^{n} K\left(\frac{\rho_n^4 \hat{d}_{it}^2}{a_n}\right) A_{tj}}{\sum_{\substack{t=1 \\ t\neq j}}^{n} K\left(\frac{\rho_n^4 \hat{d}_{it}^2}{a_n}\right)}. \tag{15}$$

$\tilde{h}_n(\xi_i, \xi_j)$ is a local weighted average which puts the highest weights on the individuals most similar to i. The bandwidth a_n controls the required level of similarity beyond which we use zero weights. Each person t with $\hat{d}_{it}$ sufficiently close to zero can be seen as a counterfactual to i, someone with a very similar linking function (i.e. a small $h_{0,n}(\xi_i, \cdot) - h_{0,n}(\xi_t, \cdot)$). The proportion of people similar to i who are linked to j gives an estimate of the link probability between i and j. The observation with $t = j$ is excluded because we assume there are no self-link: $A_{jj} = 0$ is not defined in terms of $h_{0,n}$. Adding a non-zero weight on this observation would introduce bias.

To get $\hat{h}_n(\xi_i, \xi_j)$ we take advantage of the symmetry of links, we repeat the estimation swapping the roles of i and j and take an average of the two estimates.

We show that this estimator is uniformly consistent for $h_{0,n}$ in Theorem 1.

Remark. *The estimator $\hat{h}_n$ has been proposed, but not analysed by Zeleneev (2020). We could also use a related estimator:*

$$\hat{h}_n^{(K2)}(\xi_i, \xi_j) = \frac{\sum_{\substack{t=1 \\ t\neq s}}^{n} \sum_{s=1}^{n} K\left(\frac{\rho_n^4 \hat{d}_{it}^2}{a_n}\right) K\left(\frac{\rho_n^4 \hat{d}_{js}^2}{a_n}\right) A_{ts}}{\sum_{\substack{t=1 \\ t\neq s}}^{n} \sum_{s=1}^{n} K\left(\frac{\rho_n^4 \hat{d}_{it}^2}{a_n}\right) K\left(\frac{\rho_n^4 \hat{d}_{js}^2}{a_n}\right)}.$$

Or one by Zhang, Levina, and Zhu (2017), which uses the nearest-neighbour idea:

$$\hat{h}_n^{(NN1)}(\xi_i, \xi_j) = \frac{\tilde{h}_n^{(NN1)}(\xi_i, \xi_j) + \tilde{h}_n^{(NN1)}(\xi_j, \xi_i)}{2} \quad \textit{where } \tilde{h}_n^{(NN1)}(\xi_i, \xi_j) = \frac{\sum_{t \in \mathcal{N}_i} A_{tj}}{\|\mathcal{N}_i\|}$$

where $\mathcal{n}_i$ denotes the set of neighbours of i. They show that the optimal size of the neighbourhood, $\|\mathcal{n}_i\|$, should grow at the rate of $(n\ln(n))^{1/2}$. Or, we could use a nearest-neighbour approach in both inputs simultaneously (again, mentioned, and this time analysed, by Zeleneev (2020)):

$$\hat{h}_n^{(NN2)}(\xi_i,\xi_j) = \frac{\sum_{t\in\mathcal{n}_i}\sum_{s\in\mathcal{n}_j} A_{ts}}{\|\mathcal{n}_i\|\|\mathcal{n}_j\|}.$$

In simulations all of these estimators perform similarly.

3.5 Optimal bandwidth

The linking function estimator relies on a bandwidth parameter chosen by the researcher. We propose a cross-validation procedure which allows choosing the bandwidth in an optimal way.

The idea is to choose a bandwidth for which $\hat{h}_n$ best explains the observed network A, *if we leave out A_{ij} when estimating A_{ij}*. The reason for leaving out A_{ij} is that if we don't, we are trying to estimate A_{ij} using a set of observations which include A_{ij}, hence we can estimate it perfectly. We just need to choose $a_n \simeq 0$, this puts weight one on A_{ij} and zero on all other observations, leading to a perfect prediction of A but a poor choice of bandwidth. This issue of overfitting can be avoided by removing the observation A_{ij} from the model predicting A_{ij}.

We firstly define a leave-one-out version of $\hat{h}_n$:

$$\tilde{h}_n^-(\xi_i,\xi_j) = \frac{\sum_{\substack{t=1\\t\neq i,j}}^n K\left(\frac{\rho_n^4\hat{d}_{it}^2}{a_n}\right)A_{tj}}{\sum_{\substack{t=1\\t\neq i,j}}^n K\left(\frac{\rho_n^4\hat{d}_{it}^2}{a_n}\right)}$$

$$\hat{h}_n^-(\xi_i,\xi_j) = \frac{\tilde{h}_n^-(\xi_i,\xi_j)+\tilde{h}_n^-(\xi_j,\xi_i)}{2}.$$

and then use it to obtain an estimate for the log-likelihood:

$$\ell(A,a_n) = \sum_{i=1}^n\sum_{j=1}^n A_{ij}\log\left(\hat{h}_n^-(\xi_i,\xi_j)\right) + (1-A_{ij})\log\left(1-\hat{h}_n^-(\xi_i,\xi_j)\right). \tag{16}$$

We choose a_n which maximises the above expression to be our optimal bandwidth:

$$a^{(opt)} = \max_{a_n}\ell(A,a_n). \tag{17}$$

3.6 Empirical distribution function estimator

The formation of matrix A is determined by an initial sample of ξ from F_0 and a linking probability between any pair of elements from ξ. We have defined a way to estimate the linking probabilities, but we still need a way to recreate the formation of ξ. This follows a very standard procedure, with one twist. Since the elements of ξ are i.i.d. from F_0, we should be able to use a standard bootstrap (resample from the values from the original sample, with replacement) to create a bootstrap equivalent. The non-standard part is that ξ_i are unobserved. We get around it by resampling not directly from the set of ξ_i, but from the set of original nodes: we let each bootstrap node correspond to one of the original nodes and we assign the set of characteristics of a bootstrap node (ξ_i^*) to be equal to the set of characteristics of the resampled original node. The resulting distribution is an empirical distribution function $\hat{F}_n$ defined as the CDF which corresponds to the probability mass function[11]:

$$P(\xi_i^* = x) = \begin{cases} \frac{1}{n} & \text{if } x \in \{\xi_1, \ldots, \xi_n\} \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Each bootstrap node corresponds to one of the original nodes and inherits its characteristics. The result is the same as if we formed the set of bootstrap characteristics ξ^* by resampling from the original set of characteristics ξ, with replacement.

3.7 Nonparametric network bootstrap procedure

We now describe the bootstrap procedure for $f_n(A(h_{0,n}(\xi), \eta), \rho_n, F_0)$. Just for this section, we introduce simplified notation for the matrix of estimated distances (D) and the matrix of estimated linking probabilities (H).

1. Calculate the distance between each pair of nodes $i, j \in \{1, 2, \ldots, n\}$:

$$D_{ij} = \frac{1}{n} \sum_{t=1}^{n} \left(\frac{1}{n} \sum_{s=1}^{n} A_{ts} \left(A_{is} - A_{js} \right) \right)^2 .$$

2. Calculate the optimal bandwidth parameter $a^{(opt)}$ as described in Eq. (17).

3. Calculate the probability of a link between each pair of nodes $i, j \in \{1, 2, \ldots, n\}$:

$$\hat{h}_n(\xi_i, \xi_j) = \frac{1}{2} \left(\frac{\sum_{\substack{t=1 \\ t \neq j}}^{n} K\left(\frac{D_{it}}{a^{(opt)}}\right) A_{tj}}{\sum_{\substack{t=1 \\ t \neq j}}^{n} K\left(\frac{D_{it}}{a^{(opt)}}\right)} + \frac{\sum_{\substack{t=1 \\ t \neq i}}^{n} K\left(\frac{D_{jt}}{a^{(opt)}}\right) A_{ti}}{\sum_{\substack{t=1 \\ t \neq i}}^{n} K\left(\frac{D_{jt}}{a^{(opt)}}\right)} \right)$$

11. We would like to use the standard definition of an empirical distribution function:

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\left(\xi_i < x\right)$$

but unfortunately ξ_i are not observed, and may in general not be scalar, hence this notation doesn't apply.

4. Calculate the density estimate of the original graph: $\hat{\rho}_n$ as described in Eq. (5).

5. For each $b = 1, \dots, B$:

   (a) draw an i.i.d. sample $\{\xi^*_{b,i}\}_{i=1}^n$ of size n from $\hat{F}_n$, i.e. resample from the original set of nodes $\{1, 2, \dots, n\}$ with equal probabilities and with replacement, then assign the unobserved characteristics of the bootstrap node to be the same as the unobserved characteristics of its corresponding original node. Let $\xi^*_b = \left(\xi^*_{b,1}, \dots, \xi^*_{b,n}\right)'$.

   (b) draw $\eta^*_{b,ij} \overset{ind}{\sim} \mathcal{U}[0,1]$ for $1 \le i \le j \le n$.

   (c) form the bootstrap adjacency matrix A^*_b:

$$A^*_{b,ij} = A^*_{b,ji} = \mathbb{1}\left(\hat{h}_n(\xi^*_{b,i}, \xi^*_{b,j}) \ge \eta^*_{b,ij}\right) \tag{19}$$

$$A^*_{b,ii} = 0. \tag{20}$$

   (e.g. if $\xi^*_{b,i} = \xi_t$, $\xi^*_{b,j} = \xi_s$ then $\hat{h}_n(\xi^*_{b,i}, \xi^*_{b,j}) = \hat{h}_n(\xi_t, \xi_s)$).

   (d) calculate the object of interest on the bootstrap adjacency matrix:

$$f_n(A^*_b) \equiv f_n\left(A\left(\hat{h}_n(\xi^*_b), \eta^*_b\right), \hat{\rho}_n, \hat{F}_n\right). \tag{21}$$

6. Form a $(1-\alpha)\%$ confidence interval for $f_n(A(h_{0,n}(\xi), \eta), \rho_n, F_0)$ by taking the interval between $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$ quantiles of $\{f_n\left(A^*_b\right)\}_{b=1}^B$.

For a description of how we used this procedure in simulations see Section 5. For the codes used in simulation see Appendix B.1.

4 Main results

In this section we state our main results which characterise the conditions under which the linking function estimator and our entire bootstrap procedure are consistent.

4.1 Consistency of the linking function estimator

We start by showing the uniform consistency of the linking function estimator.

Assumption 1 (The Assumptions for Uniform Consistency of the Linking Function Estimator)**.** *We make the following assumptions:*

1.1 $\frac{1}{\rho_n} = o\left(\sqrt{\frac{n}{\log(n)}}\right)$.

1.2 Let $N(\xi_j,\delta)=\left\{\xi_k:\sup_{\xi_t}|w_0(\xi_t,\xi_k)-w_0(\xi_t,\xi_j)|<\delta\right\}$ *denote the neighbourhood of* ξ_j *of size* δ *and let* $\omega(\delta)=\inf_{\xi_j\in Supp(\xi_j)}P\left(\xi_k\in N(\xi_j,\delta)|\,\xi_j\right)$*. There exist some* $\alpha,C>0$ *such that* $\omega(\delta)\geq\left(\frac{\delta}{C}\right)^{\frac{1}{\alpha}}$ *for all* $\delta>0$*.*

1.3 $K(\cdot)$ *is a kernel function which is*

- *a continuous bounded probability density function (non-negative:* $K(u)\geq 0$*, integrates to 1:* $\int K(u)du=1$*),*
- *non-zero on a bounded support: there exists a* $D\in\mathbb{R}$ *such that* $\forall|u|>D:K(u)=0$*,*
- *positive close to 0: there exist positive constants* C_1,C_2 *such that* $K(u)\geq C_1$ *whenever* $|u|\leq C_2$*,*
- *Lipschitz continuous: there exists* $C>0$ *such that* $|K(u)-K(v)|\leq C|u-v|$*.*

1.4 The bandwidth can be written as $a_n=\rho_n^4 b_n$ *for some* $b_n=o\left(1\right)$ *and* $\frac{1}{b_n}=o\left(\left(\frac{n\rho_n^2}{\log(n)}\right)^{\frac{\alpha}{1+2\alpha}}\right)$*.*

We now discuss the assumption and give some intuition.

Assumption 1.1 is our sparsity assumption. It gives a lower bound on how sparse a model can be for our estimator to remain consistent. Intuitively, the only informative observations are the links, and their number grows at a slower rate than the sample size: we expect on average $n\rho_n$ links in a sample of n individuals. Our model works well if the number of links increases at a rate faster than $\sqrt{n\log(n)}$. This is analogous to the assumption in Zhang, Levina, and Zhu (2017), with the exception that they model the increasing difficulty in estimation with n by allowing $\delta(n)\to 0$ instead of having a sparsity parameter $\rho_n\to 0$.

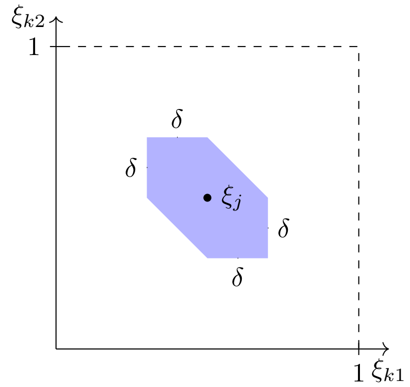
Assumption 1.2 ensures that the neighbourhoods for all observations are sufficiently large. We can think of it as a "continuity" condition for w_0, analogous to that assumed by Auerbach (2022): for all $\delta>0$:

$$\inf_{\xi_j\in Supp(\xi_j)}P_{\xi_k\sim F_0}\left(\sup_{\xi_t}|w_0\left(\xi_t,\xi_k\right)-w_0\left(\xi_t,\xi_j\right)|<\delta\middle|\xi_j\right)\geq\left(\frac{\delta}{C}\right)^{\frac{1}{\alpha}}.$$

i.e., for each $\xi_j\in Supp(\xi_j)$ there exists a sufficiently large positive measure of ξ_k with very similar friendship groups: such that $|w_0\left(\xi_t,\xi_k\right)-w_0\left(\xi_t,\xi_j\right)|<\delta$ holds for all ξ_t. The consequences of this assumption are similar to those of the Piecewise-Lipschitz assumption in Definition 2 of Zhang, Levina, and Zhu (2017) (in the proof of Theorem 2 we show that under Assumption 1.2 $\forall\varepsilon>0\exists K<\infty$ such that $Supp(\xi_i)$ can be split into K disjoint regions, each of size at least $\left(\frac{\varepsilon}{C}\right)^{\frac{1}{\alpha}}$, such that for any two points u,v which fall in the same region we have $\sup_{\xi_t}|w_0\left(\xi_t,u\right)-w_0\left(\xi_t,v\right)|\leq 2\varepsilon$ and there exists a set of points a_k, each from a different region, such that if $k\neq j$ we have $\sup_{\xi_t}|w_0\left(\xi_t,a_k\right)-w_0\left(\xi_t,a_j\right)|>\varepsilon$).

The example below shows that we can also think of this assumption as ensuring that the distribution of ξ_i is bounded away from zero while w_0 is sufficiently smooth:

Example. *For an example of Assumption 1.2 and some intuition on what α means, suppose that $\xi_i \in [0,1]^2$ and $w_0(\xi_i, \xi_j) = \xi_i \cdot \xi_j$. ξ_k satisfies $\sup_{\xi_t} |w_0(\xi_t, \xi_k) - w_0(\xi_t, \xi_j)| < \delta$ if it falls within a region in $[0,1]^2$ centred at ξ_j with a radius proportional to δ and area proportional to δ^2. If ξ_j is at least δ away from all boundaries of the support, the region has the following shape:*



and area $3\delta^2$. If the point ξ_j is closer to the boundary, we get the subset of this region which overlaps with $[0,1]^2$. The smallest area possible is $\frac{\delta^2}{2}$. If the distribution of ξ_i is uniformly bounded away from zero, the measure of ξ_j that satisfy the condition is at least proportional to δ^2, hence $\omega(\delta) \geq \left(\frac{\delta}{C}\right)^2$. A sufficient condition for our assumption is if the distribution of ξ_i is uniformly bounded away from zero and w_0 is piecewise Lipschitz. If w_0 is a well-behaved function and Supp$(\xi_i) \subset \mathbb{R}^d$, we would expect $\frac{1}{\alpha} = d$. We can think of $\frac{1}{\alpha}$ as a measure of complexity of the feature space: the more complex ξ_i are the harder the estimation.

Assumption 1.3 gives a list of fairly standard assumptions on the form of the kernel function. These are not restrictive as the kernel is chosen by the researcher and many of the standard kernels (e.g. the Epanechnikov kernel: $K(u) = \frac{3}{4}(1-u^2)\mathbb{1}(|u| < 1)$ or the triangular kernel: $K(u) = (1-|u|)\mathbb{1}(|u| < 1)$) satisfy all the requirements.

Assumption 1.4 specifies the range of bandwidths for which we can guarantee the correct asymptotic behaviour. The bandwidth a_n is a product of ρ_n^4, which cancels out the normalisation in $\left(\rho_n^2 \hat{d}_{it}\right)^2$, and $b_n \to 0$ which ensures that $\frac{\rho_n^4 \hat{d}_{it}^2}{a_n} = \frac{\hat{d}_{it}^2}{b_n} \to \infty$ for all $i \neq t$. ρ_n can be estimated and b_n is chosen by the researcher. As the effective dimension of the support of ξ_i increases, i.e. α decreases, the estimation becomes more difficult and we need b_n to go to zero at a slower rate.

Theorem 1. *Under Assumption 1:*

$$\max_{i,j}\left|\frac{\hat{h}_n(\xi_i,\xi_j)-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right| \xrightarrow{p} 0.$$

Remark. *Notice that we can write* $h_{0,n}(\xi_i,\xi_j)=\rho_n w_0(\xi_i,\xi_j)$*, decomposing the linking function into a bounded function* w_0 *which does not depend on* n *and the sparsity* $\rho_n \to 0$*. Without the normalisation by* $\frac{1}{\rho_n}$*, the difference* $\hat{h}_n - h_{0,n}$ *would trivially go to zero because both components go to zero at the rate* ρ_n*. In the statement of Theorem 1 we normalise by* $\frac{1}{\rho_n}$ *to show that, even after removing the trend to zero, the estimate of the linking function approaches its true value.*

Remark. *Note on notation: we use* $\max_{i,j}$ *to refer to maximising over indices in a specific sample of size* n*: it is a shorthand notation for* $\max_{i,j\in\{1,2,\dots,n\}}$*. We alter use* $\max_{\xi_i}$ *which refers to maximising over all* $\xi_i \in Supp(\xi_i)$*, i.e. all possible values in the support, not the set of realised values in a specific sample.*

Remark. *Theorem 1 shows uniform convergence of* $\hat{h}_n(\xi_i,\xi_j)$ *to* $h(\xi_i,\xi_j)$*, where "uniform" refers to convergence over all pairs of nodes in the original graph. The reason why we do not look at uniformity over all general points in the underlying sample space of* ξ_i[12] *is because our estimator of distance* $\hat{d}_{ij}^{(2)}$ *is defined in terms of similarity of friendship groups, hence it can only be estimated for one of the observed individuals. In our procedure we never estimate* ξ_i *directly, we don't put strong assumptions on the space it comes from, and we don't have a way of estimating* $\hat{d}$*, and hence* $\hat{h}_n$*, at a general point* (u,v) *outside of our realised set of observed individuals.*

However, the results we show later in Theorem 2 can be seen as an extension of Theorem 1 to the whole support of ξ_i*: under the assumption Assumption 1.2, for any* $\xi_i \in Supp(\xi_i)$*, if* n *is high enough, with high probability we can observer* ξ_j *similar enough to* ξ_i *that* $\hat{h}_n$ *evaluated at* ξ_j *provides a good approximation to* ξ_i *and the frequencies with which we observe different values of* $\hat{h}_n$ *is representative of the frequencies of similar values of the true* $h_{0,n}$ *over the support of* ξ_i*.*

One may be interested in using an alternative notion of distance. In the result below we the characterise conditions a distance needs to satisfy to get the conclusions of Theorem 1.

Lemma 1. *Let* $\tilde{d}_{ij}$ *be a distance between* ξ_i,ξ_j *such that there exist constants* $C_1, C_2 < \infty$*,* $\beta > 0$*,* $\gamma > 0$ *for which* $\forall \xi_i,\xi_t \in Supp(\xi_i)$*:*

$$C_1\tilde{d}_{it}^{\beta} \le \sup_{\xi_j}\left|w_0\left(\xi_t,\xi_j\right)-w_0\left(\xi_i,\xi_j\right)\right| \le C_2\tilde{d}_{it}^{\gamma}.$$

12. For example, if $\xi_i \sim \mathcal{U}[0,1]$ we could be interested in showing $sup_{u,v\in[0,1]}\left|\frac{\hat{h}_n(u,v)-h_{0,n}(u,v)}{\rho_n}\right| \xrightarrow{p} 0$.

Under Assumption 1 with the final condition in Assumption 1.4 replaced with: $\frac{1}{b_n} = o\left(\left(\frac{n\rho_n^2}{\log(n)}\right)^{\frac{\alpha}{2\beta+2\alpha}}\right)$ *we get the conclusions of Theorem 1 for* $\hat{h}_n$ *based on a consistent estimate of* $\tilde{d}_{ij}$ *instead of* $\hat{d}_{ij}$.

The only change in the assumptions is to the rate of convergence of the bandwidth b_n chosen by the researcher, the substantive assumptions remain unchanged.

4.2 Consistency of the bootstrap procedure in terms of Wasserstein distance

To show that the distribution of the bootstrap network approaches that of the original network we follow the approach from Levin and Levina (2019) (see their Section 4 for more details and motivation). We start by defining an appropriate notion of convergence between network distributions. Firstly, let the graph matching distance be the proportion of edges that differ between two graphs after their vertices have been aligned to minimise the number of such differences:

Definition 1 (Graph matching distance)**.** *Let* A_1*,* A_2 *be two* $n \times n$ *adjacency matrices,* Π_n *be the set of* $n \times n$ *permutation matrices and let* $\|A\|_{1,1} = \sum_{i=1}^n \sum_{j=1}^n |A_{ij}|$*. The graph matching distance is:*

$$d_{GM}(A_1, A_2) = \min_{P \in \Pi_n} \binom{n}{2}^{-1} \frac{\|A_1 - PA_2P'\|_{1,1}}{2}. \tag{22}$$

Equipped with a distance between graphs we can define a distance between two distributions over graphs by using the Wasserstein distance:

Definition 2. *Let* A_1*,* A_2 *be the adjacency matrices of two random graphs on* n *vertices and let* $\Gamma(A_1, A_2)$ *be the set of all couplings of* A_1 *and* A_2 *(i.e. all joint distributions with marginal distributions matching those of* A_1 *and* A_2*). For* $p \geq 1$ *the Wasserstein p-distance is given by:*

$$W_p(A_1, A_2) = \inf_{\nu \in \Gamma(A_1, A_2)} \left(\int d_{GM}^p(A_1, A_2) d\nu\right)^{\frac{1}{p}}. \tag{23}$$

Theorem 2. *Let* A *be the observed adjacency matrix, let* H *be another adjacency matrix drawn independently from the distribution of* A *and let* A^* *be a bootstrap adjacency matrix derived from* A*. Under Assumption 1:*

$$W_p^p(A^*, H) = o_p(\rho_n). \tag{24}$$

The graph matching distance is an upper bound on the cut metric, which in turn metrises convergence of subgraph densities, hence Theorem 2 implies that all subgraph densities of A^* converge to the same limit as those of H, proving that the bootstrap network distribution converges to the original network's distribution.

Remark. *As noted by Levin and Levina (2019), this notion of convergence is not sufficient to ensure that $f(A^*)$ converges to the same distribution as $f(H)$ for a general function $f(\cdot)$.*

4.3 Consistency of the bootstrap procedure for U-statistics

Because the result of Theorem 2 is not sufficient to guarantee that the distribution of a function of A^* is close to the distribution of the same function of A, we show this directly for an important class of functions with known limiting distributions.

Throughout this argument we use stars to denote the bootstrap equivalent, e.g. $\xi_i^* \sim \hat{F}_n$.

4.3.1 Appropriate notions of convergence

We start by introducing the definitions used in this section.

We choose f_n that has a distribution limit (usually a normal random variable), i.e. we assume $J_n(t, h_{0,n}, F_0) \Rightarrow J(t, w_0, F_0)$ for some non-degenerate distribution $J(t, w_0, F_0)$, where "$\Rightarrow$" denotes weak convergence. One convenient way to characterise weak convergence is though the following distance between measures: let P and Q be probability measures on a common metric space S equipped with a distance d_S and let

$$f(S) = \left\{ f : S \to \mathbb{R} : |f(x) - f(y)| \leq d_S(x, y), \sup_{x \in S} |f(x)| \leq 1 \right\}$$

be the set of (Lipschitz) continuous and bounded real-valued functions on S, then:

$$d_W(P, Q) \equiv \sup_{f \in f(S)} \left| \int f(x) dP(x) - \int f(x) dQ(x) \right|.$$

It can be shown[13] that $P_n \Rightarrow P$ if and only if $d_W(P_n, P) \to 0$ as $n \to \infty$.

In order to prove consistency of the bootstrap procedure we would like to show that the distribution $J_n\left(t, \hat{h}_n, \hat{F}_n\right)$, the bootstrap equivalent of $J_n(t, h_{0,n}, F_0)$, achieves the same asymptotic distribution $J(t, w_0, F_0)$. Unfortunately, the concept of weak convergence cannot be applied directly to the bootstrap statistic because both the bootstrap distribution $\hat{F}_n$ and the estimator $\hat{h}_n$ are random functions depending on the realisation of ξ, hence $J_n\left(t, \hat{h}_n, \hat{F}_n\right)$ and $d\left(J_n(t, \hat{h}_n, \hat{F}_n), J(t, w_0, F_0)\right)$

13. See e.g. Proposition (M) in Chapter I of Hahn (1993)

are also random. To proceed, we define two new concepts which generalise weak convergence to account for this randomness:

Definition 3. *We say that* P_n converges weakly *to* P almost surely*, denoted by* $P_n \overset{a.s.}{\Rightarrow} P$*, if* $d_W(P_n, P) \xrightarrow{a.s.} 0$*. For* $X_n \sim P_n$*,* $X \sim P$ *we write* $X_n \xrightarrow{d} X$ almost surely.

Analogously,

Definition 4. *we say that* P_n converges weakly *to* P in probability*, denoted by* $P_n \overset{p}{\Rightarrow} P$*, if* $d_W(P_n, P) \xrightarrow{p} 0$*. For* $X_n \sim P_n$*,* $X \sim P$ *we write* $X_n \xrightarrow{d} X$ in probability.

Although this is a weaker requirement than regular convergence in distribution, Gine and Zinn (1990), who introduced the concept of weak convergence in probability, show that this notion is sufficient for the construction of asymptotically correct confidence intervals.

4.3.2 Distributions of intermediate terms

Since the elements of A exhibit dependence (e.g. A_{ij} and A_{jk} both depend on ξ_j), it is relatively difficult to work with $f_n(A(h_{0,n}(\xi), \eta), \rho_n, F_0)$ directly. We can instead consider its expectation taken with respect to the Bernoulli trials with probabilities determined by $h_{0,n}(\xi)$:

$$\tilde{f}_n\left(h_{0,n}\left(\xi\right), \rho_n, F_0\right) \equiv E_{h_{0,n}}(f_n(A(h_{0,n}(\xi), \eta), \rho_n, F_0)|\xi) \tag{25}$$

where we have taken the expectation over η and the remaining object becomes a function of the i.i.d. ξ_i.

Let $\tilde{J}_n$ denote the distribution of $\tilde{f}_n$:

$$\tilde{J}_n(t, h_{0,n}, F_0) = P_{F_0}\left(\tilde{f}_n(h_{0,n}(\xi), \rho_n, F_0) \leq t\right). \tag{26}$$

The limit of $\tilde{J}_n$ is easier to find than that of J_n, and the limits coincide if we can show that $f_n - \tilde{f}_n$ is negligible.

Remark. *We often work with conditional expectations and switch between variables that follow different distributions (e.g. the true distribution* F_0 *and the estimated empirical distribution* $\hat{F}_n$*). When we think it is beneficial to clarify, we add subscripts to the expectation operator indicating with respect to which distribution we are taking the expectation. For example,* $E_{h_{0,n}}(f_n(A(h_{0,n}(\xi), \eta), \rho_n, F_0)|\xi) = \int f_n(A(h_{0,n}(\xi), \eta), \rho_n, F_0)d\eta$ *indicates that we are taking expectation with respect to the independent Bernoulli trials with probabilities determined by* $h_{0,n}$ *while* $E_{h_{0,n},F_0}(f_n(A(h_{0,n}(\xi), \eta), \rho_n, F_0))$ *denotes the expectation with respect to both the Bernoulli* *trials and the true distribution of* ξ*. The latter can also be written as* $E_{F_0}\left(\tilde{f}_n(h_{0,n}(\xi), \rho_n, F_0)\right)$*, where* $\tilde{f}_n(h_{0,n}(\xi), \rho_n, F_0)$ *has already been integrated over the Bernoulli trials, hence its randomness only comes from* F_0*, the true distribution of* ξ*.*

To illustrate the need for these intermediate terms we introduce an important class of statistics for which $\tilde{f}_n$ take the form of a U-statistic.

Definition 5. *Let* ι *be a set of* m *unique nodes,* ξ *be an* n*-dimensional vector of i.i.d. draws from a distribution* F *and* η *an* n*-dimensional vector of independent draws from* $\mathcal{U}[0,1]$*, and* $\rho \in [0,1]$ *be the sparsity level. Denote the adjacency matrix on the subgraph with nodes in* ι *by* $A(h(\xi(\iota)), \eta(\iota))$*.*

Let $g(A(\iota)) : \{0,1\}^{\binom{m}{2}} \to \mathbb{R}$ *be a non-degenerate symmetric function from a subgraph on* $m < \infty$ *nodes to the real line such that* $E_{h_{0,n},F_0}(g(A(\iota))) = \theta$*, where* θ *is a parameter of interest. We can estimate* θ *on the whole network* A *by*

$$\hat{\theta} = \frac{1}{\binom{n}{m}} \sum_{1 \le \iota_1 < \iota_2 < \cdots < \iota_m \le n} g(A(\iota)). \tag{27}$$

To get the corresponding $f_n(A(h(\xi), \eta), \rho, F)$ *with a well-defined distribution we recentre and normalise the above expression:*

$$f_n^U(A(h(\xi), \eta), \rho, F) = \frac{\sqrt{n}}{\binom{n}{m}\rho^{\tau(g)}} \sum_{1 \le \iota_1 < \iota_2 < \cdots < \iota_m \le n} \left(g(A(h(\xi(\iota)), \eta(\iota))) - E_{h,F}(g(A(h(\xi(\iota)), \eta(\iota))))\right) \tag{28}$$

and

$$\tilde{f}_n^U(h(\xi), \rho, F) = \frac{\sqrt{n}}{\binom{n}{m}\rho^{\tau(g)}} \sum_{1 \le \iota_1 < \iota_2 < \cdots < \iota_m \le n} \left(\tilde{g}(h(\iota)) - E_F(\tilde{g}(h(\iota)))\right) \tag{29}$$

where $\tilde{g}(h(\xi(\iota))) \equiv E_h(g(A(h(\xi(\iota)), \eta(\iota)))|\xi(\iota))$ *and we choose* $\tau(g)$ *to get a normalisation for which there exists a non-degenerate bounded function* $\tilde{\tilde{g}}$ *such that*

$$\frac{\tilde{g}(h_{0,n}(\iota))}{\rho_n^{\tau(g)}} = \tilde{\tilde{g}}(w_0(\iota)) + O(\rho_n).$$

The choice of $\tau(g)$ is quite simple: it is the smallest number of ones such that $g(\cdot)$ evaluated at a vector of $\tau(g)$ ones and $\binom{m}{2} - \tau(g)$ zeros is non-zero. More importantly, the normalisation is not important for practical applications. We introduce it in the definition because it is necessary to get a well-defined asymptotic distribution (see Theorem 3, without the normalisation the limiting value of the $\hat{\theta}$ would be 0), but we do not need it if our interest is in constructing a confidence interval for θ. To see this, suppose θ and $\hat{\theta}$ are as above and let the bootstrap equivalent of the estimator be $\hat{\theta}_b^* = \frac{1}{\binom{n}{m}} \sum_{1 \le \iota_1 < \cdots < \iota_m \le n} g(A_b^*(\iota))$. We can calculate the estimator for B bootstrap adjacency matrices and find a confidence interval for θ as $\left[\hat{\theta}_L^*, \hat{\theta}_U^*\right]$ where $\hat{\theta}_L^* = q_{\frac{\alpha}{2}}\left(\left\{\hat{\theta}_b^*\right\}_{b=1}^B\right)$ and $\hat{\theta}_U^* = q_{1-\frac{\alpha}{2}}\left(\left\{\hat{\theta}_b^*\right\}_{b=1}^B\right)$. This way we get:

$$
\begin{aligned}
1-\alpha &\simeq P\left(\hat{\theta}_L^* < \theta < \hat{\theta}_U^*\right) \\
&= P\left(\frac{\sqrt{n}}{\hat{\rho}_n^{\tau(g)}}\left(\hat{\theta} - \hat{\theta}_U^*\right) < \frac{\sqrt{n}}{\hat{\rho}_n^{\tau(g)}}\left(\hat{\theta} - \theta\right) < \frac{\sqrt{n}}{\hat{\rho}_n^{\tau(g)}}\left(\hat{\theta} - \hat{\theta}_L^*\right)\right) \\
&\simeq P\left(\frac{\sqrt{n}}{\hat{\rho}_n^{*\tau(g)}}\left(\hat{\theta} - \hat{\theta}_U^*\right) < \frac{\sqrt{n}}{\hat{\rho}_n^{\tau(g)}}\left(\hat{\theta} - \theta\right) < \frac{\sqrt{n}}{\hat{\rho}_n^{*\tau(g)}}\left(\hat{\theta} - \hat{\theta}_L^*\right)\right).
\end{aligned}
$$

The above confidence interval for θ is equivalent to a confidence interval for $\frac{\sqrt{n}}{\hat{\rho}_n^{\tau(g)}}\left(\hat{\theta} - \theta\right)$ of the form $\left[\frac{\sqrt{n}}{\hat{\rho}_n^{\tau(g)}}\left(\hat{\theta} - \hat{\theta}_U^*\right), \frac{\sqrt{n}}{\hat{\rho}_n^{\tau(g)}}\left(\hat{\theta} - \hat{\theta}_L^*\right)\right]$ and is closely approximated by $\left[\frac{\sqrt{n}}{\hat{\rho}_n^{*\tau(g)}}\left(\hat{\theta} - \hat{\theta}_U^*\right), \frac{\sqrt{n}}{\hat{\rho}_n^{*\tau(g)}}\left(\hat{\theta} - \hat{\theta}_L^*\right)\right]$ (we could also use the quantiles of $f_n^U\left(A\left(\hat{h}_n\left(\xi^*\right), \eta^*\right), \hat{\rho}_n, \hat{F}_n\right)$ or $f_n^U\left(A\left(\hat{h}_n\left(\xi^*\right), \eta^*\right), \hat{\rho}_n^*, \hat{F}_n\right)$ directly). The consistency of $\hat{\rho}_n$ and $\hat{\rho}_n^*$ for ρ_n follows from the proof of Theorem A.2.

Example. *We now show the relation between f_n and $\tilde{f}_n$ on an example. Suppose $m = 3$, e.g. $\iota = (1, 2, 3)$, and the function g only depends on two entries in A*[14]*: $g(A(\iota)) = g(A_{1,2}, A_{2,3})$.*

Conditional on ξ, the Bernoulli trials that determine the entries of A are independent. Hence, for example, $P(A_{ij} = 1, A_{jk} = 1|\xi) = P(A_{ij} = 1|\xi)P(A_{jk} = 1|\xi) = h_{0,n}(\xi_i, \xi_j)h_{0,n}(\xi_j, \xi_k)$. It follows that

$$
g(A_{1,2}, A_{2,3})|\xi = \begin{cases} g(0,0) & \text{with probability } (1 - h_{0,n}(\xi_1, \xi_2))(1 - h_{0,n}(\xi_2, \xi_3)) \\ g(0,1) & \text{with probability } (1 - h_{0,n}(\xi_1, \xi_2))h_{0,n}(\xi_2, \xi_3) \\ g(1,0) & \text{with probability } h_{0,n}(\xi_1, \xi_2)(1 - h_{0,n}(\xi_2, \xi_3)) \\ g(1,1) & \text{with probability } h_{0,n}(\xi_1, \xi_2)h_{0,n}(\xi_2, \xi_3). \end{cases}
$$

The conditional expectation is a function of $h_{0,n}(\xi(\iota))$:

$$
\begin{aligned}
E(g(A_{1,2}, A_{2,3})|\xi) &\equiv \tilde{g}(h_{0,n}(\xi_1, \xi_2), h_{0,n}(\xi_2, \xi_3)) \\
&= g(0,0)(1 - h_{0,n}(\xi_1, \xi_2))(1 - h_{0,n}(\xi_2, \xi_3)) + g(0,1)(1 - h_{0,n}(\xi_1, \xi_2))h_{0,n}(\xi_2, \xi_3) \\
&+ g(1,0)h_{0,n}(\xi_1, \xi_2)(1 - h_{0,n}(\xi_2, \xi_3)) + g(1,1)h_{0,n}(\xi_1, \xi_2)h_{0,n}(\xi_2, \xi_3).
\end{aligned}
$$

If $g(0,0) \neq 0$, the first term on the right is $O(1)$ and dominates over the next terms. In this case we

14. For simplicity in this example we used a function which is not necessarily symmetric. Before plugging it into Eq. (28) we should symmetrise it in the following way:

$$
\bar{g}(A(\iota)) = \frac{g(A_{1,2}, A_{2,3}) + g(A_{1,2}, A_{1,3}) + g(A_{1,3}, A_{2,3}) + g(A_{2,3}, A_{1,2}) + g(A_{1,3}, A_{1,2}) + g(A_{2,3}, A_{1,3})}{6}.
$$

choose $\tau(g) = 0$. If $g(0,0) = 0$ but $g(0,1) \neq 0$ or $g(1,0) \neq 0$, the dominating term is proportional to $(1 - h_{0,n}(\xi_i, \xi_j))h_{0,n}(\xi_j, \xi_k) = (1 - \rho_n w_0(\xi_i, \xi_j))\rho_n w_0(\xi_j, \xi_k) = O(\rho_n)$, hence we choose $\tau(g) = 1$ to normalise it. If $g(0,0) = g(0,1) = g(1,0) = 0$ but $g(1,1) \neq 0$, the dominating term is proportional to $h_{0,n}(\xi_i, \xi_j)h_{0,n}(\xi_j, \xi_k) = \rho_n^2 w_0(\xi_i, \xi_j)w_0(\xi_j, \xi_k) = O(\rho_n^2)$, hence the correct normalisation is $\tau(g) = 2$. When $\rho_n \to 0$, only the dominating term influences the limiting behaviour. We call this dominating term $\tilde{\tilde{g}}$. In this example:

$$
\begin{aligned}
&\tilde{\tilde{g}}(h_{0,n}(\xi_1, \xi_2), h_{0,n}(\xi_2, \xi_3)) = \\
&\qquad = \begin{cases} g(0,0) & \text{if } g(0,0) \neq 0 \\ g(0,1)h_{0,n}(\xi_2, \xi_3) + g(1,0)h_{0,n}(\xi_1, \xi_2) & \text{if } g(0,0) = 0, g(0,1) \neq 0, g(1,0) \neq 0 \\ g(1,1)h_{0,n}(\xi_1, \xi_2)h_{0,n}(\xi_2, \xi_3) & \text{if } g(0,0) = g(0,1) = g(1,0) = 0, g(1,1) \neq 0. \end{cases}
\end{aligned}
$$

In all cases we have:

$$\frac{\tilde{g}(h_{0,n}(\xi_1, \xi_2), h_{0,n}(\xi_2, \xi_3))}{\rho_n^{\tau(g)}} = \tilde{\tilde{g}}(w_0(\xi_1, \xi_2), w_0(\xi_2, \xi_3)) + O(\rho_n)$$

where the first term is $O(1)$ and does not depend on the sample size n.

We will later do Taylor expansion of $\tilde{g}$, for which it is interesting to note that regardless of whether g is a nice function (continuous, differentiable, etc) of A, $\tilde{g}$ is (infinitely many times) continuously differentiable in $h_{0,n}$ and has bounded derivatives. Taking a derivative of $\tilde{g}$ with respect to $h_{0,n}$ lowers the power on the $h_{0,n}$ terms by one, hence if $\tilde{g} \sim \rho_n^{\tau(g)}$ then $\tilde{g}' \sim \rho_n^{\tau(g)-1}$ and $\frac{\tilde{g}'}{\rho_n^{\tau(g)}} = \frac{1}{\rho_n}$.

Then

$$\tilde{f}_n(h_{0,n}(\xi), \rho_n, F_0) = \frac{\sqrt{n}}{\binom{n}{m}\rho_n^{\tau(g)}} \sum_{1 \leq \iota_1 < \iota_2 < \cdots < \iota_m \leq n} \left(\bar{\tilde{g}}(h_{0,n}(\iota)) - E_{F_0}(\bar{\tilde{g}}(h_{0,n}(\iota)))\right). \tag{30}$$

We can think of the above as a function of the i.i.d. ξ. If g is symmetric, so is $\tilde{g}(h_{0,n}(\cdot))$, and the $\tilde{f}_n(h_{0,n}(\xi), \rho_n, F_0)$ takes the form of a (normalised) U-statistic, for which we have results such as LLN and CLT. Hence it's much easier to work with than the original $f_n(A(h_{0,n}(\xi), \eta), \rho_n, F_0)$ (which was not a U-statistic due to the dependence in A).

Remark. *It is usually not the case that $\tilde{f}_n$ has the same form as f_n with A_{ij} replaced with $h_{0,n}(\xi_i, \xi_j)$, but it can happen in some special cases. One such example are motif densities, for which the g function is a product of terms of the form A_{ij} (if the motif has an edge between nodes i and j) and $(1 - A_{ij})$ (if the edge is supposed to be missing). For example, if the motif of interest* *is a triangle, we have* $g(A_{ij}, A_{jk}, A_{ki}) = A_{ij}A_{jk}A_{ki}$. *This is 1 if all inputs are equal to 1 and 0 in all other cases, hence* $\tilde{g}(h_{0,n}(\xi_i,\xi_j), h_{0,n}(\xi_j,\xi_k), h_{0,n}(\xi_k,\xi_i)) = h_{0,n}(\xi_i,\xi_j)h_{0,n}(\xi_j,\xi_k)h_{0,n}(\xi_k,\xi_i)$.

Remark. *Because in the proof of Theorem 3 we rely on a CLT for U-statistics applied to* $\tilde{f}_n$ *instead of* f_n*, the normalisation by* $\frac{1}{\rho_n^{\tau(g)}}$ *is chosen to balance the rate of growth of the variance of* $\tilde{g}$ *rather than* g. *Take a simple example of* $g(A_{ij}) = A_{ij}$. *Then* $\tau(g) = 1$ *and:*

$$E\left(\left(\frac{g(A_{ij})}{\rho_n^{\tau(g)}}\right)^2\right) = \frac{E\left(h_{0,n}(\xi_i,\xi_j)\right)}{\rho_n^{2\tau(g)}} = \frac{1}{\rho_n^{\tau(g)}} \to \infty$$

$$E\left(\left(\frac{E\left(g(A_{ij})|\,\xi\right)}{\rho_n^{\tau(g)}}\right)^2\right) = \frac{\left(E\left(h_{0,n}(\xi_i,\xi_j)\right)\right)^2}{\rho_n^{2\tau(g)}} = O(1).$$

The following notation simplifies the statement of the theorem:

Definition 6. *Set* $\mathcal{M}_m$: *Let* $\mathcal{M}_m$ *be the set of all possible multisets*[15] *of cardinality* m *with elements from* $\{1, 2, \ldots, m\}$.

That is, $\mathcal{M}_m$ contains all possible combinations of index numbers from 1 to m that are of length m and can be all unique or have any value repeated any number of times.

We are now ready to state the final main result, which shows that for statistics which can be represented as in Definition 5 and are non-degenerate (i.e. $\sigma_1^2 \neq 0$): 1. the limiting distribution in probability of the bootstrap statistic is asymptotically normal and the same as the limiting distribution of the original statistic and 2. the bootstrap is consistent, in the sense that the finite-sample distribution of the bootstrap statistic approaches the finite-sample distribution of the original statistic as the sample size increases.

Theorem 3. *Let* $f_n^U(A(h(\xi), \eta), \rho, F)$ *be as in Eq.* (28). *There exists a normalisation* $\tau(g)$[16] *and a function* $\tilde{\tilde{g}} : Supp(\xi)^m \to \mathbb{R}$ *such that* $\frac{\tilde{g}(h_{0,n}(\xi(\iota)))}{\rho_n^{\tau(g)}} = \tilde{\tilde{g}}(w_0(\xi(\iota))) + O(\rho_n)$ *and* $0 < E\left(|\tilde{\tilde{g}}(w_0(\xi(j)))|\right) < \infty$ *for all* $j \in \mathcal{M}_m$. *If Assumption 1 holds and:*

$$Var_{F_0}(E_{F_0}(\tilde{\tilde{g}}(w_0(\xi(\iota)))|\xi_{\iota_1})) \equiv \sigma_1^2 > 0$$

$$\frac{n}{\binom{n}{m}\rho_n^{\tau(g)}} \to 0$$

then

15. A multiset is like a set but allows for repeated elements.

16. For $m = 2$, if $g(0) \neq 0$ we set $\rho_n^{-\tau(g)} = 1$, $\tilde{\tilde{g}}(w_0(\xi_i, \xi_j)) = g(0)$ and if $g(0) = 0$ but $g(1) \neq 0$ we set $\rho_n^{-\tau(g)} = \frac{1}{\rho_n}$ and $\tilde{\tilde{g}}(w_0(\xi_i, \xi_j)) = g(1)w_0(\xi_i, \xi_j)$. More generally, for $m \geq 2$, $\rho_n^{-\tau(g)} = \frac{1}{\rho_n^k}$ where k is the smallest number of ones such that $g(\cdot)$ evaluated at a vector of k ones and $\binom{m}{2} - k$ zeros is non-zero.

1. $f_n^U(A(h_{0,n}(\xi),\eta),\rho_n,F_0) \xrightarrow{d} N(0,m^2\sigma_1^2)$ *and*
   $f_n^U\left(A\left(\hat{h}_n\left(\xi^*\right),\eta^*\right),\hat{\rho}_n,\hat{F}_n\right) \xrightarrow{d} N(0,m^2\sigma_1^2)$ *in probability.*
2. $\sup_t \left|P\left(f_n^U\left(A\left(\hat{h}_n\left(\xi^*\right),\eta^*\right),\hat{\rho}_n,\hat{F}_n\right)\leq t\right) - P\left(f_n^U\left(A\left(h_{0,n}\left(\xi\right),\eta\right),\rho_n,F_0\right)\leq t\right)\right| \xrightarrow{p} 0.$

In the above theorem we add two new assumptions on top of those in Assumption 1. The first one, $\sigma_1^2 \neq 0$, restricts our attention to non-degenerate U-statistics. Levin and Levina (2019) claim that in the case of degenerate U-statistics the approximation error is of a comparable size to the leading term, implying that their bootstrap cannot recover the distributions of degenerate U-statistics. As explained in Serfling (2009) section 5.5, in the degenerate case the correct normalisation would be of the form $\frac{n^{\frac{c}{2}}}{\binom{n}{m}\rho_n^2}$ for some $c \geq 2$ and we would expect a more complicated limiting distribution than normal. In that case $f_n - \tilde{f}_n = O\left(\sqrt{\frac{n^c}{\binom{n}{m}\rho_n^2}}\right)$, which could go to zero sufficiently fast to remain negligible. We suspect that recovering distributions of degenerate U-statistics could still be possible with our method but we leave the detailed analysis for future work.

The other condition: $\frac{n}{\binom{n}{m}\rho_n^{\tau(g)}} \to 0$ gives a restriction on the allowed level of sparsity. We require $\frac{1}{\rho_n} = o\left(n^{\frac{m-1}{\tau(g)}}\right)$, where $\tau(g) \in \{0,1,\ldots,\binom{m}{2}\}$. For sufficiently large $\tau(g)$ this condition may be stronger than Assumption 1.1. This is not surprising: large $\tau(g)$ means that the function g takes non-zero values only for very rare events, and these events are even less common in sparser graphs. Hence to be able to maintain consistency we need to restrict the allowed level of sparsity.

This condition is needed to ensure that the $f_n - \tilde{f}_n$ term does not affect the limiting distribution. If $\frac{n}{\binom{n}{m}\rho_n^{\tau(g)}} = O(1)$, the limit of this term would affect the resulting distribution and the overall limit would be the current one plus the limit of this adjustment term. If $\frac{n}{\binom{n}{m}\rho_n^{\tau(g)}} \to \infty$ this adjustment term would dominate the asymptotic behaviour. In that case we would need to use normalisation by $\frac{n}{\binom{n}{m}\rho_n^{\frac{\tau(g)}{2}}}$. The currently dominating term under the new normalisation would go to zero. Deriving the distribution of the new dominating term is difficult due to a high level of dependence between the elements of A.

Remark. *Green and Shalizi (2022) specify the maximal allowed level of sparsity in two cases: when the motif is acyclic they assume* $\frac{1}{\rho_n} = o(n)$ *and for a general motif they require* $\frac{1}{\rho_n} = o\left(n^{\frac{1}{2m}}\right)$ *for the empirical graphon and the weaker condition of* $\frac{1}{\rho_n} = o\left(n^{\frac{2}{m}}\right)$ *for a general linking function estimator, e.g. their histogram graphon.*

These conditions are weakly stronger than our $\frac{1}{\rho_n} = o\left(n^{\frac{m-1}{\tau(g)}}\right)$*: when* $g(\cdot)$ *corresponds to an acyclic motif we have* $\tau(g) \leq m-1$[17]*, hence* $\frac{m-1}{\tau(g)} \geq 1$*; when* $g(\cdot)$ *corresponds to a general motif we have* $\tau(g) \leq \binom{m}{2}$[18]*, hence* $\frac{m-1}{\tau(g)} \geq \frac{2}{m} \geq \frac{1}{2m}$*.*

17. $\tau(g)$ corresponds to the number of edges in the motif and m denotes the number of vertices. The maximal number of edges in an undirected acyclic graph on m nodes is $m-1$.
18. The maximal number of edges in an undirected graph on m nodes is $\binom{m}{2}$.

However, for consistency of our linking function estimator we require $\frac{1}{\rho_n} = o\left(\sqrt{\frac{n}{\log(n)}}\right)$, *which is stronger than the* $\frac{1}{\rho_n} = o(n)$ *condition for acyclic motifs. For general motifs, our condition is always weaker than the condition needed for the empirical graphon. In comparison with* $\frac{1}{\rho_n} = o\left(n^{\frac{2}{m}}\right)$ *for histogram graphon, our condition is weaker when* $m > 4$ *and stronger for* $m \leq 4$.

One of the motivations for looking at this class of functions on networks is that it contains subgraph densities, which can be viewed as 'network moments,' in the sense that if two networks match on the densities of all subgraphs they come from the same network generating distribution. Theorem 3 implicitly shows that the bootstrap network distribution converges to the distribution of the original network. However, as the subgraphs become more complicated we need to impose stronger conditions on sparsity, meaning that full convergence of all subgraphs would only follow for dense models in which ρ_n does not go to 0.

There are other linking function estimators (e.g. Zhang, Levina, and Zhu (2017)) and alternative ways to resample nodes. The next result characterises the conditions needed for consistency of the class of functions considered in Theorem 3 when we replace the $(\hat{h}_n, \hat{F})$ in our procedure with alternative estimators of $(h_{0,n}, F_0)$.

Lemma 2. *Theorem 3 holds for any estimators* (h_n, F_n) *of* $(h_{0,n}, F_0)$ *which satisfy:*

1. $E_{F_n}\left(\left(\frac{1}{\rho_n}\left(h_n(\xi_i^*, \xi_j^*) - h_{0,n}(\xi_i^*, \xi_j^*)\right)\right)^2\right) \xrightarrow{p} 0.$
2. $E_{F_n}\left(f\left(\xi^*(\iota)\right)\right) \xrightarrow{p} E_{F_0}\left(f\left(\xi(\iota)\right)\right)$ *for all* $f : Supp(\xi)^k \to \mathbb{R}$ *such that* $E_{F_0}\left(\left|f\left(\xi(\iota)\right)\right|\right) < \infty$ *for all* $\iota \in \mathcal{m}_k$, *for any* $k \leq 2m-1$.

In the proofs in Appendix A we restate Theorem 3 in a generalised way which incorporates the conditions given in Lemma 2.

A consequence of Theorem 3 is that the bootstrap procedure can consistently recover critical values and asymptotically valid confidence intervals, as stated in Corollary 1. In order to be able to define the confidence intervals and comment on their coverage we need an inverse of the bootstrap distribution, but $J_n\left(t, \hat{h}_n, \hat{F}_n\right)$ may not necessarily be continuous or strictly increasing in t, hence it may not be invertible in the standard sense. Because of this we define the inverse of a distribution in the following way:

Definition 7. *Let:*

$$J^{-1}(\alpha, h, F) \equiv \inf\left\{t : J(t, h, F) \geq \alpha\right\} \tag{31}$$

be the α*th quantile of the distribution* $J(t, h, F)$.

Corollary 1. *Under the conditions of Theorem 3:*

1. $J_n^{-1}\left(1-\alpha,\hat{h}_n,\hat{F}_n\right)\xrightarrow{p} c_{1-\alpha}$*, where* $c_{1-\alpha}$ *is the* $1-\alpha$ *critical value from* $N(0,m^2\sigma_1^2)$[19].

2. *If* F_0 *doesn't enter the function* f_n^U *directly but only through a parameter* θ[20]*:*

$$f_n^U(A(h_{0,n}(\xi),\eta),\rho_n,\theta),$$

then the $(1-\alpha)$ *confidence interval for* θ *constructed as:*

$$CI_n\left(1-\alpha,A,\hat{h}_n,\hat{F}_n\right)=\left\{\theta : J_n^{-1}\left(\frac{\alpha}{2},\hat{h}_n,\hat{F}_n\right)\leq f_n^U(A,\hat{\rho}_n,\theta)\leq J_n^{-1}\left(1-\frac{\alpha}{2},\hat{h}_n,\hat{F}_n\right)\right\} \tag{32}$$

is asymptotically valid:

$$P_{h_{0,n},F_0}\left(\theta\in CI_n\left(1-\alpha,A,\hat{h}_n,\hat{F}_n\right)\right)\xrightarrow{p}1-\alpha. \tag{33}$$

In defining the bootstrap statistic and forming confidence intervals we do not normalise the $f_n(\cdot)$ by the estimated variance. This is in part because the variance estimators may not be readily available, and even when they are, they tend to be complicated (e.g. for the subclass of motif densities Green and Shalizi (2022) Lemma 2 gives an expression for an estimator of variance. It is derived combinatorially by considering all motifs that can be achieved by merging two copies of the motif of interest on partially overlapping sets of nodes). Another reason is that, as pointed out by Hahn (1993), convergence weakly in probability ensures convergence of moments over the set of bounded and Lipschitz continuous functions, which does not include $f(x)=x^2$, meaning that weak convergence in probability of our bootstrap estimator does not guarantee the consistency of its variance. When properties of the variance estimate are unknown, it is safer to use the percentile method for the construction of confidence intervals.

However, when a reliable variance estimate is known, normalising the statistic of interest by the estimate of its standard deviation could improve the performance of the bootstrap procedure. While we do not analyse the rates theoretically, the logic should be close to the case of standard bootstrap, where Edgeworth expansion arguments show that a normalised bootstrap with a pivotal limiting distribution can achieve a faster rate of convergence, see e.g. Hansen (2014) sections 10.8-10.11.

19. I.e. $\Phi\left(\frac{c_{1-\alpha}}{m^2\sigma_1^2}\right)=1-\alpha$ where $\Phi(\cdot)$ denotes the CDF of $N(0,1)$.
20. For example in equation (28) we have $\theta=E_{h_{0,n},F_0}(g(A(\iota)))$.

5 Simulations

We test the performance of our procedure using Monte Carlo simulations. We simulate the true adjacency matrices for $\xi_i \overset{iid}{\sim} \mathcal{U}[0,1]$ and one of the following linking functions:

1. dot product function: $h(\xi_i, \xi_j) = \rho_n \xi_i \xi_j$. This is the parametric form assumed by Levin and Levina (2019), it's a relatively simple function and a good benchmark.

2. horseshoe function: $h(\xi_i, \xi_j) = \frac{\rho_n}{2}\left(e^{-200\left(\xi_i - \xi_j^2\right)^2} + e^{-200\left(\xi_j - \xi_i^2\right)^2}\right)$. This function was also used by Green and Shalizi (2022). They borrow it from Wang (2016), who described it as "a challenging example for graphon estimation."

3. high-density function:
   $h(\xi_i, \xi_j) = \frac{\rho_n}{0.975}\left(1 - \mathbb{1}\left(\left|\frac{1}{2} - \xi_i\right| \leq 0.05\right)\mathbb{1}\left(\left|\frac{1}{2} - \xi_j\right| \leq 0.05\right)\right)\left(1 - \frac{1}{2}\left(\left|\frac{1}{2} - \xi_i\right| + \left|\frac{1}{2} - \xi_j\right|\right)\right)$.
   The previous two functions had relatively low density (by construction, $\rho_n \leq 0.25$ for the dot product function and $\rho_n \leq 0.113$ for the horseshoe function). This final function has $\rho_n \leq 0.759$, allowing us to test the performance with higher density levels.

The plots of these functions are included in Appendix B.3.

In the estimation procedure we use the normal kernel: $K(u) = e^{-\frac{u^2}{2}}$ and the bandwidth $a^{(opt)}$ chosen by maximising $\ell(A, a_n)$, as described in Section 3.5.

We test the performance of the algorithm for a range of statistics which have economic interpretation, some of them are covered by our Theorem 3 while other are not, including some which are much more complicated to compute.

- density: $f_n(A) = \frac{1}{\binom{n}{2}} \sum\sum_{1 \leq i < j \leq n} A_{ij}$, i.e. the number of edges divided by the number of possible edges. This function is useful for normalisation and is an example of a U-statistic.

- triangle density: $f_n(A) = \binom{n}{3}^{-1} \sum\sum\sum_{1 \leq i < j < k \leq n} A_{ij} A_{jk} A_{ik}$, i.e. the proportion of all subsets of 3 nodes that are fully connected. This is another example of a U-statistic, but of a more complicated form. It can be used to measure clustering.

- transitivity: $f_n(A) = 3\frac{\#triangles}{\#triads} = \frac{tr\left(A^3\right)}{\sum_{i=1}^n \sum_{j=1}^n (A^2)_{ij} - tr(A^2)}$, i.e. the ratio of fully connected triples to connected triples. This statistic can be seen as the extent of triadic closure (the tendency of people who have a common friend to become friends with each other) and can be used as a measure of the ability of a group of people to maintain cooperation.

- kth largest eigenvalue of the adjacency matrix: $f_n(A) = \lambda_k(A)$. The eigenvector of the largest eigenvalue can be used as a measure of centrality, or the level of influence of individuals in a network. The other eigenvalues are also informative, e.g. the second eigenvalue of a normalised adjacency matrix, known as spectral homophily, can be seen as a measure of cohesiveness (Chetty et al. 2022), similar to transitivity.

- maximal betweenness centrality: $f_n(A) = \max_i \sum_{j,k} \frac{\#\ shortest\ paths\ between\ j\ and\ k\ through\ i}{\#\ shortest\ paths\ between\ j\ and\ k}$. This is another measure of how influential a person is, how well they are connected, which is an important determinant of, for example, how effective they would be at spreading information (Banerjee, Chandrasekhar, Duflo, and Jackson 2019) or holding someone accountable in their saving goal (Breza and Chandrasekhar 2019).

- modularity of the Louvain community detection algorithm: for a given partition of nodes into communities, modularity is defined as the proportion of edges within communities minus the proportion if the edges were distributed at random. The Louvain community detection algorithm aims to find a partition which maximises modularity by iteratively moving nodes to communities and aggregating communities until no further improvement is possible. We used functions 'louvain_communities' and 'modularity' from the Python networkx package, see their documentation for precise definitions.

A difficulty in running Monte Carlo simulations is that for each confidence interval coverage we need to generate 1000 true graphs, and for each of those we need 1000 bootstrap graphs, hence for each data point in our plots and tables we need to evaluate the statistic of interest a million times. Some of the above statistics take a bit of time to estimate, making the simulation process slow. To overcome this issue and to speed up the simulations we obtain the confidence interval coverage using the WARP procedure from Giacomini, Politis, and White (2013). It's a clever trick which allows us to only generate one bootstrap graph for each true graph, making the computation time close to 500 times faster:

1. Generate S true adjacency matrices on n nodes using the same true linking function (usually $S = 1000$, n between 25 and 1000).

2. For each true adjacency matrix A_s:

   (a) Find the optimal bandwidth $a_s^{(opt)} = \max_a \ell(A_s, a)$.

   (b) Calculate the matrix $\hat{h}_{n,s}$ based on A_s with bandwidth $a_s^{(opt)}$.

   (c) Resample n nodes of A_s to form the nodes of the bootstrap graph.

   (d) Generate a single bootstrap adjacency matrix $A^*_{s,1}$ by adding an edges between nodes with probabilities determined by $\hat{h}_{n,s}$.

Note that the number of bootstrap replications is $B = 1$.

3. Estimate the true value by the average of the statistic evaluated for the true graphs: $f_n^{(true)} = \frac{1}{S}\sum_{s=1}^{S} f_n(A_s)$ (or use the theoretical true value, if known).

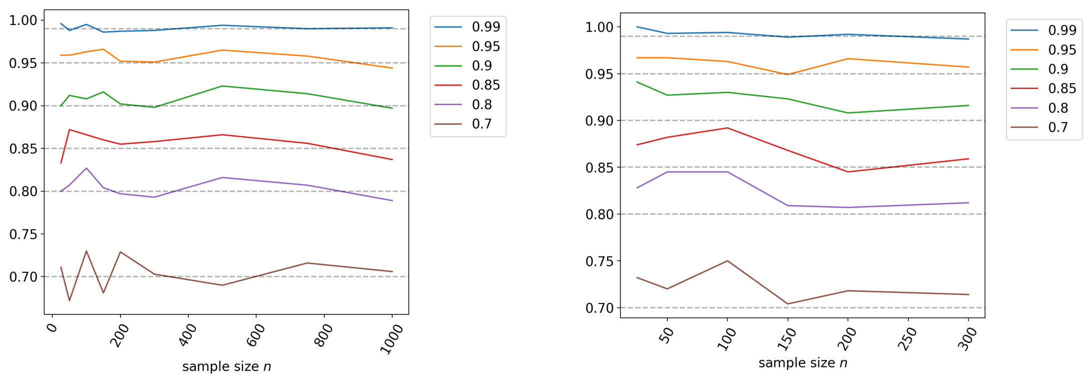
4. Calculate the deviation of the statistic in the bootstrapped graph from the statistic evaluated for the corresponding true graph: $f_n(A_{s,1}^*) - f_n(A_s)$ for all $s \in \{1, \dots, S\}$. Denote the αth quantile of the empirical distribution of this set by $\hat{q}_\alpha\left(\left\{f_n(A_{s,1}^*) - f_n(A_s)\right\}_{i=1}^{S}\right)$.

5. Calculate the confidence intervals as: $CL_s = [CL_s^l, CL_s^u]$, where

$$CL_s^l = f_n(A_s) - \hat{q}_{1-\frac{\alpha}{2}}\left(\left\{f_n(A_{s,1}^*) - f_n(A_s)\right\}_{i=1}^{S}\right)$$
$$CL_s^u = f_n(A_s) - \hat{q}_{\frac{\alpha}{2}}\left(\left\{f_n(A_{s,1}^*) - f_n(A_s)\right\}_{i=1}^{S}\right).$$

6. Store the empirical coverage, i.e. the proportion of confidence intervals which cover the true value: $1 - \alpha^{(emp)} = \frac{1}{S}\sum_{s=1}^{S} \mathbb{1}\left(CL_s^l \leq f_n^{(true)} \leq CL_s^u\right)$.

The following plots and tables show results of some of our simulations. The code used in the simulations can be found in Appendix B.1 and tables with more results are in Appendix B.2.



(a) Confidence interval coverage for density.

(b) Confidence interval coverage for λ_1.

Figure 1: Confidence interval coverage for different sample sizes n based on Monte Carlo simulations using the product generating function and $\rho_n = 0.1875$.

We start by looking at the confidence interval coverage for different values of n, ρ_n and α. From Fig. 1 we can see that performance at different α is quite similar, but larger values have proportionally larger deviations and allow us to see the trend more clearly. This is why, although in practice we tend to be most interested in the 95% confidence intervals, we present results for 70% or 80% in most of our plots. From Fig. 2 we can see that at a constant density level (no sparsity, ρ_n doesn't decrease with n) the performance improves with sample size. For $n \geq 250$ all statistics achieve good confidence interval coverage levels, although not always perfect: we may get coverage of e.g. 60% instead of 70%. As the sparsity level increases ($\rho_n \to 0$), the performance tends to get worse, but remains close to desired for statistics which are easier to estimate (e.g. density, triangle density, or the highest eigenvalue) while it gets significantly worse for more complicated statistics (e.g. λ_{10}). To some extent, we may be able to overcome these issues by changing the choice of bandwidth, as we describe below. The bottom two panels check performance for sparsity levels at which our theoretical results don't give any performance guarantees: as expected, the performance is poor in those cases.

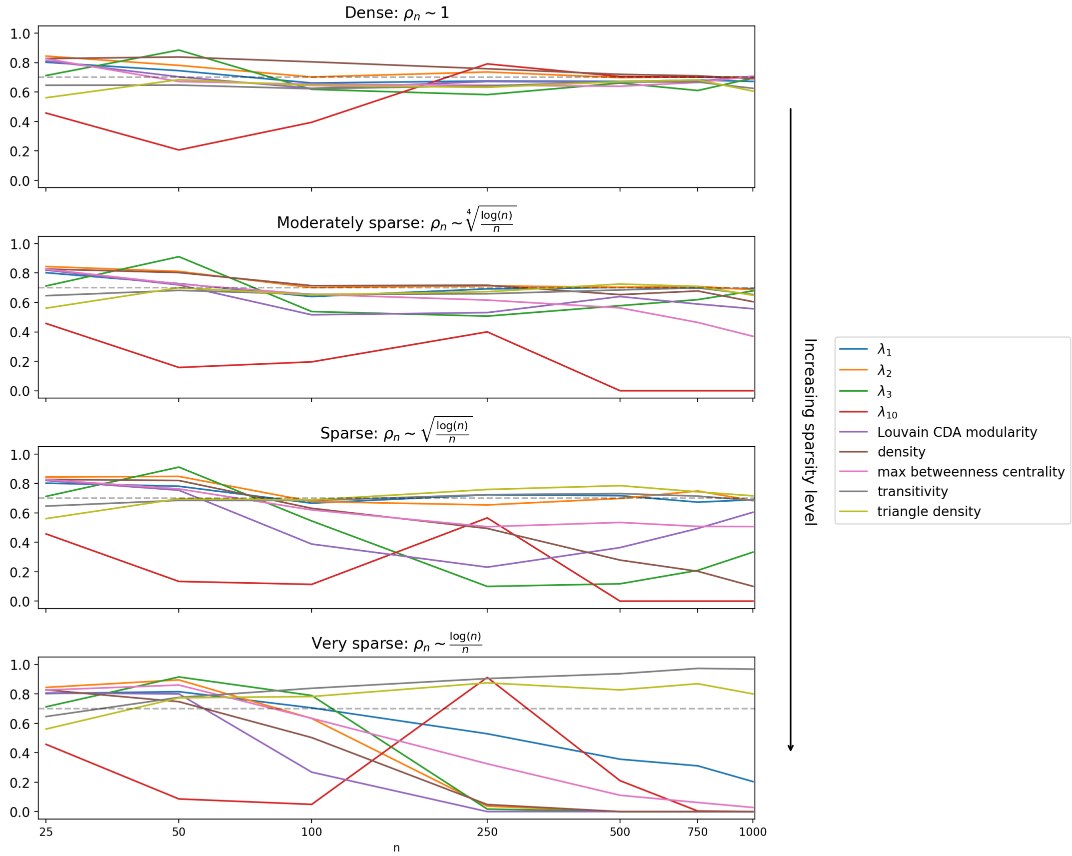
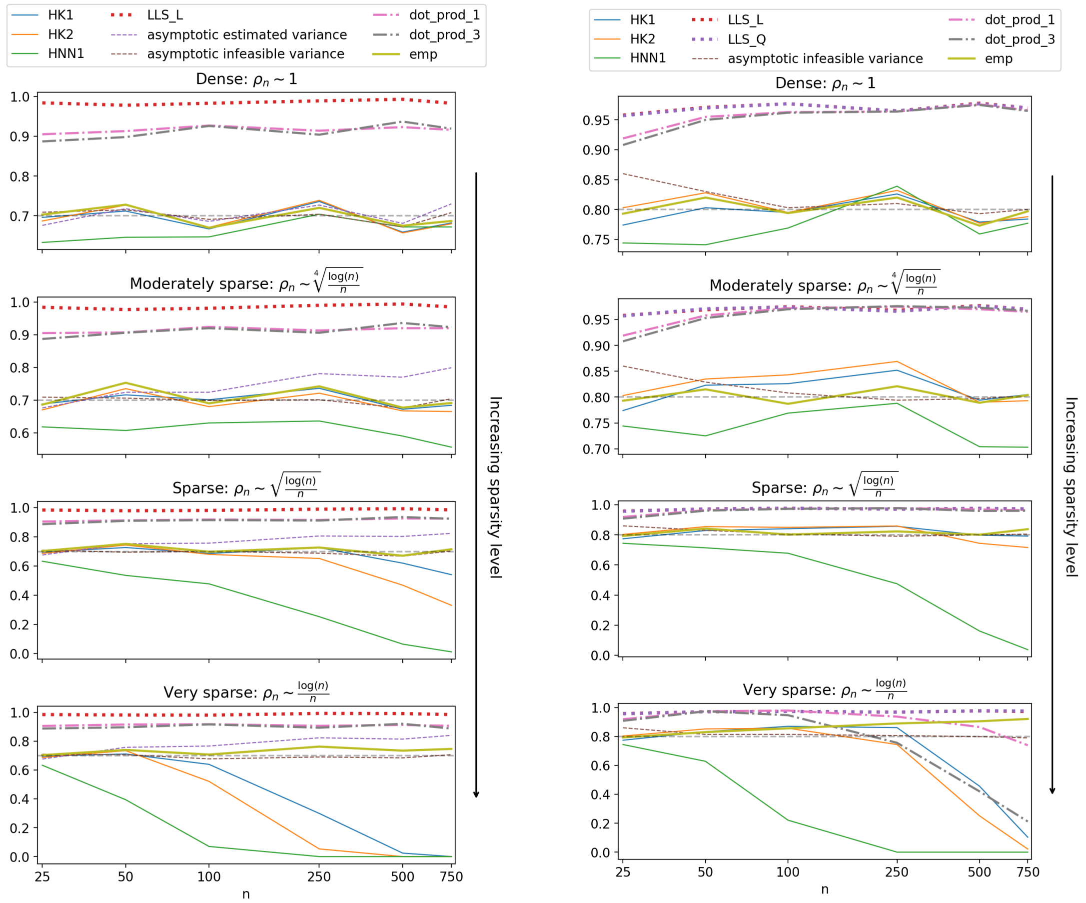


Figure 2: 70% confidence interval coverage for a range of statistics for the horseshoe generating function at different levels of sparsity.

Next we compare the performance of different variations of our method and some of the existing competitor methods. Fig. 3a shows that variations of our method (HK1 and HK2) perform very similarly: we choose to use HK1 as the main method since HK2 is more computationally intensive and HK1 shows slight advantage for sparse graphs. Our method with Zhang, Levina, and Zhu (2017) linking function estimator (HNN1) performs slightly worse and its performance drops more significantly for sparser graphs. We believe this is mostly due to the choice of bandwidth (the estimator in Zhang, Levina, and Zhu (2017) relies on the theoretical optimal bandwidth instead of our numerically chosen $a^{(opt)}$). For U-statistics the empirical bootstrap of Green and Shalizi

(2022) performs very well and remains good even at sparsity levels our methods can't handle. This suggests that for sparse graphs we may want to consider lower bandwidth choice than $a^{(opt)}$, which would make our method more similar to the empirical bootstrap. The dot product bootstrap of Levin and Levina (2019) is presented for the correctly specified case of $k = 1$, as well as for $k = 3$. This method should have an advantage over the other ones since it is parametric and for $k = 1$ it is based on a correctly specified functional form of the linking function. However, it achieves coverage which is too high, over 90% instead of the required 70%. In simulations, we have seen that many estimates of linking probabilities ended up outside of the $[0, 1]$ region, which we believe is the reason why Levin and Levina (2019) estimators ended up biased. Their method should work better at larger sample sizes. The linear and quadratic methods from Lin, Lunde, and Sarkar (2020) are very similar to each other and suffer from the same issue of giving confidence intervals with higher coverage than desired.



(a) Density: 70% confidence interval coverage across different methods.

(b) Triangle density: 80% confidence interval coverage across different methods.

Figure 3: Confidence interval coverage for different methods using the product generating function. We compare: HK1 (our main method based on $\hat{h} \equiv \hat{h}^{(K1)}$ with $a^{(opt)}$), HK2 (our bootstrap method but using the linking function estimator $\hat{h}^{(K2)}$ with $a^{(optK2)}$ based on $\hat{h}^{(K2)}$), HNN1 (our bootstrap method but but using the linking function estimator $\hat{h}^{(NN1)}$ from Zhang, Levina, and Zhu (2017) with their optimal choice of neighbourhood size), emp (empirical bootstrap from Green and Shalizi (2022)), dot_prod_k (the bootstrap method from Levin and Levina (2019) based on assuming a k-dimensional ξ_i), asymptotic estimated variance (the asymptotic distribution from Bickel, Chen, and Levina (2011) with variance estimated according to the formula in Green and Shalizi (2022)), asymptotic infeasible variance (the asymptotic distribution from Bickel, Chen, and Levina (2011) with the true theoretical variance), LLS_L and LLS_Q (the linear and quadratic methods from Lin, Lunde, and Sarkar (2020)).

Most of the competitor methods can only be applied to U-statistics. An exception is the empirical bootstrap, which can be seen as a limiting case of our procedure with bandwidth close to 0. From Fig. 4 and Fig. 5 we can see that the empirical bootstrap would perform very poorly for statistics such as lower eigenvalues, max betweenness centrality and Louvain CDA modularity. There is also a version of the dot product bootstrap Levin and Levina (2019) which applies to more general functions, but, probably due to our bad coding or relatively small sample sizes, we were getting many estimated probabilities outside of $[0, 1]$ which led to very bad performance, likely not representative of the quality of their method, and is hence not presented here.

We have seen poor coverage for some of the more complicated statistics of interest, and in the final set of simulations we explore if this could be improved by choosing a different bandwidth than $a^{(opt)}$. We look at $ca^{(opt)}$ for a range of constants c. In Fig. 4 the confidence interval coverage for simple statistics such as density, triangle density or the largest eigenvalue is good at the default bandwidth and not too sensitive to the bandwidth choice: the performance remains good between $0.01a^{(opt)}$ to $4a^{(opt)}$. However, other statistics, such as eigenvalues below the largest one (e.g. λ_2, λ_3 and λ_{10} in Fig. 4) have poor coverage for the default choice of with $c = 1$, and low values of c. Luckily, in those cases we can fix the problem by choosing a wider bandwidth: the confidence interval coverage for $c \simeq 2$ remains very good. Hence for more complicated statistics, we need to be careful and check the performance not only at the default bandwidth choice but also at e.g. half or twice the default bandwidth.

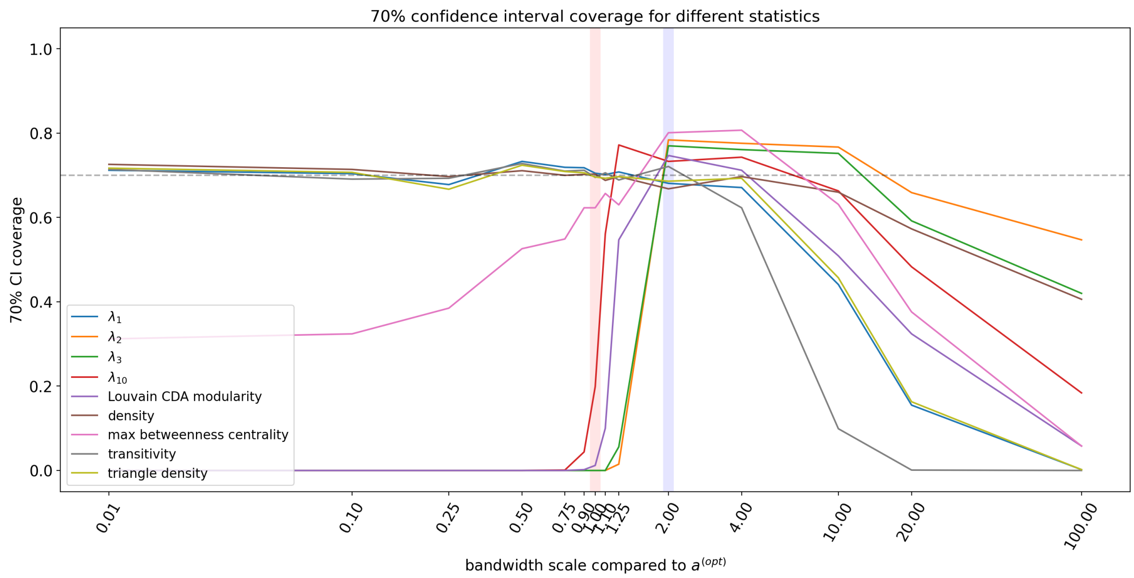


Figure 4: Confidence interval coverage for different bandwidths $c \times a^{(opt)}$: comparison of different statistics at $\alpha = 0.3$, $n = 300$ and $\rho_n = 0.1875$ based on Monte Carlo simulations using the product generating function.

These issues don't always arise: Fig. 5 shows an example when all statistics perform well for our default choice. For most statistics using a smaller bandwidth is not an issue: they do reasonably well for $c \leq 1$, even as small as $c = 0.01$, although they don't perform as well as for c close to 1. However, this is not universally true: some statistics, such as λ_{10} and maximal betweenness centrality, have very poor coverage outside of the region of $0.9 \leq c \leq 1.25$. The coverage for all statistics gets significantly worse when we use wider bandwidths ($c \geq 10$).

When the graph is sufficiently large and dense, c close to 1 gives good performance of most

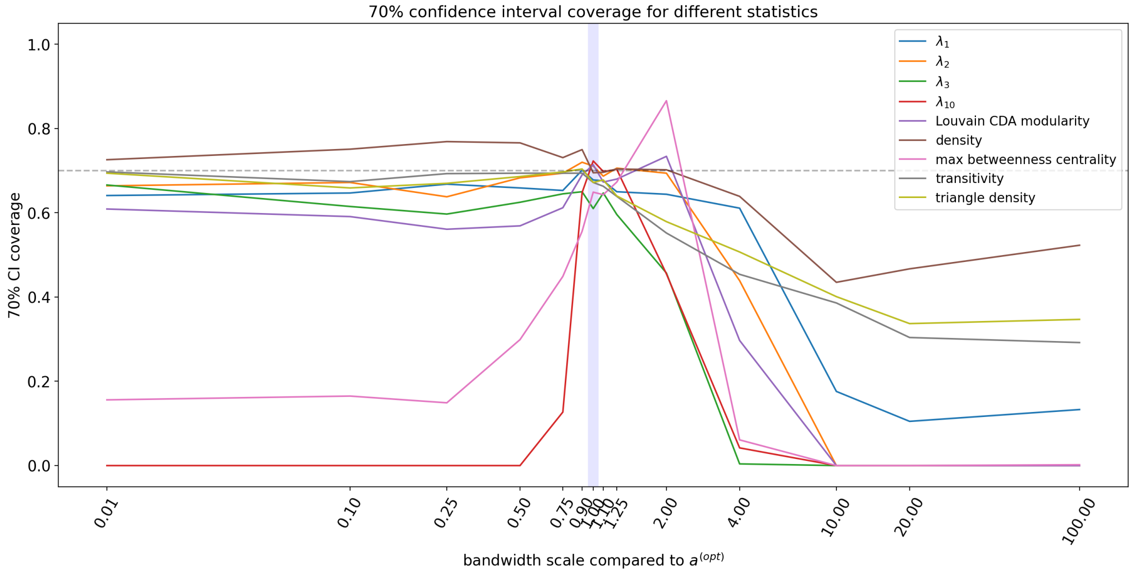
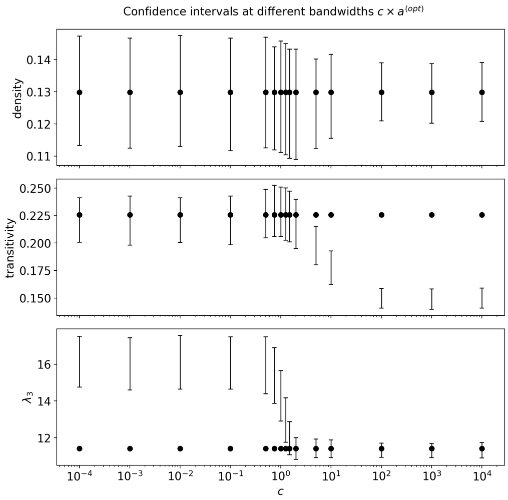


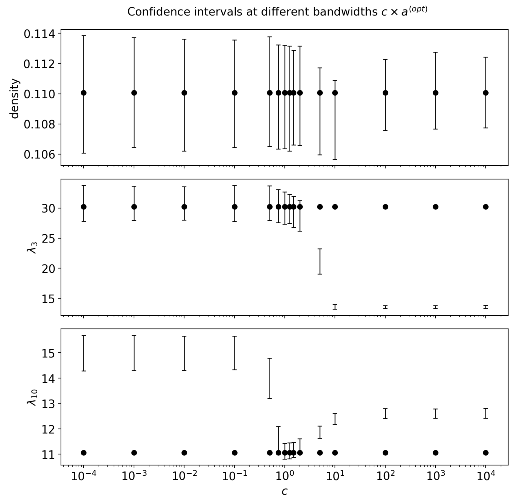
Figure 5: Confidence interval coverage for different bandwidths $c \times a^{(opt)}$: comparison of different statistics at $\alpha = 0.3$, $n = 500$ and $\rho_n = 0.1125$ based on Monte Carlo simulations using the horseshoe generating function.

statistics we have checked. However, the performance does depend on the bandwidth choice and some statistics may be estimated poorly with the default bandwidth, especially when the graph is relatively sparse and the statistic is more complicated. In those cases, experimenting with different values of c could give more reliable results. This is one advantage of our method over the empirical bootstrap (which can be seen as a limiting case of our model with $c = 0$): while at the default setting both algorithms may be bad at estimating confidence intervals for lower eigenvalues, the performance of our method can be improved by selecting a larger bandwidth (oversmoothing) while the empirical bootstrap does not depend on any parameters that could be tweaked in a similar way.

The possibility of a poor performance at $a^{(opt)}$ raises a question: how can we choose the best bandwidth in an application, when we only have one observed network and no way to run a Monte Carlo simulation confirming the coverage? Luckily, there is an easy rule-of-thumb way to verify our choice. Fig. 6 shows an example of bootstrap confidence intervals formed from $B = 1000$ bootstrap replications for different statistics of a specific single true graph A estimated using different bandwidths. We can use it in the following way: if the statistic estimate from the original graph is in the middle of the confidence interval formed by bootstrapped graphs, the choice of the bandwidth is good. In Fig. 6a we see that density is always estimated relatively well, transitivity remains well-estimated for smaller than optimal bandwidths but is underestimated when the bandwidth is too large, and λ_3 is overestimated for smaller than optimal bandwidths but remains well esti-



(a) Confidence intervals for different statistics for the product generating function at $n = 300$ and $\rho_n = 0.125$.



(b) Confidence intervals for different statistics for the horseshoe generating function at $n = 500$ and $\rho_n = 0.1125$

Figure 6: Confidence intervals for different statistics and for bandwidths $c \times a^{(opt)}$ based on $B = 1000$ bootstrap graphs.

mated for larger than optimal bandwidth. In simulations, we have noticed a pattern that when the true value is above the estimated confidence interval lowering the bandwidth tends to improve the performance, while when the value from the original graph is below the confidence interval increasing the bandwidth often solves the problem. However, this is not always true: the lowest panel in Fig. 6b shows that λ_{10} is overestimated when bandwidth is either too low or too high compared to the optimal one. The middle panel also shows that the choice of a statistic doesn't determine the behaviour: for the horseshoe function λ_3 is better estimated for lower bandwidths and underestimated for higher ones, which is a different pattern than that of λ_3 from the product generating function in the bottom of Fig. 6a.

6 Application: the Diffusion of Microfinance

For our application, we use the data from Banerjee, Chandrasekhar, Duflo, and Jackson (2013), a paper which analyses how information about microfinance spreads through social networks in 43 villages in India.

Prior to the introduction of a microfinance program they surveyed households in these villages and formed a network of connections based on 12 binary signals indicating if households knew each other (e.g. did they visit each other's homes, lend each other money, etc.). Each of these variables could be seen as a our A matrix, or we could combine them by taking a union, like in the original paper, to get an overall adjacency matrix. This is compatible with out framework: the closer two households are, the higher the probability that they will report each other as connected, hence we can view the reporting of a connection ($A_{ij} = 1$) as a signal from a Bernoulli distribution with probability of success proportional to the closeness of their friendship ($h_{0,n}(\xi_i, \xi_j)$).

Once the microfinance program entered the villages, they observed a set of first-informed villagers (injection points, chosen because they were village leaders who tend to be well-connected) and subsequent participation by households over a number of years.

One of the goals of the paper is to understand how the information about the program was spreading through the villages. This is modelled by a parametric diffusion model. Firstly, the probability p_{it} of household i with characteristics X_i participating when first informed is estimated from the logistic function:

$$\log\left(\frac{p_{it}}{1-p_{it}}\right) = X_i'\beta.$$

The parameter $\hat{\beta}$ is estimated using the information about the leaders only. The aim is to estimate the probability of transferring information about a microfinance program by people who participate in it themselves (q^P) and by those who know about it but do not participate (q^N). This is done by simulating the information spreading over time discretised into T periods (trimesters). In each period the newly informed decide whether to participate ($m_{it} = 1$) with probability p_{it}, then each informed household spreads the information to its neighbours with probability $m_{it}q^P + (1 - m_{it})q^N$.

For each village v and for each set of discretised parameter values $\left(q^P, q^N\right)$ they simulate the spread of information and adoption decisions, and then calculate moments $m_{sim,v}\left(q^P, q^N\right)$ based on the final set of participating households (e.g. the fraction of households that have no participating neighbours but participate themselves, or the covariance of households participating in the program with the share of second neighbours that are participating). The average of these simulated moments across S simulations is compared to the observed empirical moments for the given village, $m_{emp,v}$. They then choose the parameter values which minimise the average of a function of deviation of simulated moments from empirical moments across all villages:

$$(\hat{q}^P, \hat{q}^N) = \arg\min_{q^P, q^N} \left(\frac{1}{43}\sum_{v=1}^{43} m_{sim,v}\left(q^P, q^N\right) - m_{emp,v}\right)' \hat{W} \left(\frac{1}{43}\sum_{v=1}^{43} m_{sim,v}\left(q^P, q^N\right) - m_{emp,v}\right).$$

where $\hat{W} = \frac{1}{43}\sum_{v=1}^{43}\left(m_{sim,v}\left(\tilde{q}^P, \tilde{q}^N\right) - m_{emp,v}\right)\left(m_{sim,v}\left(\tilde{q}^P, \tilde{q}^N\right) - m_{emp,v}\right)'$ for a first-stage estimates $\tilde{q}^P, \tilde{q}^N$ obtained by using I as the weighting matrix. To form confidence intervals they use bootstrap which resamples whole villages. The resulting estimates are shown as the "Original" ones in Fig. 7.

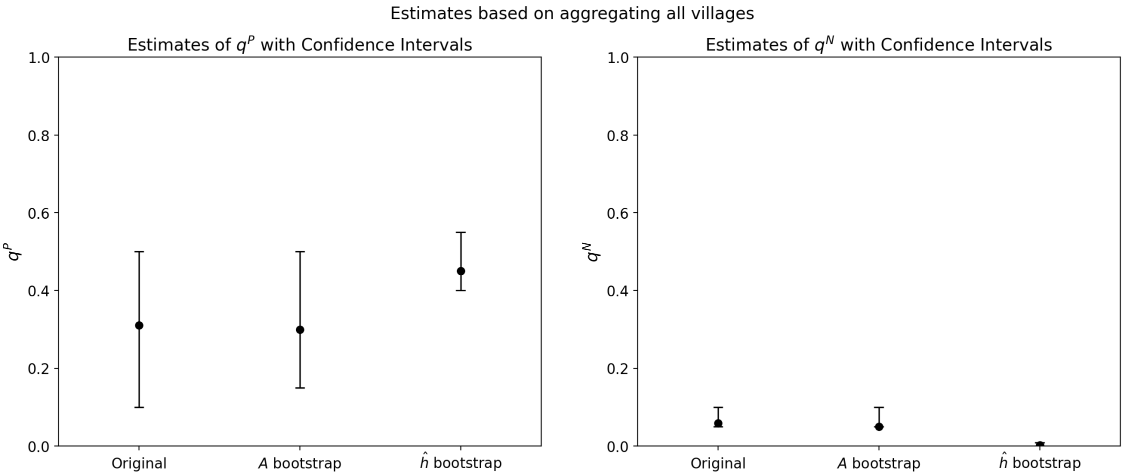


Figure 7: Estimates of q^P (left) and q^N (right) with 95% confidence intervals based on aggregating all villages: a comparison of the original result from Banerjee, Chandrasekhar, Duflo, and Jackson (2013) and our two methods.

The original paper considers a few variations of the model, including one which allows for endorsement effects. We only consider the information model without endorsement because it's less computationally demanding (the parameters are identified using a grid search and increasing the dimension of the parameter space by one leads to an exponential growth in the number of required simulations) and the original paper did not find evidence of a significant endorsement effect.

In our replication we use the same procedure for finding the parameters but we use our bootstrap to form confidence intervals. Instead of resampling whole villages we can estimate the matrix $\hat{h}_n$ for each of the villages[21] and use it to generate $B = 1000$ new sets of 43 villages with structures similar to the original ones. We can then repeat the whole estimation procedure for each new set of villages and obtain bootstrap estimates $(\hat{q}_b^{*P}, \hat{q}_b^{*N})$. The confidence intervals are formed by taking the $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$ quantiles for the distributions of $\hat{q}_b^{*P}$ and $\hat{q}_b^{*N}$. These estimates are presented in Fig. 7 as the "A bootstrap". They are very similar to the original estimates, with slightly narrower confidence intervals[22].

The replication of the original result indicates that our method performs well, though it would not be advised in this situation because it is much more computationally demanding than the original bootstrap. However, with our setup we can do more. It's reasonable to assume that the spread of information is more likely between households which have a stronger connection (higher $h_{0,n}$), i.e.

21. The villages are assumed to be independent due to relatively large geographical distances between them. If they were not independent we could treat all households as belonging to one larger network.

22. Note that the confidence intervals for q^N can't get much narrower because of the discretisation of the parameter space.

we can view the current model as an approximation to the true model in which the probability of spreading information depends not on the binary connection status A_{ij}, but on the actual strength of connection $h_{0,n}(\xi_i, \xi_j)$. Since $\hat{h}_n$ estimated as part of our procedure is a consistent estimate of $h_{0,n}$, we can repeat the simulations using a diffusion model based on $\hat{h}_n$ rather than on A (the imperfect signal about $h_{0,n}$). We assume that in any period $t \in \{1, \dots, T\}$ the informed individual i spreads the information to another individual j with probability $\hat{h}_n(\xi_i, \xi_j)\left(m_{it}q^P + (1 - m_{it})q^N\right)$. The rest of the estimation procedure remains unchanged. The resulting estimates are reported as the "$\hat{h}$ bootstrap" estimates in Fig. 7. The confidence intervals are now much narrower than in the previous two cases and the conclusions differ as well: q^P is estimated to be higher (point estimate 0.45, 95% confidence interval $[0.4, 0.55]$) while q^N is essentially zero (point estimate 0.002, 95% confidence interval $[0, 0.009]$).

This contrasts with the findings of Banerjee, Chandrasekhar, Duflo, and Jackson (2013) who highlight the importance of non-participants in the diffusion process by showing that constraining q^N to be equal to zero leads to simulated participation dropping from 20.0% to 13.97%[23]. Our model shows that if we use a diffusion model based on $\hat{h}_n$ instead of A the estimated value of q^N is not distinguishable from zero and the simulated participation drops from 18.46% at the optimal values to 18.21% when we restrict q^N to 0, a drop of one seventy-fifth instead of one third. Using the more realistic assumption that the likelihood of spreading information depends on how well the households know each other removes the need for information spreading by non-participants.

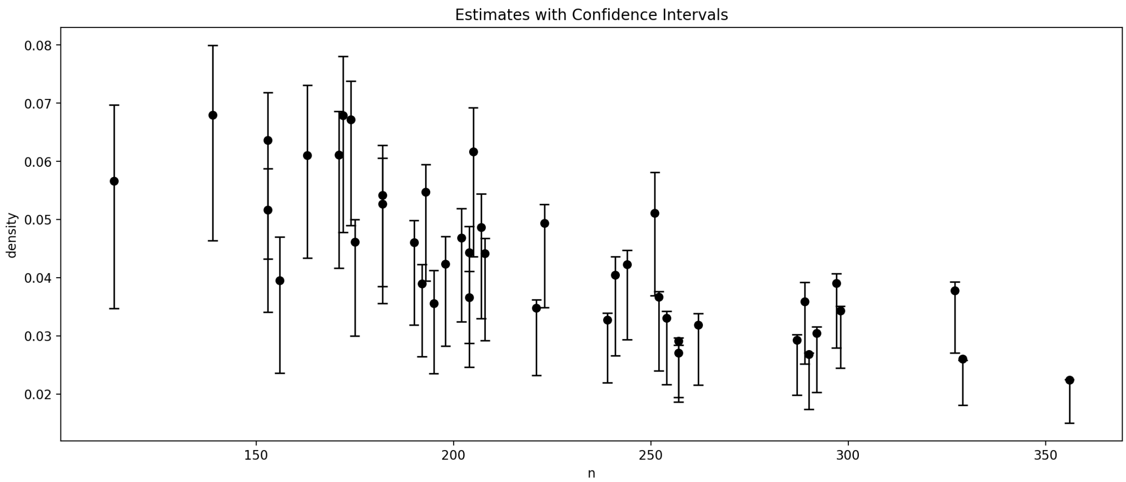


Figure 8: Densities of all 43 villages with bootstrap confidence intervals plotted against village size on the horizontal axis.

Another extension made possible by our model is performing the analysis on individual village

23. The actual observed participation rate was 19.38%. It is not used as one of the moments matched in the parameter estimation.

level. So far, we have assumed that the parameters are common across all villages, and with the original bootstrap resampling whole villages there was no way to form confidence intervals on at the village-level. With our bootstrap method, instead of minimising a (weighted) average of deviations of simulated moments from empirical moments across all villages, we can minimise them for each individual village. This allows us to:

1. **Check if all the villages come from the same network generating distribution.** We bootstrap each village separately, find confidence intervals for some network statistics (e.g. density, or the largest eigenvalue) and see if these intervals overlap for all villages.

2. **Estimate the q^P and q^N parameters (and their confidence intervals) for each village separately, see if there are systematic differences between villages.** If there are differences, see if the hypotheses ($q^N > 0$ and $q^P > q^N$) hold in each village.

We firstly look at densities of all the 43 villages. Our bootstrap method allows us to not only obtain their point estimates but also add confidence intervals to see if the villages are systematically different from each other. Fig. 8 shows that the villages become more sparse as their size increases (consistent with the assumption that $\rho_n \to 0$ as $n \to \infty$). The confidence intervals in this graph are formed using the same bootstrapped villages that were used for estimating the model parameters.

Moving on to the model parameters, we have repeated the estimation using the original diffusion model based on the adjacency matrix A (Fig. 9) and the new diffusion model based on the linking probabilities $\hat{h}_n$ (Fig. 10). We can see that the two methods produce similar though not identical results. For some villages the estimation is very imprecise, leading to very wide confidence intervals. At least half of the villages have q^N precisely estimated to be zero, even in the model based on A: this may suggest that it's the imprecisely measured villages which drive the aggregate estimate to be positive.

In the last panels we can see that one of the predictions of the model, $q^P - q^N > 0$, cannot be concluded for most of the villages as we can't reject the hypothesis that $q^P - q^N = 0$ (mostly due to imprecise measurements).

A practical extension of the current analysis would be to identify the village characteristics which help predict the estimated ranges of q^P and q^N. This would help policymakers choose villages in which the microfinance programs would have the highest chance of success or personalise the way in which the initial group of informed leaders is chosen depending on the information transmission characteristics in a given village.

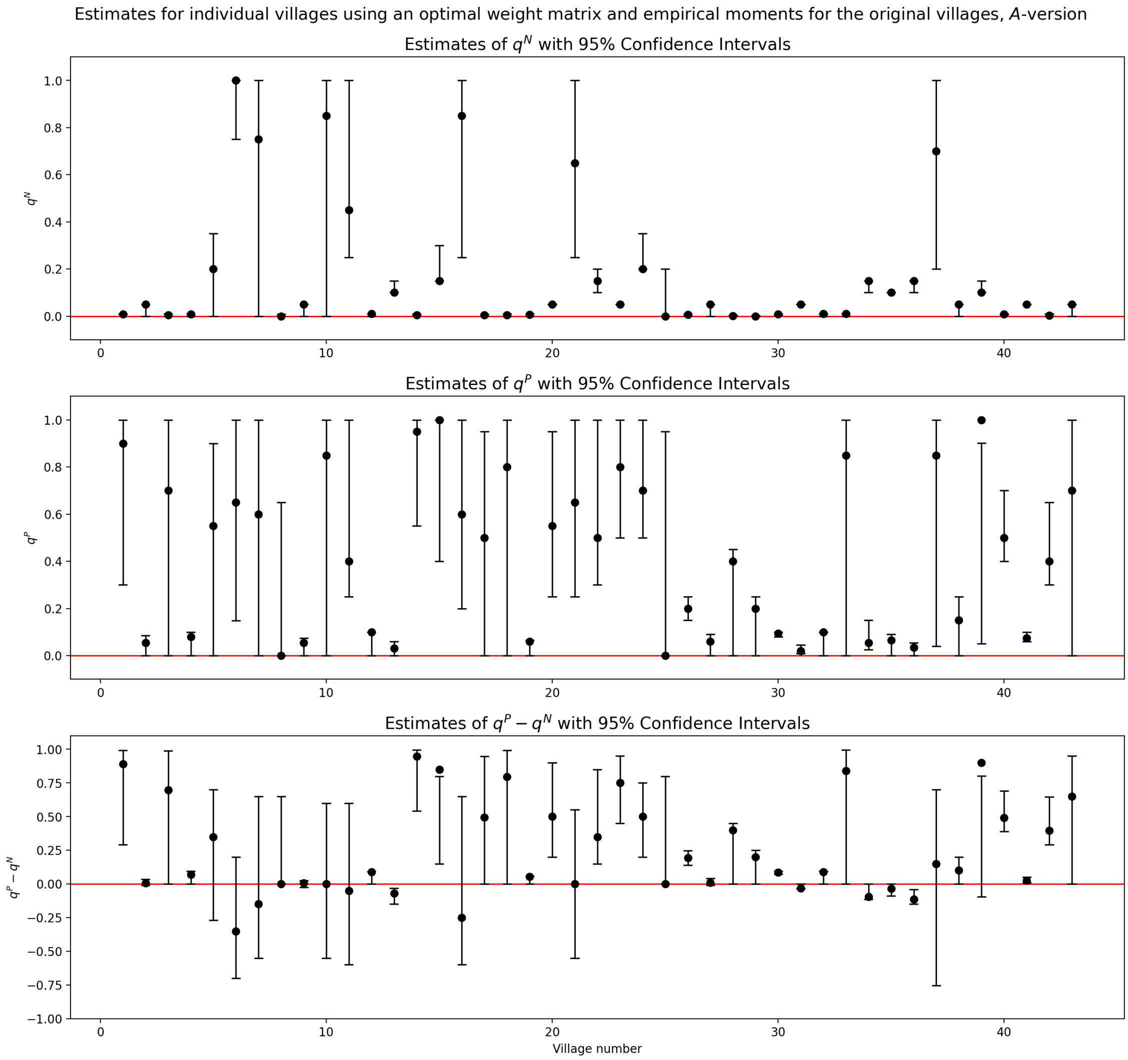


Figure 9: Estimates of q^N (top), q^P (middle) and $q^N - q^P$ (bottom) with 95% confidence intervals for individual villages using an optimal weight matrix and empirical moments for the original villages, A-version.

7 Conclusion and Extensions

In this paper we have proposed a nonparametric linking function estimator and a related network bootstrap procedure and we have provided conditions under which both achieve some notions of consistency.

In the future projects (or the future versions of this project) we aim to provide a theoretical justification for consistency of our bootstrap method over a wider class of statistics, which is suggested by the promising results in our simulations. Most importantly, we would like to extend our results to regression models in which the outcome depends, possibly in a complicated way, on the entire adjacency matrix (e.g. spillover effects from neighbours). Unfortunately, it looks like in these cases the behaviour is not well approximated by that of an average of i.i.d. random

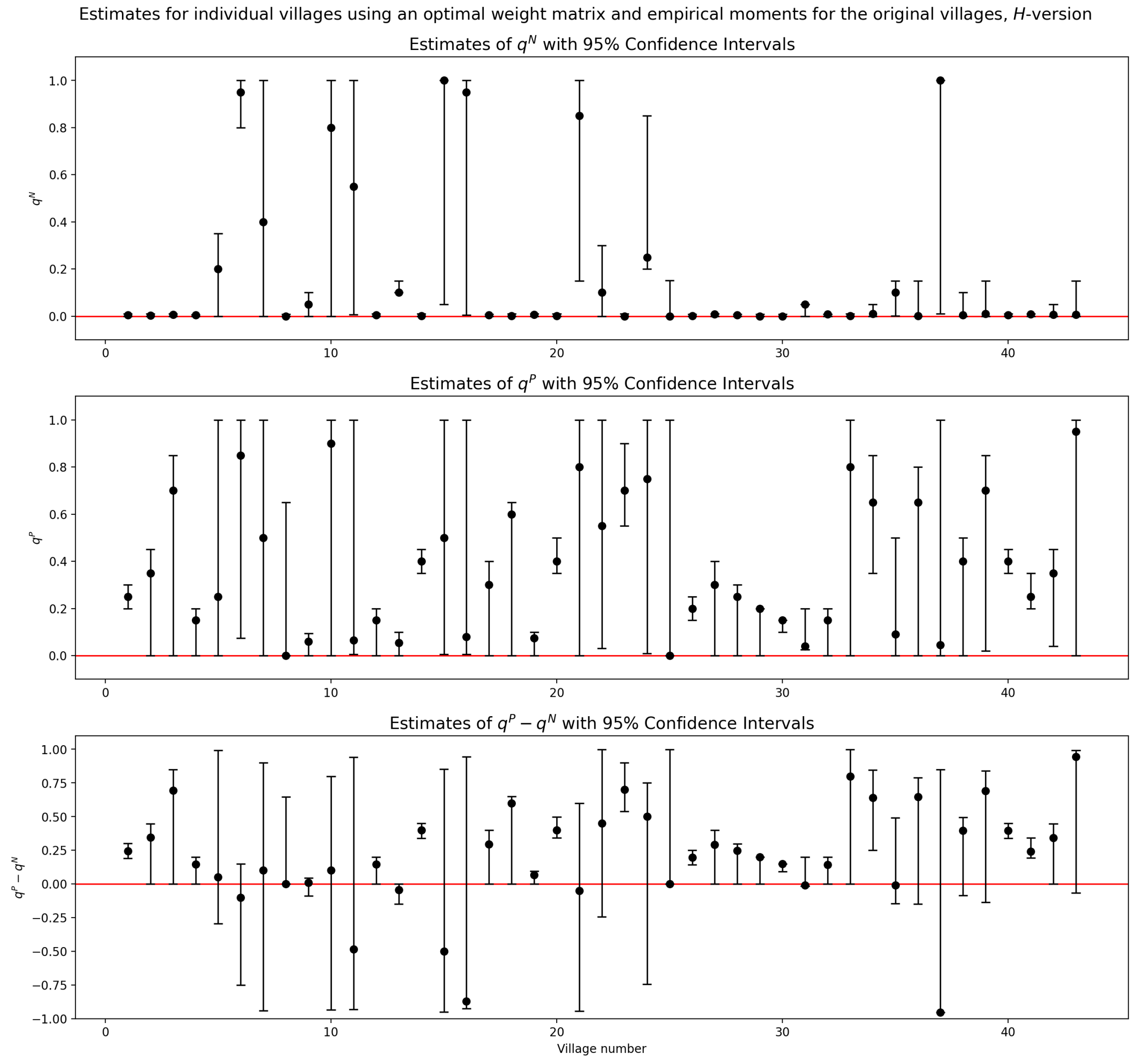


Figure 10: Estimates of q^N (top), q^P (middle) and $q^N - q^P$ (bottom) with 95% confidence intervals for individual villages using an optimal weight matrix and empirical moments for the original villages, H-version.

variables, which makes deriving asymptotic results tricky. This is both an obstacle in proving bootstrap consistency and a reason why bootstrap methods are particularly needed when it comes to strongly dependent data structures such as networks.

One way in which we may be able to get around this issue is by looking for another notion of network distance than the Wasserstein metric proposed by Levin and Levina (2019). More specifically, one which would be sufficient for proving that the convergence is preserved after a transformation.

A less closely related future project inspired by this paper could be formulating a fast numerical procedure which allows for the selection of an objective-specific optimal bandwidth in two-step procedures. As in our setup, suppose we have a two-step estimation procedure where in the first step we estimate some parameter dependent on a tuning parameter (like $\hat{h}_n$ based on a_n), which we then plug into a (possibly random) second-step estimation (e.g. density in the resulting network). We ultimately care about the result of the second step (for us, the bootstrap confidence interval coverage of the statistic estimated in the second step) and we wish to optimise its performance by choosing an optimal tuning parameter in the first step. In this paper we have relied on optimising the first-step estimation ($a^{(opt)}$ was chosen to optimise the estimation of $\hat{h}_n$), but this was shown not to provide the best results in the second step across all second-step statistics. We have considered some algorithms which do rely on the second step performance (e.g. the approximate average MSE approach), but which are too slow to be useful in practice.

Finally, we believe that forming models which use $\hat{h}_n$ as a better proxy for the strength of connections between individuals than the adjacency matrix A opens up exciting opportunities for empirical research.

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List of all notation

The notation in this file can get a bit heavy so we provide this list for reference.

- n – sample size, number of individuals in the network.
- A – an $n \times n$ adjacency matrix. Binary, symmetric, observed.
- A_{ij} – i, jth entry of the matrix A: 1 if i, j are connected (are neighbours), 0 if they are not.
- i, j, k, s, t – usually used to refer to one of the n individuals.
- ξ_i – vector of characteristics of individual i, enters the linking function.
- F_0 – distribution of ξ_i.
- $h_{0,n}$ – linking function, takes characteristics ξ_i, ξ_j as inputs and outputs the probability with which individuals i and j are linked. If the inputs are vectors $\xi(\iota) = (\xi_{\iota 1}, \xi_{\iota 2}, \ldots, \xi_{\iota m})$ of characteristics of multiple individuals it outputs the matrix of linking probabilities.
- ρ_n – density/sparsity parameter. Density in the sense that it is the expected edge density, sparsity in the sense that as $n \to \infty$ the density of edges decreases: $\rho_n \to 0$.
- w_0 – underlying linking probability before accounting for sparsity: $\rho_n w_0(\xi_i, \xi_j) = h_{0,n}(\xi_i, \xi_j)$.
- $\varphi(\xi_i, \xi_t) = E\left(\frac{A_{is}A_{ts}}{\rho_n^2} | \xi_i, \xi_t\right)$ – a function measuring the probability of a common friend between i and j normalised by the sparsity level.
- $d_{ij} = \sqrt{E\left(E\left(w_0\left(\xi_t, \xi_s\right)\left(w_0\left(\xi_i, \xi_s\right) - w_0\left(\xi_j, \xi_s\right)\right) \middle| \xi_i, \xi_j, \xi_t\right)^2 \middle| \xi_i, \xi_j\right)}$ – theoretical distance between i and j.
- $\hat{d}_{ij} = \frac{1}{\rho_n^2}\sqrt{\frac{1}{n}\sum_{t=1}^n \left(\frac{1}{n}\sum_{s=1}^n A_{ts}\left(A_{is} - A_{js}\right)\right)^2}$ – estimated distance between i and j.
- D_{ij} – shorthand notation for $\rho_n^4 \hat{d}_{ij}^2$ used in the description of the bootstrap procedure.
- $\hat{h}_n$ – estimated linking function:

$$\hat{h}_n(\xi_i, \xi_j) = \frac{\tilde{h}_n(\xi_i, \xi_j) + \tilde{h}_n(\xi_j, \xi_i)}{2} \quad \text{where} \quad \tilde{h}_n(\xi_i, \xi_j) = \frac{\sum_{\substack{t=1 \\ t \neq j}}^n K\left(\frac{\rho_n^4 \hat{d}_{it}^2}{a_n}\right) A_{tj}}{\sum_{\substack{t=1 \\ t \neq j}}^n K\left(\frac{\rho_n^4 \hat{d}_{it}^2}{a_n}\right)}$$

- K – kernel function used in estimating linking probability.
- a_n – a bandwidth parameter, chosen by the researcher.

- $\hat{F}_n$ – the empirical distribution function of ξ_i; assigns equal probability to each of the original observations.
- * – a bootstrap equivalent, e.g. $\xi_i^* \sim \hat{F}_n$ is the bootstrap version of $\xi_i \sim F_0$.
- $\hat{}$ – an estimate.
- $\max_{i,j} \equiv \max_{i,j\in\{1,2,...,n\}}$ – maximum over indices in a specific sample of size n.
- $\max_{\xi_i} \equiv \max_{\xi_i \in Supp(\xi_i)}$ – maximum over all $\xi_i \in Supp(\xi_i)$.
- $N(\xi_j, \delta) = \left\{\xi_k : \sup_{\xi_t} |w_0(\xi_t, \xi_k) - w_0(\xi_t, \xi_j)| < \delta\right\}$ – the neighbourhood of ξ_j of size δ.
- $\omega(\delta) = \inf_{\xi_j \in Supp(\xi_j)} P\left(\xi_k \in N(\xi_j, \delta) | \, \xi_j\right)$ – the infimum over all possible ξ_i of the measures of their neighbourhoods of size δ.
- $b_n = \frac{a_n}{\rho_n^4}$ – a bandwidth parameter normalised by sparsity; the effective bandwidth size after accounting for the rate at which density goes to zero.
- $\hat{h}_n^-$ – leave-one-out version of $\hat{h}_n$, evaluated in the same way as $\hat{h}_n$ but without the observations $t = i, j$. Used for numerically choosing the optimal bandwidth.
- $\ell(A, a_n)$ – log-likelihood used for numerically choosing the optimal bandwidth. Defined in Eq. (16).
- $a^{(opt)}$ – numerically chosen optimal bandwidth. Defined in Eq. (17).
- B – number of bootstrap replications.
- $f_n(A_n(h_{0,n}(\xi), \eta), \rho_n, F_0)$ – a function whose distribution we are interested in.
- η – a vector of random variables which together with the linking function determine the realised links in A. We assume $\eta_{ij} \overset{ind}{\sim} \mathcal{U}[0,1]$ for $1 \leq i \leq j \leq n$ and η independent of ξ.
- $\tilde{f}_n\left(h_{0,n}\left(\xi\right), \rho_n, F\right) \equiv E(f_n(A_n(h_{0,n}(\xi), \eta), \rho_n, F_0)|\xi)$ – a function whose distribution we are interested in after averaging out the variation due to observing A instead of $h_{0,n}$.
- $E_{h_{0,n}}$, e.g. in $E_{h_{0,n}}(f_n(A_n(h_{0,n}(\xi), \eta), \rho_n, F_0)|\xi) = \int f_n(A_n(h_{0,n}(\xi), \eta), \rho_n, F_0) d\eta$ – expectation taken with respect to the independent Bernoulli trials with probabilities determined by $h_{0,n}$.
- $E_{h_{0,n}, F_0}$ – expectation taken with respect to both the Bernoulli trials and the true distribution of ξ.

- ι – a vector of m nodes from $\{1, \ldots, n\}$.
- $A(\iota)$ – the adjacency matrix of the subgraph with nodes ι (i.e. A from which we remove $n-m$ rows and columns not in ι).
- m – usually denotes the size of a subgraph or an order of U-statistic.
- g – a kernel function (in the U-statistic sense); a function of a subset of A.
- $\tilde{g}$ – a kernel function (in the U-statistic sense); a function of a subset of $h_{0,n}$. Equal to g after averaging out the variation due to observing A instead of $h_{0,n}$.
- $\tau(g)$ – a normalisation chosen to ensure $\frac{E_{h_{0,n},F_0}(g(A_n(\iota)))}{\rho_n^{\tau(g)}} = O_p(1)$.
- $\tilde{\tilde{g}}$ – the leading term in the normalised $\tilde{g}$; a function of a subset of w_0. $O_p(1)$.
- $\tilde{J}_n$ – the distribution of $\tilde{f}_n$.
- J_n – the distribution of f_n.
- $\hat{J}_{n,B}$ – an estimate of the distribution of f_n based on B bootstrap samples.
- J – limiting distribution of J_n as $n \xrightarrow{\infty}$: $J_n(t, h_{0,n}, F_0) \Rightarrow J(t, w_0, F_0)$.
- $\Rightarrow$ – weak convergence.
- $\overset{a.s.}{\Rightarrow}$ – weak convergence almost surely, see Definition 3.
- $\overset{p}{\Rightarrow}$ – weak convergence in probability, see Definition 4.
- d_W – distance between measures which metrises weak convergence.
- $f(S) = \{f : S \to \mathbb{R} : |f(x) - f(y)| \leq d_S(x,y), \sup_{x\in S} |f(x)| \leq 1\}$ – the set of Lipschitz continuous and bounded real-valued functions on a metric space S equipped with distance d_S.
- $C_{w,F,\rho}$ – the set of non-random sequences of pairs of functions and distributions $\{(h_n, F_n)\}_{n=1}^{\infty}$ which satisfy a set of conditions on convergence of moments, see Definition A.1.
- CI_n – a bootstrap confidence interval as defined in Eq. (32).
- $\mathcal{M}_m$ – the set of all possible multisets of cardinality m with elements from $\{1, 2, \ldots, m\}$
- $\hat{\rho}_n$ – estimator of density; the density of the observed adjacency matrix A.
- $\lambda_k(A)$ or λ_k – the kth largest eigenvalue of matrix A.
- $\hat{q}_\alpha$ – the estimate of αth quantile.

- q^P – in the application, the probability of transferring information about a microfinance program by program's participants

- q^N – in the application, the probability of transferring information about a microfinance program by those who don't participate the program themselves.

- C – generic positive constant, its value may change between different expressions in which it is used.

- C_ε – a positive constant which depends on $\varepsilon > 0$. Its value may change between different expressions in which it is used.

- T_n – a remainder term used in the proof of Theorem 1.

- M_w – an upper bound on the value of w_0: $\sup_{\xi_i,\xi_j} |w_0(\xi_i,\xi_j)| \leq M_w$.

- $r_n(i) = E\left(K\left(\frac{d_{it}^2}{b_n}\right)\middle|\xi_i\right)$ – the shorthand notation for the expected kernel weights based on the distance between i and other individuals used in the estimation of $\hat{h}_n(\xi_i,\xi_j)$.

- $\hat{r}_n(i) = \frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^n K\left(\frac{\hat{d}_{it}^2}{b_n}\right)$ – the estimate of $r_n(i)$.

- $r_n = \inf_{\xi_i} r_n(i)$ – the smallest possible expected kernel weight. We need to ensure it's not too small or we would not be able to successfully estimate $h_{0,n}(\xi_i,\xi_j)$.

A Appendix: proofs

A.1 Proof of uniform consistency of the linking function estimator

Proof of Theorem 1. Throughout this argument we use C_ε to denote a positive constant which depends on $\varepsilon > 0$. The value of C_ε may change between different expressions in which it is used.

By definition,

$$\max_{i,j}\left|\frac{\hat{h}_n(\xi_i,\xi_j) - h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right| = \max_{i,j}\left|\frac{\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^n K\left(\frac{\hat{d}_{it}^2}{b_n}\right)\left(\frac{A_{tj}-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right)}{\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^n K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}\right|$$

$$= \max_{i,j}\left|\frac{\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^n K\left(\frac{\hat{d}_{it}^2}{b_n}\right)\left(\frac{A_{tj}-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right)}{E\left(K\left(\frac{d_{it}^2}{b_n}\right)\middle|\xi_i\right) + \left(\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^n K\left(\frac{\hat{d}_{it}^2}{b_n}\right) - E\left(K\left(\frac{d_{it}^2}{b_n}\right)\middle|\xi_i\right)\right)}\right|$$

$$\leq \left(\max_{i,j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^n \frac{K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}{r_n(i)}\left(\frac{A_{tj}-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right)\right|\right)\left(1+\max_{i,j}\left|1-\frac{E\left(K\left(\frac{d_{it}^2}{b_n}\right)\middle|\xi_i\right)}{\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^n K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}\right|\right) \quad (34)$$

The inequality follows from $\max\left|\frac{a}{b+c}\right| \leq \max\left|\frac{a}{b}\right| + \max\left|\frac{ac}{b(b+c)}\right| \leq \left(\max\left|\frac{a}{b}\right|\right)\left(1+\max\left|\frac{c}{b+c}\right|\right)$, where $b = E\left(K\left(\frac{d_{it}^2}{b_n}\right)\Big|\,\xi_i\right)$. In Lemma A.1 we show that the second factor converges almost surely to one.

We now focus on the first factor. Since $K(\cdot)$ is Lipschitz continuous with a Lipschitz constant C (by Assumption 1.3):

$$K\left(\frac{\hat{d}_{it}^2}{b_n}\right) \leq K\left(\frac{d_{it}^2}{b_n}\right) + \left|K\left(\frac{\hat{d}_{it}^2}{b_n}\right) - K\left(\frac{d_{it}^2}{b_n}\right)\right| \leq K\left(\frac{d_{it}^2}{b_n}\right) + C\left|\frac{\hat{d}_{it}^2 - d_{it}^2}{b_n}\right|.$$

It follows that

$$\begin{aligned}
&\max_{i,j}\left|\frac{1}{(n-1)r_n(i)}\sum_{\substack{t=1\\t\neq j}}^{n} K\left(\frac{\hat{d}_{it}^2}{b_n}\right)\left(\frac{A_{tj}-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right)\right| \\
&\leq \max_{i,j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\left(\frac{A_{tj}-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right) + C\left|\frac{\hat{d}_{it}^2 - d_{it}^2}{b_n r_n(i)}\right|\left(\frac{A_{tj}-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right)\right| \\
&\leq \max_{i,j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\left(\frac{A_{tj}-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right)\right| \\
&\quad + C\max_{i,j,t,t\neq j}\left|\frac{\hat{d}_{it}^2 - d_{it}^2}{b_n r_n(i)}\right| \underbrace{\max_{i,j}\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\left|\frac{A_{tj}-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right|}_{\leq 2M_w + \underbrace{T_n}_{\xrightarrow{a.s.} 0}}.
\end{aligned}$$

$M_w < \infty$ is defined in Lemma A.1. In Lemma A.2 we define T_n and show that the last factor in the last expression is almost surely bounded, hence

$$\begin{aligned}
&\max_{i,j}\left|\frac{1}{(n-1)\rho_n r_n(i)}\sum_{\substack{t=1\\t\neq j}}^{n}K\left(\frac{\hat{d}_{it}^2}{b_n}\right)(A_{tj}-h_{0,n}(\xi_i,\xi_j))\right|\\
&\quad\leq \max_{i,j}\left|\frac{1}{(n-1)\rho_n r_n(i)}\sum_{\substack{t=1\\t\neq j}}^{n}K\left(\frac{d_{it}^2}{b_n}\right)(A_{tj}-h_{0,n}(\xi_t,\xi_j)+h_{0,n}(\xi_t,\xi_j)-h_{0,n}(\xi_i,\xi_j))\right|\\
&\qquad + C(2M_w+T_n)\frac{1}{b_n r_n}\left(\max_{i,j}\left|\hat{d}_{ij}^2-d_{ij}^2\right|\right)\\
&\quad\leq \max_{i,j}\left|\frac{\sum_{\substack{t=1\\t\neq j}}^{n}K\left(\frac{d_{it}^2}{b_n}\right)(A_{tj}-h_{0,n}(\xi_t,\xi_j))}{(n-1)\rho_n r_n(i)}\right| + \max_{i,j}\left|\frac{\sum_{\substack{t=1\\t\neq j}}^{n}K\left(\frac{d_{it}^2}{b_n}\right)(h_{0,n}(\xi_t,\xi_j)-h_{0,n}(\xi_i,\xi_j))}{(n-1)\rho_n r_n(i)}\right|\\
&\qquad + C(2M_w+T_n)\frac{1}{b_n r_n}\left(\max_{i,j}\left|\hat{d}_{ij}^2-d_{ij}^2\right|\right).
\end{aligned}$$

where $T_n \xrightarrow{a.s.} 0$. We complete the proof by showing that the three terms go to zero in probability in Lemma A.3, Lemma A.4 and Lemma A.5. □

Lemma A.1. *Under the assumptions of Theorem 1, for any $\varepsilon > 0$:*

$$\begin{aligned}
&\sum_{n=3}^{\infty}P\left(\left|\max_{i,j}\left|1-\frac{E\left(K\left(\frac{d_{it}^2}{b_n}\right)\Big|\,\xi_i\right)}{\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}\right|\right|>\varepsilon\right)\\
&=O\left(\sum_{n=3}^{\infty}n^2\exp\left(-nr_nC_\varepsilon\right)+\sum_{n=3}^{\infty}n^4\exp\left(-nb_n^2r_n^2\rho_n^2C_\varepsilon\right)\right)=O(1).
\end{aligned}$$

hence

$$\max_{i,j}\left|1-\frac{E\left(K\left(\frac{d_{it}^2}{b_n}\right)\Big|\,\xi_i\right)}{\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}\right|\xrightarrow{a.s.}0.$$

Proof. Take any $\varepsilon > 0$. We start by using a union bound:

$$P\left(\max_{i,j}\left|1-\frac{E\left(K\left(\frac{d_{it}^2}{b_n}\right)\Big|\,\xi_i\right)}{\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}\right|>\varepsilon\right)\leq n^2P\left(\left|1-\frac{E\left(K\left(\frac{d_{it}^2}{b_n}\right)|\xi_i,\xi_j\right)}{\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}\right|>\varepsilon\right)$$

Let $r_n(i) = E\left(K\left(\frac{d_{it}^2}{b_n}\right)\Big|\xi_i\right) \geq r_n \geq 0$ and $\hat{r}_n(i) = \frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n} K\left(\frac{\hat{d}_{it}^2}{b_n}\right)$. We have:

$$\begin{aligned}
P\left(\left|1-\frac{r_n(i)}{\hat{r}_n(i)}\right| > \varepsilon\right) &\leq P\left(|\hat{r}_n(i) - r_n(i)| > \varepsilon|\hat{r}_n(i)|\right) \\
&\leq P\left(|\hat{r}_n(i) - r_n(i)| > \varepsilon|\hat{r}_n(i)| \text{ and } |\hat{r}_n(i)| \geq \frac{r_n(i)}{2}\right) \\
&\quad + P\left(|\hat{r}_n(i) - r_n(i)| > \varepsilon|\hat{r}_n(i)| \text{ and } |\hat{r}_n(i)| < \frac{r_n(i)}{2}\right) \\
&\leq P\left(|\hat{r}_n(i) - r_n(i)| > \varepsilon\frac{r_n(i)}{2}\right) + P\left(|\hat{r}_n(i)| < \frac{r_n(i)}{2}\right) \\
&\leq P\left(\left|\frac{\hat{r}_n(i)}{r_n(i)} - 1\right| > \frac{\varepsilon}{2}\right) + P\left(\left|\frac{\hat{r}_n(i)}{r_n(i)} - 1\right| > \frac{1}{2}\right)
\end{aligned}$$

where the last line follows from:

$$\begin{aligned}
P\left(|\hat{r}_n(i)| < \frac{r_n(i)}{2}\right) &\leq P\left(\hat{r}_n(i) < \frac{r_n(i)}{2}\right) = P\left(\hat{r}_n(i) - r_n(i) < -\frac{r_n(i)}{2}\right) \\
&= P\left(r_n(i) - \hat{r}_n(i) > \frac{r_n(i)}{2}\right) \leq P\left(|r_n(i) - \hat{r}_n(i)| > \frac{r_n(i)}{2}\right) \\
&\leq P\left(\left|\frac{\hat{r}_n(i)}{r_n(i)} - 1\right| > \frac{1}{2}\right).
\end{aligned}$$

We use the above derivation and the law of iterated expectations to get am upper bound of the form:

$$\begin{aligned}
P\left(\max_{i,j}\left|1 - \frac{E\left(K\left(\frac{d_{it}^2}{b_n}\right)\Big|\xi_i\right)}{\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n} K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}\right| > \varepsilon\right) &\leq n^2 E\left(P\left(\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}{r_n(i)} - 1\right| > \frac{\varepsilon}{2}\Bigg|\xi_i, \xi_j\right)\right) \\
&\quad + n^2 E\left(P\left(\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}{r_n(i)} - 1\right| > \frac{1}{2}\Bigg|\xi_i, \xi_j\right)\right).
\end{aligned}$$

The last two terms are identical, up to the value of ε. We use $K(\cdot)$ Lipschitz continuous and separate out the terms with $t = i, j$, so that the remaining average is of i.i.d terms that only depend on t.

$$
\begin{aligned}
& n^2 E\left(P\left(\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}{r_n(i)}-1\right|>\frac{\varepsilon}{2}\middle|\xi_i,\xi_j\right)\right)\\
& \leq n^2 E\left(P\left(\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}-1+\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}C\left|\frac{\hat{d}_{it}^2-d_{it}^2}{r_n(i)b_n}\right|\right|>\frac{\varepsilon}{2}\middle|\xi_i,\xi_j\right)\right)\\
& \leq n^2 E\left(P\left(\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}-1\right|>\frac{\varepsilon}{4}\middle|\xi_i,\xi_j\right)\right)\\
& \qquad + n^2 E\left(P\left(\left|\max_{i,t}\left(\frac{\hat{d}_{it}^2-d_{it}^2}{r_n(i)b_n}\right)\right|>\frac{\varepsilon}{4C}\middle|\xi_i,\xi_j\right)\right)\\
& \leq n^2 E\left(P\left(\left|\frac{1}{n-2}\sum_{\substack{t=1\\t\neq i,j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}-1\right|>\frac{\varepsilon}{4}-\frac{2C}{(n-2)r_n}\middle|\xi_i,\xi_j\right)\right)+O\left(n^4\exp\left(-nb_n^2r_n^2\rho_n^2C_\varepsilon\right)\right)
\end{aligned}
$$

where the last rate follows from Lemma A.5. For the first term we apply Bernstein's inequality: conditional on ξ_i, ξ_j, $\frac{1}{r_n(i)}K\left(\frac{d_{it}^2}{b_n}\right)-1$ are i.i.d., mean zero, bounded by $\frac{2C}{r_n}$, with variance $O\left(\frac{1}{r_n}\right)$:

$$
\begin{aligned}
Var\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}-1\middle|\xi_i\right) &\leq \frac{E\left(\left(K\left(\frac{d_{it}^2}{b_n}\right)\right)^2\middle|\xi_i\right)}{r_n(i)^2}\leq\frac{CE\left(K\left(\frac{d_{it}^2}{b_n}\right)\middle|\xi_i\right)}{\left(E\left(K\left(\frac{d_{it}^2}{b_n}\right)\middle|\xi_i\right)\right)^2}\\
&=\frac{C}{E\left(K\left(\frac{d_{it}^2}{b_n}\right)\middle|\xi_i\right)}\leq\frac{C}{r_n}=O\left(\frac{1}{r_n}\right),
\end{aligned}
$$

hence for any $\varepsilon > 0$:

$$
\begin{aligned}
& n^2 P\left(\left|\frac{1}{n-2}\sum_{\substack{t=1\\t\neq i,j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}-1\right|>\frac{\varepsilon}{4}-\frac{2C}{(n-2)r_n}\middle|\xi_i,\xi_j\right)\\
& \leq 2n^2\exp\left(-\frac{(n-2)\left(\frac{\varepsilon}{4}-\frac{2C}{(n-2)r_n}\right)^2}{2\left(O\left(\frac{1}{r_n}\right)+\frac{1}{3}\frac{C}{r_n}\left(\frac{\varepsilon}{4}-\frac{2C}{(n-2)r_n}\right)\right)}\right)\\
& \leq n^2\exp\left(-nr_nC_\varepsilon\right).
\end{aligned}
$$

where $C_\varepsilon > 0$ is some constant dependent on ε. Note that the final value does not depend on the choice of ξ_i, ξ_j, hence it does not change when we take expectation over ξ_i, ξ_j.

It remains to show that the following expression:

$$\sum_{n=3}^{\infty} P\left(\max_{i,j}\left|1-\frac{E\left(K\left(\frac{d_{it}^2}{b_n}\right)\Big|\,\xi_i\right)}{\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n} K\left(\frac{\hat{d}_{it}^2}{b_n}\right)}\right|>\varepsilon\right)$$
$$\leq O\left(\sum_{n=3}^{\infty} n^2\exp\left(-nr_nC_\varepsilon\right)+\sum_{n=3}^{\infty} n^4\exp\left(-nb_n^2r_n^2\rho_n^2C_\varepsilon\right)\right)$$

is bounded. The last sum is bounded by arguments shown in Lemma A.5. For the first sum we have:

$$\sum_{n=3}^{\infty} n^4 e^{-nr_nC_\varepsilon}=\sum_{n=3}^{\infty} n^2 e^{-nr_nC_\varepsilon\log(n)\frac{1}{\log(n)}}=\sum_{n=3}^{\infty} n^2\left(e^{\log(n)}\right)^{-C_\varepsilon\frac{nr_n}{\log(n)}}=\sum_{n=3}^{\infty} n^{2-C_\varepsilon\frac{nr_n}{\log(n)}}.$$

It remains to show $\frac{nr_n}{\log(n)}\to\infty$. We start by showing $r_n\geq Cb_n^{\frac{1}{2\alpha}}$:

$$r_n=\inf_{\xi_i} r_n(i)=\inf_{\xi_i} E\left(K\left(\frac{d_{it}^2}{b_n}\right)\Big|\,\xi_i\right)\geq C_1\inf_{\xi_i} P\left(\frac{d_{it}^2}{b_n}\leq C_2\Big|\,\xi_i\right)$$
$$\geq C_1\inf_{\xi_i} P\left(E_s\left(\left(w_0\left(\xi_i,\xi_s\right)-w_0\left(\xi_t,\xi_s\right)\right)^2\Big|\,\xi_i,\xi_t\right)\leq\frac{C_2}{M_w^2}b_n\Big|\,\xi_i\right)$$
$$\geq C_1\inf_{\xi_i} P\left(\xi_t\in N\left(\xi_i,\sqrt{\frac{C_2}{M_w^2}b_n}\right)\Big|\,\xi_i\right)=C_1\omega\left(\sqrt{\frac{C_2}{M_w^2}b_n}\right)\geq Cb_n^{\frac{1}{2\alpha}}.$$

In the first inequality we use the part of Assumption 1.3 which says the kernel is separated from 0 for input values sufficiently close to 0. The second inequality comes from:

$$d_{ij}^2=E_t\left(\left(E_s\left(w_0\left(\xi_t,\xi_s\right)\left(w_0\left(\xi_i,\xi_s\right)-w_0\left(\xi_j,\xi_s\right)\right)\right|\xi_i,\xi_j,\xi_t\right)\right)^2\Big|\,\xi_i,\xi_j\right)$$
$$\leq E_t\left(E_s\left(w^2\left(\xi_t,\xi_s\right)\left(w_0\left(\xi_i,\xi_s\right)-w_0\left(\xi_j,\xi_s\right)\right)^2\Big|\,\xi_i,\xi_j,\xi_t\right)\Big|\,\xi_i,\xi_j\right)$$
$$\leq M_w^2E_s\left(\left(w_0\left(\xi_i,\xi_s\right)-w_0\left(\xi_j,\xi_s\right)\right)^2\Big|\,\xi_i,\xi_j\right)\leq M_w^4<\infty$$

where the first inequality is due to Jensen's inequality and the second follows from the fact that for any $\xi_i,\xi_j\in Supp(\xi)$ $w_0(\xi_i,\xi_j)$ is bounded; we denote the bound by $M_w<\infty$. To see this, recall that $\rho_n w_0(u,v)=h_{0,n}(u,v)\in[0,1]$, hence we have $w_0(u,v)\in\left[0,\frac{1}{\rho_n}\right]$ for all $n\in\mathbb{N}$. Then also $w_0(u,v)\in\bigcap_{n=1}^{\infty}\left[0,\frac{1}{\rho_n}\right]\subset\left[0,\frac{1}{\sup_n\rho_n}\right]$. $\sup_n\rho_n$ exists since ρ_n, which can be interpreted as the marginal probability of an edge, is bounded above by 1. Let $M_w=\frac{1}{\sup_n\rho_n}$ denote the upper bound on the size of w_0, i.e. for any $\xi_i,\xi_j\in Supp(\xi)$ we have $|w_0(\xi_i,\xi_j)|\leq M_w$.

The third inequality follows from the fact that if for some ξ_t we have $\sup_{\xi_s}|w_0(\xi_t,\xi_s)-w_0(\xi_i,\xi_s)|<\delta$, then $E_s\left(\left(w_0\left(\xi_i,\xi_s\right)-w_0\left(\xi_t,\xi_s\right)\right)^2\Big|\,\xi_i,\xi_t\right)<\delta^2$, i.e. $\xi_t\in N\left(\xi_i,\delta\right)$. For the final steps we use Assumption 1.2. The required divergence follows from Assumption 1.4:

$$\frac{nr_n}{\log(n)} \geq C \frac{nb_n^{\frac{1}{2\alpha}}}{\log(n)} \to \infty.$$

□

Lemma A.2. *Under the assumptions of Theorem 1, there exists a sequence of random variables T_n such that*

$$\sum_{n=3}^{\infty} P(|T_n| > \varepsilon) = O\left(\sum_{n=3}^{\infty} n^2 \exp(-n\rho_n C_\varepsilon)\right) = O(1) \quad \textit{hence} \quad T_n \xrightarrow{a.s.} 0$$

and

$$\max_{i,j} \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \left| \frac{A_{tj} - h_{0,n}(\xi_i, \xi_j)}{\rho_n} \right| \leq 2M_w + T_n.$$

Proof. We start by looking at a representative term inside the summation.

$$\begin{aligned}
\left| \frac{A_{tj} - h_{0,n}(\xi_i, \xi_j)}{\rho_n} \right| &\leq \left| \frac{A_{tj}}{\rho_n} \right| + \left| \frac{h_{0,n}(\xi_i, \xi_j)}{\rho_n} \right| \\
&= \frac{A_{tj}}{\rho_n} + w_0(\xi_i, \xi_j) \\
&= \frac{A_{tj}}{\rho_n} - w_0(\xi_t, \xi_j) + w_0(\xi_t, \xi_j) + w_0(\xi_i, \xi_j) \\
&\leq \frac{A_{tj}}{\rho_n} - w_0(\xi_t, \xi_j) + 2M_w.
\end{aligned}$$

We use triangle inequality, the fact that A_{tj} and $h_{0,n}(\xi_i, \xi_j)$ are non-negative and the definition of $w_0(\xi_i, \xi_j)$. We add and subtract $w_0(\xi_t, \xi_j)$ and use the fact that all possible values of w_0 are bounded by M_w.

Going back to the sum:

$$\max_{i,j} \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \left| \frac{A_{tj} - h_{0,n}(\xi_i, \xi_j)}{\rho_n} \right| \leq 2M_w + \max_{i,j} \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \frac{A_{tj}}{\rho_n} - w_0(\xi_t, \xi_j)$$

$$\leq 2M_w + \max_{i,j,i \neq j} \left| \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \frac{A_{tj}}{\rho_n} - w_0(\xi_t, \xi_j) \right| + \max_{i} \left| \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq i}}^{n} \frac{A_{ti}}{\rho_n} - w_0(\xi_t, \xi_i) \right| = 2M_w + T_n.$$

In the second step we split into cases with $i \neq j$ and $i = j$. We apply union bound and Bernstein's theorem to the averages. For the first one, we separate out the term with $t = i$ (we later condition on ξ_i, ξ_j, we want the remaining terms in the sum to be i.i.d. after conditioning). $\frac{A_{tj}}{\rho_n} - w_0(\xi_t, \xi_j)$ for $t \neq i, j$ are, conditional on ξ_i, ξ_j, independent, zero mean:

$$
\begin{aligned}
&E\left(\frac{A_{tj}}{\rho_n} - w_0(\xi_t,\xi_j)\bigg|\, \xi_i,\xi_j\right) = E\left(E\left(\frac{A_{tj}}{\rho_n} - w_0(\xi_t,\xi_j)\bigg|\, \xi_i,\xi_j,\xi_t\right)\bigg|\, \xi_i,\xi_j\right) \\
&= E\left(E\left(\frac{A_{tj}}{\rho_n}\bigg|\, \xi_i,\xi_j,\xi_t\right) - w_0(\xi_t,\xi_j)\bigg|\, \xi_i,\xi_j\right) = E\left(w_0(\xi_t,\xi_j) - w_0(\xi_t,\xi_j)|\, \xi_i,\xi_j\right) = 0
\end{aligned}
$$

and bounded by $\frac{1}{\rho_n}$: since A and $h_{0,n}$ take values in $[0,1]$, we have $\left|\frac{A_{tj}}{\rho_n} - w_0(\xi_t,\xi_j)\right| = \left|\frac{A_{tj} - h_{0,n}(\xi_t,\xi_j)}{\rho_n}\right| \leq \frac{1}{\rho_n}$. The second moments are $O\left(\frac{1}{\rho_n}\right)$:

$$
\begin{aligned}
&Var\left(\frac{1}{\rho_n}\left(A_{tj} - h_{0,n}(\xi_t,\xi_j)\right)\bigg|\, \xi_i,\xi_j\right) = E\left(\left(\frac{1}{\rho_n}\left(A_{tj} - h_{0,n}(\xi_t,\xi_j)\right)\right)^2\bigg|\, \xi_i,\xi_j\right) \\
&= E\left(\frac{1}{\rho_n^2}\left(h_{0,n}(\xi_t,\xi_j)\left(1 - h_{0,n}(\xi_t,\xi_j)\right)\right)\bigg|\, \xi_i,\xi_j\right) = E\left(w_0(\xi_t,\xi_j)\left(\frac{1}{\rho_n} - w_0(\xi_t,\xi_j)\right)\bigg|\, \xi_j\right) \\
&= O\left(\frac{1}{\rho_n}\right) + O(1) = O\left(\frac{1}{\rho_n}\right).
\end{aligned}
$$

For any $\varepsilon > 0$:

$$
\begin{aligned}
&P\left(\max_{i,j,i\neq j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{A_{tj}}{\rho_n} - w_0(\xi_t,\xi_j)\right| > \varepsilon\right) \\
&\leq P\left(\max_{i,j,i\neq j}\left|\frac{1}{n-2}\sum_{\substack{t=1\\t\neq i,j}}^{n}\frac{A_{tj}}{\rho_n} - w_0(\xi_t,\xi_j)\right| > \varepsilon - \frac{1}{(n-2)\rho_n}\right) \\
&\leq n(n-1)E\left(P\left(\left|\frac{1}{n-2}\sum_{\substack{t=1\\t\neq i,j}}^{n}\frac{A_{tj}}{\rho_n} - w_0(\xi_t,\xi_j)\right| > \varepsilon - \frac{1}{(n-2)\rho_n}\bigg|\, \xi_i,\xi_j\right)\right) \\
&\leq 2n(n-1)\exp\left(-\frac{(n-2)\left(\varepsilon - \frac{1}{(n-2)\rho_n}\right)^2}{2\left(O\left(\frac{1}{\rho_n}\right) + \frac{1}{3}\frac{1}{\rho_n}\left(\varepsilon - \frac{1}{(n-2)\rho_n}\right)\right)}\right) \\
&\leq n^2\exp\left(-n\rho_n C_\varepsilon\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&P\left(\max_i \left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq i}}^{n}\frac{A_{ti}}{\rho_n} - w_0(\xi_t,\xi_i)\right| > \varepsilon\right)\\
&\leq nE\left(P\left(\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq i}}^{n}\frac{A_{ti}}{\rho_n} - w_0(\xi_t,\xi_i)\right| > \varepsilon \,\middle|\, \xi_i\right)\right)\\
&\leq 2n\exp\left(-\frac{(n-1)\varepsilon^2}{2\left(O\left(\frac{1}{\rho_n}\right)+\frac{1}{3}\frac{1}{\rho_n}\varepsilon\right)}\right)\\
&\leq n\exp\left(-n\rho_n C_\varepsilon\right).
\end{aligned}
$$

This is dominated by the previous term. Combining the above results, for any $\varepsilon > 0$:

$$
\begin{aligned}
\sum_{n=3}^{\infty} P\left(\max_{i,j}\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\left|\frac{A_{tj}-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right| > 2M_w+\varepsilon\right) &\leq \sum_{n=3}^{\infty} P\left(|T_n| > \varepsilon\right)\\
&\leq O\left(\sum_{n=3}^{\infty} n^2\exp\left(-n\rho_n C_\varepsilon\right)\right) < \infty.
\end{aligned}
$$

For the last claim, note that under Assumption 1.1 we have $\frac{\log(n)}{\rho_n n} \to 0$. Then:

$$
\sum_{n=3}^{\infty} n^2 e^{-n\rho_n C_\varepsilon} = \sum_{n=3}^{\infty} n^2 e^{-n\rho_n C_\varepsilon \log(n)\frac{1}{log(n)}} = \sum_{n=3}^{\infty} n^2\left(e^{\log(n)}\right)^{-C_\varepsilon\frac{\rho_n n}{\log(n)}} = \sum_{n=3}^{\infty} n^{2-C_\varepsilon\frac{\rho_n n}{\log(n)}} < \infty
$$

for any $C_\varepsilon > 0$, since $2 - C_\varepsilon\frac{\rho_n n}{\log(n)} \to -\infty$. Hence $T_n \xrightarrow{a.s} 0$ and the term of interest is almost surely bounded above by $2M_w$.

□

Lemma A.3. *Under the assumptions of Theorem 1, for any $\varepsilon > 0$:*

$$
\sum_{n=3}^{\infty} P\left(\max_{i,j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\left(\frac{A_{tj}-h_{0,n}(\xi_t,\xi_j)}{\rho_n}\right)\right| > \varepsilon\right) \leq O\left(\sum_{n=3}^{\infty} n^2\exp\left(-nr_n\rho_n C_\varepsilon\right)\right) < \infty
$$

hence

$$
\max_{i,j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\left(\frac{A_{tj}-h_{0,n}(\xi_t,\xi_j)}{\rho_n}\right)\right| \xrightarrow{a.s.} 0.
$$

Proof. We start by separating the cases when $i \neq j$ and $i = j$:

$$\max_{i,j} \left| \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \left(\frac{A_{tj} - h_{0,n}(\xi_t, \xi_j)}{\rho_n} \right) \right| \leq \max_{i,j,i \neq j} \left| \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \left(\frac{A_{tj} - h_{0,n}(\xi_t, \xi_j)}{\rho_n} \right) \right|$$
$$+ \max_{i} \left| \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq i}}^{n} \frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \left(\frac{A_{ti} - h_{0,n}(\xi_t, \xi_i)}{\rho_n} \right) \right|.$$

We split the first sum into the term with $t = i$ and the rest the rest (which is i.i.d. over t), use the triangle inequality and the fact that A_{ij} and $h_{0,n}(\xi_i, \xi_j)$ take values in $[0, 1]$ for any choice of i, j while the kernel function is absolutely bounded.

$$\max_{i,j,i \neq j} \left| \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \left(\frac{A_{tj} - h_{0,n}(\xi_t, \xi_j)}{\rho_n} \right) \right|$$
$$\leq \max_{i,j,i \neq j} \left| \frac{n-2}{n-1} \frac{1}{n-2} \sum_{\substack{t=1 \\ t \neq i,j}}^{n} \frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \frac{A_{tj} - h_{0,n}(\xi_t, \xi_j)}{\rho_n} \right|$$
$$+ \max_{i,j,i \neq j} \frac{1}{n-1} \left(\underbrace{\left| \frac{K\left(\frac{d_{ii}^2}{b_n}\right)}{r_n(i)} \right|}_{\leq \frac{C}{r_n}} \underbrace{\left| \frac{A_{ij} - h_{0,n}(\xi_i, \xi_j)}{\rho_n} \right|}_{\leq \frac{1}{\rho_n}} \right)$$
$$\leq \max_{i,j,i \neq j} \frac{n-2}{n-1} \left| \frac{1}{n-2} \sum_{\substack{t=1 \\ t \neq i,j}}^{n} \frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \frac{A_{tj} - h_{0,n}(\xi_t, \xi_j)}{\rho_n} \right| + \frac{C}{(n-1) r_n \rho_n}.$$

The expression inside the sum in the first term is bounded by $\frac{C}{r_n \rho_n}$, hence after conditioning on ξ_i, ξ_j we can apply the Bernstein's inequality for bounded i.i.d. random variables. The conditional expectation of that term is zero:

$$E\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \frac{A_{tj} - h_{0,n}(\xi_t, \xi_j)}{\rho_n} \middle| \xi_i, \xi_j \right) = E\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \left(\frac{1}{\rho_n} E\left(A_{tj} \middle| \xi_i, \xi_j, \xi_t \right) - w_0(\xi_t, \xi_j) \right) \middle| \xi_i, \xi_j \right)$$
$$= E\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \left(w_0(\xi_t, \xi_j) - w_0(\xi_t, \xi_j) \right) \middle| \xi_i, \xi_j \right) = 0$$

where the first equality is due to the law of iterated expectations, the second uses the fact that d_{it}^2, $r_n(i)$ and $h_{0,n}(\xi_t, \xi_j)$ are not random after conditioning on ξ_i, ξ_j, ξ_t. A_{tj} is independent of ξ_i, hence $E\left(A_{tj} \middle| \xi_i, \xi_j, \xi_t \right) = E\left(A_{tj} \middle| \xi_j, \xi_t \right)$ which by definition equals $h_{0,n}(\xi_t, \xi_j) = \rho_n w_0(\xi_t, \xi_j)$. The conditional variance is $O\left(\frac{1}{r_n\rho_n}\right)$:

$$\begin{aligned}
&Var\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\frac{A_{tj}-h_{0,n}(\xi_t,\xi_j)}{\rho_n}\middle|\xi_i,\xi_j\right)\\
&=E\left(\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\right)^2E\left(\left(\frac{A_{tj}-h_{0,n}(\xi_t,\xi_j)}{\rho_n}\right)^2\middle|\xi_i,\xi_j,\xi_t\right)\middle|\xi_i,\xi_j\right)\\
&=E\left(\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\right)^2\left(w_0(\xi_t,\xi_j)\left(\frac{1}{\rho_n}-w_0(\xi_t,\xi_j)\right)\right)\middle|\xi_i,\xi_j\right)\\
&\leq\frac{M_w}{\rho_n}\underbrace{\frac{E\left(\left(K\left(\frac{d_{it}^2}{b_n}\right)\right)^2\middle|\xi_i\right)}{r_n(i)^2}}_{=O\left(\frac{1}{r_n}\right)}=O\left(\frac{1}{r_n\rho_n}\right)
\end{aligned}$$

where in the last line we use that the kernel function is bounded ($K(\cdot)\leq C$ by Assumption 1.3) and hence

$$\frac{E\left(\left(K\left(\frac{d_{it}^2}{b_n}\right)\right)^2\middle|\xi_i\right)}{r_n(i)^2}\leq\frac{CE\left(K\left(\frac{d_{it}^2}{b_n}\right)\middle|\xi_i\right)}{\left(E\left(K\left(\frac{d_{it}^2}{b_n}\right)\middle|\xi_i\right)\right)^2}=\frac{C}{E\left(K\left(\frac{d_{it}^2}{b_n}\right)\middle|\xi_i\right)}\leq\frac{C}{r_n}=O\left(\frac{1}{r_n}\right).$$

By union bound and Bernstein's inequality, for any $\varepsilon>0$ and $n\geq 3$:

$$\begin{aligned}
&P\left(\max_{i,j,i\neq j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\left(\frac{A_{tj}-h_{0,n}(\xi_t,\xi_j)}{\rho_n}\right)\right|>\varepsilon\right)\\
&\leq n(n-1)E\left(P\left(\left|\frac{1}{n-2}\sum_{\substack{t=1\\t\neq i,j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\frac{A_{tj}-h_{0,n}(\xi_t,\xi_j)}{\rho_n}\right|>\varepsilon-\frac{C}{(n-2)\rho_nr_n}\middle|\xi_i,\xi_j\right)\right)\\
&\leq 2n(n-1)\exp\left(\frac{-(n-2)\left(\varepsilon-\frac{C}{(n-2)\rho_nr_n}\right)^2}{2\left(O\left(\frac{1}{r_n\rho_n}\right)+\frac{C}{3r_n\rho_n}\left(\varepsilon-\frac{C}{(n-2)\rho_nr_n}\right)\right)}\right)\\
&\leq n^2\exp\left(-nr_n\rho_nC_\varepsilon\right)
\end{aligned}$$

for some $C_\varepsilon>0$. We can proceed in a very similar way for the case of $i=j$ to get:

$$P\left(\max_{i}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq i}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\frac{A_{ti}-h_{0,n}(\xi_t,\xi_i)}{\rho_n}\right|>\varepsilon\right)\leq O\left(n\exp\left(-nr_n\rho_nC_\varepsilon\right)\right).$$

Combining all the terms gives the required result: for any $\varepsilon>0$

$$\sum_{n=3}^{\infty} P\left(\max_{i,j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\left(\frac{A_{tj}-h_{0,n}(\xi_t,\xi_j)}{\rho_n}\right)\right|>\varepsilon\right)\leq O\left(\sum_{n=3}^{\infty} n^2\exp\left(-nr_n\rho_nC_\varepsilon\right)\right)<\infty$$

under Assumption 1.1 and Assumption 1.4 which, by derivation similar to that at the end of the proof of A.1, give:

$$\frac{n\rho_n r_n}{\log(n)}\geq C\frac{n\rho_n b_n^{\frac{1}{2\alpha}}}{\log(n)}\to\infty.$$

Hence

$$\max_{i,j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\left(\frac{A_{tj}-h_{0,n}(\xi_t,\xi_j)}{\rho_n}\right)\right|\xrightarrow{a.s.}0.$$

□

Lemma A.4. *Under the assumptions of Theorem 1, for any $\varepsilon>0$:*

$$P\left(\left|\max_{i,j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\left(\frac{h_{0,n}(\xi_t,\xi_j)-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right)\right|\right|>\varepsilon\right)$$
$$\leq O\left(n^2\exp\left(-nr_nC_\varepsilon\right)\right)+O\left(b_n^{\frac{\alpha^2}{(2\alpha+1)^2}}\right)\to 0.$$

hence

$$\max_{i,j}\left|\frac{1}{n-1}\sum_{\substack{t=1\\t\neq j}}^{n}\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)}\left(\frac{h_{0,n}(\xi_t,\xi_j)-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right)\right|\xrightarrow{p}0.$$

Proof. Intuitively, this result holds because as n increases $\frac{d_{it}^2}{b_n}$ becomes large, and hence $K\left(\frac{d_{it}^2}{b_n}\right)$ becomes zero, unless ξ_i and ξ_t are very close to each other in the sense that their $h_{0,n}(\xi_t,\xi_j)$ and $h_{0,n}(\xi_i,\xi_j)$ are similar for all ξ_j.

We start by showing that whenever $h_{0,n}(\xi_t,\xi_j)$ and $h_{0,n}(\xi_i,\xi_j)$ are not close, their distance d_{it} will be separated away from zero.

We follow the ideas from Auerbach (2022)'s proof of Lemma 1 which shows that for any i,t,n and any $\varepsilon>0$ we can find a $\delta>0$ such that $\sqrt{E\left(\left(w_0(\xi_i,\xi_j)-w_0(\xi_t,\xi_j)\right)^2\middle|\xi_i,\xi_t\right)}\geq\varepsilon\implies d_{it}=$

$\sqrt{E\left((\varphi(\xi_i,\xi_j) - \varphi(\xi_t,\xi_j))^2 \middle| \xi_i,\xi_t \right)} \geq \delta.$

Our idea is to add an extra step at the beginning: if for given i,j,t there is a $\nu > 0$ for which we have $|w_0(\xi_t,\xi_j) - w_0(\xi_i,\xi_j)| \equiv \left| \frac{h_{0,n}(\xi_t,\xi_j) - h_{0,n}(\xi_i,\xi_j)}{\rho_n} \right| > \nu$, then there exists an $\varepsilon > 0$ such that $\sqrt{E\left((w_0(\xi_i,\xi_j) - w_0(\xi_t,\xi_j))^2 \middle| \xi_i,\xi_t \right)} \geq \varepsilon$ (which in turn implies $d_{it} \geq \delta$). In other words, $|w_0(\xi_t,\xi_j) - w_0(\xi_i,\xi_j)|$ can be large only if d_{it} is large, in which case the weight placed on that term is small.

In our case, the issue is that in $\sqrt{E\left((w_0(\xi_i,\xi_j) - w_0(\xi_t,\xi_j))^2 \middle| \xi_i,\xi_t \right)}$ we take an expectation with respect to j, but the initial statement is given for a fixed j. To get around it, we replace the fixed j with a random element of a neighbourhood of j, then take an expectation with respect to an element of that neighbourhood, and use an upper bound which takes expectation over all possible values, not just those in the neighbourhood of j.

Recall from Assumption 1.2 that $N(\xi_j,\delta) = \left\{\xi_k : \sup_{\xi_t} |w_0(\xi_t,\xi_k) - w_0(\xi_t,\xi_j)| < \delta\right\}$ denotes the neighbourhood of ξ_j of size δ. We fix i,j,t,k where $k \in N\left(\xi_j, \frac{\nu}{3}\right)$. Then

$$
\begin{aligned}
&\mathbb{1}\left(|w_0(\xi_t,\xi_j) - w_0(\xi_i,\xi_j)| > \nu\right) \\
&= \mathbb{1}\left(|w_0(\xi_t,\xi_j) - w_0(\xi_t,\xi_k) + w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k) + w_0(\xi_i,\xi_k) - w_0(\xi_i,\xi_j)| > \nu\right) \\
&\leq \underbrace{\mathbb{1}\left(|w_0(\xi_t,\xi_j) - w_0(\xi_t,\xi_k)| > \frac{\nu}{3}\right)}_{=0} + \mathbb{1}\left(|w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k)| > \frac{\nu}{3}\right) \\
&\quad + \underbrace{\mathbb{1}\left(|w_0(\xi_i,\xi_k) - w_0(\xi_i,\xi_j)| > \frac{\nu}{3}\right)}_{=0} \\
&= \mathbb{1}\left((w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k))^2 > \frac{\nu^2}{9}\right).
\end{aligned}
$$

If the above holds for any fixed i,j,t and for any $\xi_k \in N\left(\xi_j, \frac{\nu}{3}\right)$, it also holds if we take expectation over $\xi_k \in N\left(\xi_j, \frac{\nu}{3}\right)$. Recall from Assumption 1.2 that $\omega(\delta) = \inf_{\xi_j} P\left(\xi_k \in N(\xi_j,\delta) \middle| \xi_j\right)$ and $\omega(\delta) \geq$

$\left(\frac{\delta}{C}\right)^{\frac{1}{\alpha}}$ for all $\delta > 0$. We use $E(X|A) = \frac{E(\mathbb{1}_A X)}{P(A)}$.

$$\begin{aligned}
&\mathbb{1}\left(|w_0(\xi_t,\xi_j) - w_0(\xi_i,\xi_j)| > \nu\right) \\
&\leq \mathbb{1}\left(E\left((w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k))^2 \middle| \xi_k \in N\left(\xi_j, \frac{\nu}{3}\right), \xi_i, \xi_j, \xi_t\right) > \frac{\nu^2}{9}\right) \\
&= \mathbb{1}\left(\frac{E\left((w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k))^2 \mathbb{1}\left(\xi_k \in N\left(\xi_j, \frac{\nu}{3}\right)\right) \middle| \xi_i, \xi_j, \xi_t\right)}{P\left(\xi_k \in N\left(\xi_j, \frac{\nu}{3}\right) \middle| \xi_j\right)} > \frac{\nu^2}{9}\right) \\
&\leq \mathbb{1}\left(E\left((w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k))^2 \mathbb{1}\left(\xi_k \in N\left(\xi_j, \frac{\nu}{3}\right)\right) \middle| \xi_i, \xi_j, \xi_t\right) > \frac{\nu^2}{9}\omega\left(\frac{\nu}{3}\right)\right) \\
&\leq \mathbb{1}\left(E\left((w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k))^2 \middle| \xi_i, \xi_t\right) > \frac{\nu^2}{9}\omega\left(\frac{\nu}{3}\right)\right) \\
&= \mathbb{1}\left(\sqrt{E\left((w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k))^2 \middle| \xi_i, \xi_t\right)} > \frac{\nu}{3}\sqrt{\omega\left(\frac{\nu}{3}\right)}\right)
\end{aligned}$$

We set $\varepsilon = \frac{\nu}{3}\sqrt{\omega\left(\frac{\nu}{3}\right)}$, which completes the argument.

Like Auerbach (2022)[24], we assume there exist some $\alpha, C > 0$ such that for any δ we have $\omega(\delta) \geq \left(\frac{\delta}{C}\right)^{\frac{1}{\alpha}}$ (this is our Assumption 1.2). Then:

$$\begin{aligned}
\mathbb{1}\left(|w_0(\xi_t,\xi_j) - w_0(\xi_i,\xi_j)| > \nu\right) &\leq \mathbb{1}\left(\sqrt{E\left((w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k))^2 \middle| \xi_i, \xi_t\right)} > \frac{\nu}{3}\sqrt{\left(\frac{\nu}{3C}\right)^{\frac{1}{\alpha}}}\right) \\
&= \mathbb{1}\left(3C^{\frac{1}{2\alpha+1}}\left(E\left((w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k))^2 \middle| \xi_i, \xi_t\right)\right)^{\frac{\alpha}{2\alpha+1}} > \nu\right).
\end{aligned}$$

hence

$$|w_0(\xi_t,\xi_j) - w_0(\xi_i,\xi_j)| \leq 3C^{\frac{1}{2\alpha+1}}\left(E\left((w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k))^2 \middle| \xi_i, \xi_t\right)\right)^{\frac{\alpha}{2\alpha+1}}.$$

Combining with Auerbach (2022)[25] result:

$$\sqrt{E\left((w_0(\xi_t,\xi_k) - w_0(\xi_i,\xi_k))^2 \middle| \xi_i, \xi_t\right)} \leq 2C^{\frac{1}{4\alpha+2}} d_{it}^{\frac{\alpha}{2\alpha+1}}.$$

we get

$$|w_0(\xi_t,\xi_j) - w_0(\xi_i,\xi_j)| \leq \tilde{C} d_{it}^{\frac{2\alpha^2}{(2\alpha+1)^2}}$$

24. One major difference is that Auerbach (2022) does not allow for sparsity in his model, in his case $\rho_n = 1$. Hence we impose an assumption analogous to his to w_0, not $h_{0,n}$. If we were to define everything in terms of $h_{0,n}$, we would need $N_n(\xi_j, \delta) = \left\{\xi_k : \sup_{\xi_t} |h_{0,n}(\xi_t,\xi_k) - h_{0,n}(\xi_t,\xi_j)| < \delta\right\}$, $\omega_n(\delta) = \inf_{\xi_j} P\left(\xi_k \in N_n(\xi_j,\delta) | \xi_j\right)$ and $\omega_n(\delta) \geq \left(\frac{\delta}{\rho_n C}\right)^{\frac{1}{\alpha}}$.

25. This is Auerbach (2022) Lemma A1 restated in our notation. To account for the fact that Auerbach (2022) does not allow for sparsity we replace their f, which is equivalent to our $h_{0,n}$, with a w_0 and their δ with our equivalent term d.

for $\tilde{C} = 3 \times 2^{\frac{2\alpha}{2\alpha+1}} \times C^{\frac{3\alpha+1}{(2\alpha+1)^2}}$.

We can now return to the term of interest.

$$\max_{i,j} \left| \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \left(w_0(\xi_t, \xi_j) - w_0(\xi_i, \xi_j)\right) \right| \leq \max_{i,j} \left| \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \tilde{C} d_{it}^{\frac{2\alpha^2}{(2\alpha+1)^2}} \right|$$

$$\leq \tilde{C} \left(\max_{i,j} \left| \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} d_{it}^{\frac{2\alpha^2}{(2\alpha+1)^2}} - E\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} d_{it}^{\frac{2\alpha^2}{(2\alpha+1)^2}} \middle| \xi_i \right) \right| + \right.$$

$$\left. + \max_i \left| E\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} d_{it}^{\frac{2\alpha^2}{(2\alpha+1)^2}} \middle| \xi_i \right) \right| \right).$$

The first term goes to zero by the union bound and Bernstein's inequality, where, conditionally on ξ_i, ξ_j and after separating out the term with $t = i$, the terms inside the average are i.i.d., mean zero, bounded by $r_n^{-1} C M_w^{\frac{4\alpha^2}{(2\alpha+1)^2}} = O\left(r_n^{-1}\right)$ and have variance $O\left(r_n^{-1}\right)$. For any $\varepsilon > 0$:

$$P\left(\max_{i,j} \left| \frac{1}{n-1} \sum_{\substack{t=1 \\ t \neq j}}^{n} \frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} d_{it}^{\frac{2\alpha^2}{(2\alpha+1)^2}} - E\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} d_{it}^{\frac{2\alpha^2}{(2\alpha+1)^2}} \middle| \xi_i \right) \right| > \varepsilon \right)$$

$$\leq 2n(n-1) \exp\left(- \frac{(n-2)\left(\varepsilon - \frac{C M_w^{\frac{4\alpha^2}{(2\alpha+1)^2}}}{(n-2) r_n}\right)^2}{2\left(O\left(r_n^{-1}\right) + \frac{C M_w^{\frac{4\alpha^2}{(2\alpha+1)^2}}}{3 r_n} \left(\varepsilon - \frac{C M_w^{\frac{4\alpha^2}{(2\alpha+1)^2}}}{(n-2) r_n}\right)\right)} \right)$$

$$+ 2n \exp\left(- \frac{(n-1)\varepsilon^2}{2\left(O\left(r_n^{-1}\right) + \frac{C M_w^{\frac{4\alpha^2}{(2\alpha+1)^2}} \varepsilon}{3 r_n} \right)} \right)$$

$$\leq n^2 \exp\left(-n r_n C_\varepsilon\right) \to 0.$$

The last convergence was shown at the end of the proof of Lemma A.1.

It remains to show that the last term goes to zero too. By Assumption 1.3, there exists a $D \in \mathbb{R}$ such that $\forall |u| > D : K(u) = 0$. If $d_{it} \neq 0$: $\frac{d_{it}^2}{b_n} = O\left(\frac{1}{b_n}\right) \to \infty$, so eventually, as $n \to \infty$, $\frac{d_{it}^2}{b_n} > D$

and $K\left(\frac{d_{it}^2}{b_n}\right) = 0$ (and if $d_{it} = 0$ the whole term is identically equal to zero). We have:

$$\begin{aligned}
\max_i \left| E\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} d_{it}^{\frac{2\alpha^2}{(2\alpha+1)^2}} \middle| \xi_i \right) \right| &= \max_i \left| E\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} d_{it}^{\frac{2\alpha^2}{(2\alpha+1)^2}} \mathbb{1}\left(\frac{d_{it}^2}{b_n} \leq D\right) \middle| \xi_i \right) \right| \\
&\leq \max_i \left| (Db_n)^{\frac{\alpha^2}{(2\alpha+1)^2}} E\left(\frac{K\left(\frac{d_{it}^2}{b_n}\right)}{r_n(i)} \middle| \xi_i \right) \right| \\
&= \max_i \left| (Db_n)^{\frac{\alpha^2}{(2\alpha+1)^2}} \underbrace{\frac{E\left(K\left(\frac{d_{it}^2}{b_n}\right) \middle| \xi_i\right)}{r_n(i)}}_{=1} \right| \\
&\leq D^{\frac{\alpha^2}{(2\alpha+1)^2}} b_n^{\frac{\alpha^2}{(2\alpha+1)^2}} = O\left(b_n^{\frac{\alpha^2}{(2\alpha+1)^2}}\right) \to 0.
\end{aligned}$$

The last expression goes to zero by Assumption 1.4. Note however that under Assumption 1.4 the rate of convergence to zero is too slow to ensure almost sure convergence of this term. This is the reason why we only get uniform convergence in probability in Theorem 1 and convergence weakly in probability in Theorem 3.

□

Lemma A.5. *Under the assumptions of Theorem 1, for any* $\varepsilon > 0$*:*

$$\sum_{n=3}^{\infty} P\left(\frac{1}{b_n r_n} \max_{i,j} \left|\hat{d}_{ij}^2 - d_{ij}^2\right| > \varepsilon\right) = O\left(\sum_{n=3}^{\infty} n^2 \exp\left(-n b_n^2 r_n^2 \rho_n^2 C_\varepsilon\right)\right) = O(1)$$

hence

$$\frac{1}{b_n r_n} \max_{i,j} \left|\hat{d}_{ij}^2 - d_{ij}^2\right| \xrightarrow{a.s.} 0.$$

Proof. We follow the same steps as in Lemma B1 in Auerbach (2022). By definition:

$$\begin{aligned}
\hat{d}_{ij} &= \sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}}{\rho_n}\left(\frac{A_{is}-A_{js}}{\rho_n}\right)\right)^2} \\
\tilde{d}_{ij} &= \sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(\varphi(\xi_i,\xi_t)-\varphi(\xi_j,\xi_t)\right)^2} \\
d_{ij} &= \sqrt{E_t\left(\left(\varphi(\xi_i,\xi_t)-\varphi(\xi_j,\xi_t)\right)^2 \middle| \xi_i,\xi_j\right)}.
\end{aligned}$$

Take any $\varepsilon > 0$. We have:

$$\begin{aligned}
P\left(\frac{1}{b_n r_n}\max_{i,j}\left|\hat{d}_{ij}^2 - d_{ij}^2\right| > \varepsilon\right) &= P\left(\max_{i,j}\left|\hat{d}_{ij}^2 - d_{ij}^2\right| > \varepsilon b_n r_n\right) \\
&= P\left(\max_{i,j}\left|\hat{d}_{ij}^2 - \tilde{d}_{ij}^2 + \tilde{d}_{ij}^2 - d_{ij}^2\right| > \varepsilon b_n r_n\right) \\
&\leq P\left(\max_{i,j}\left|\hat{d}_{ij}^2 - \tilde{d}_{ij}^2\right| > \frac{\varepsilon b_n r_n}{2}\right) + P\left(\max_{i,j}\left|\tilde{d}_{ij}^2 - d_{ij}^2\right| > \frac{\varepsilon b_n r_n}{2}\right)
\end{aligned} \tag{35}$$

where the last inequality follows from the fact that $|a+b| > \varepsilon$ implies $|a| > \frac{\varepsilon}{2}$ or $|b| > \frac{\varepsilon}{2}$ and hence $P(|a+b| > \varepsilon) \leq P(|a| > \frac{\varepsilon}{2}) + P(|b| > \frac{\varepsilon}{2})$.

For the first term in (35), we plug in the definitions, then use $a^2 - b^2 = (a-b)(a+b)$ and the fact that the second bracket approaches a limit bounded by $4M_w^2$:

$$\begin{aligned}
&P\left(\max_{i,j}\left|\hat{d}_{ij}^2 - \tilde{d}_{ij}^2\right| > \frac{\varepsilon b_n r_n}{2}\right) \\
&= P\left(\max_{i,j}\left|\frac{1}{n}\sum_{t=1}^{n}\left(\left(\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}}{\rho_n}\left(\frac{A_{is}}{\rho_n} - \frac{A_{js}}{\rho_n}\right)\right)^2 - \left(\varphi(\xi_i,\xi_t) - \varphi(\xi_j,\xi_t)\right)^2\right)\right| > \frac{\varepsilon b_n r_n}{2}\right) \\
&= P\left(\max_{i,j}\left|\frac{1}{n}\sum_{t=1}^{n}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{is}}{\rho_n^2} - \varphi(\xi_i,\xi_t) + \varphi(\xi_j,\xi_t) - \frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{js}}{\rho_n^2}\right)\times\right.\right. \\
&\qquad\left.\left.\times\left(\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{is}}{\rho_n^2} + \varphi(\xi_i,\xi_t) - \varphi(\xi_j,\xi_t) - \frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{js}}{\rho_n^2}\right)\right| > \frac{\varepsilon b_n r_n}{2}\right) \\
&\leq P\left(\max_{i,j}\left|\frac{1}{n}\sum_{t=1}^{n}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{is}}{\rho_n^2} - \varphi(\xi_i,\xi_t) + \varphi(\xi_j,\xi_t) - \frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{js}}{\rho_n^2}\right)\right| > \frac{\varepsilon b_n r_n}{16M_w^2}\right) + \\
&\qquad + P\left(\max_{i,j,t}\left|\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{is}}{\rho_n^2} + \varphi(\xi_i,\xi_t) - \varphi(\xi_j,\xi_t) - \frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{js}}{\rho_n^2}\right| > 8M_w^2\right)
\end{aligned}$$

where the last equality follows from the fact that $ab > \varepsilon$ implies $b \geq M$ or $a > \frac{\varepsilon}{M}$. For the first term, we again note that $|a+b| > \varepsilon$ implies $|a| > \frac{\varepsilon}{2}$ or $|b| > \frac{\varepsilon}{2}$, we split the expression into a part with terms that only depend on i and a part with terms that only depend on j. We then get:

$$
\begin{aligned}
&P\left(\max_{i,j}\left|\frac{1}{n}\sum_{t=1}^{n}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{is}}{\rho_n^2}-\varphi(\xi_i,\xi_t)+\varphi(\xi_j,\xi_t)-\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{js}}{\rho_n^2}\right)\right|>\frac{\varepsilon b_n r_n}{16M_w^2}\right)\\
&\leq 2P\left(\max_{i}\left|\frac{1}{n}\sum_{t=1}^{n}\left(\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{is}}{\rho_n^2}-\varphi(\xi_i,\xi_t)\right)\right|>\frac{\varepsilon b_n r_n}{32M_w^2}\right)\\
&\leq 2nE\left(P\left(\max_{t,t\neq i}\left|\frac{1}{n-1}\sum_{s=1}^{n}\frac{A_{ts}A_{is}}{\rho_n^2}-\varphi(\xi_i,\xi_t)\right|>\frac{\varepsilon b_n r_n}{32M_w^2}-\frac{1}{(n-1)\rho_n^2}\right|\xi_i\right)\right)\\
&\leq 2n(n-1)E\left(P\left(\left|\frac{1}{n-2}\sum_{\substack{s=1\\ s\neq i,t}}^{n}\frac{A_{ts}A_{is}}{\rho_n^2}-\varphi(\xi_i,\xi_t)\right|>\frac{\varepsilon b_n r_n}{32M_w^2}-\frac{1}{(n-1)\rho_n^2}-\frac{2}{(n-2)\rho_n^2}\right|\xi_i,\xi_t\right)\right)\\
&\leq 4n(n-1)\exp\left(\frac{-(n-2)\left(\frac{\varepsilon b_n r_n}{32M_w^2}-\frac{1}{(n-1)\rho_n^2}-\frac{2}{(n-2)\rho_n^2}\right)^2}{2\left(\frac{C}{\rho_n^2}+\frac{1}{3}\frac{1}{\rho_n^2}\left(\frac{\varepsilon b_n r_n}{32M_w^2}-\frac{1}{(n-1)\rho_n^2}-\frac{2}{(n-2)\rho_n^2}\right)\right)}\right)\\
&\leq n^2\exp\left(-nb_n^2r_n^2\rho_n^2C_\varepsilon\right)
\end{aligned}
$$

where the second inequality follows from the union bound applied to $\max_i$ and the fact that $\frac{1}{n}\sum_{t=1}^{n}x_t \leq \frac{n-1}{n}\frac{1}{n-1}\sum_{\substack{t=1\\ t\neq i}}^{n}\max_{t,t\neq i}x_t+\frac{1}{n}x_i = \frac{n-1}{n}\left(\max_{t,t\neq i}x_t+\frac{1}{n-1}x_i\right)$. In this case $x_i = \frac{1}{n}\sum_{s=1}^{n}\frac{A_{is}^2}{\rho_n^2}-\varphi(\xi_i,\xi_i)$ and $|x_i|\leq\frac{1}{\rho_n^2}$ (since A and $\rho_n^2\varphi$ both belong to $[0,1]$). We also use the fact that $\frac{n-1}{n}|a|>\varepsilon$ implies $|a|>\varepsilon$. Next, notice that $\left|a\pm\frac{1}{(n-1)\rho_n^2}\right|>\varepsilon$ implies that either $|a|\geq a>\varepsilon\pm\frac{1}{(n-1)\rho_n^2}>\varepsilon-\frac{1}{(n-1)\rho_n^2}$ or $|a|\geq -a>\varepsilon\pm\frac{1}{(n-1)\rho_n^2}>\varepsilon-\frac{1}{(n-1)\rho_n^2}$, so in either case we get $|a|>\varepsilon-\frac{1}{(n-1)\rho_n^2}$. For the third inequality, we again apply the union bound, this time over $t\neq i$, and separate out the terms with $s=i$ or $s=t$, similarly to the previous step. The final inequality follows from Bernstein's inequality with $\left|\frac{A_{ts}A_{is}}{\rho_n^2}-\varphi(\xi_i,\xi_t)\right|\leq\frac{1}{\rho_n^2}$ and $Var\left(\frac{A_{ts}A_{is}}{\rho_n^2}-\varphi(\xi_i,\xi_t)\right)=E\left(\varphi(\xi_i,\xi_t)\left(\frac{1}{\rho_n^2}-\varphi(\xi_i,\xi_t)\right)\right)\leq\frac{M_w^2}{\rho_n^2}$.

For the second term, take any $0<\varepsilon<2M_w^2$ and use similar arguments.

$$
\begin{aligned}
&P\left(\max_{i,j,t}\left|\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{is}}{\rho_n^2}+\varphi(\xi_i,\xi_t)-\varphi(\xi_j,\xi_t)-\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{js}}{\rho_n^2}\right|>8M_w^2\right)\\
&\leq 2P\left(\max_{i,t}\left|\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{is}}{\rho_n^2}+\varphi(\xi_i,\xi_t)\right|>4M_w^2\right)\\
&\leq 2\Bigg(\underbrace{P\left(\max_{i,t}\left|\frac{1}{n}\sum_{s=1}^{n}\frac{A_{ts}A_{is}}{\rho_n^2}-\varphi(\xi_i,\xi_t)\right|>\varepsilon\right)}_{\leq 2n^2\exp\left(\frac{-(n-2)\left(\varepsilon-\frac{2}{(n-1)\rho_n^2}\right)^2}{2\left(O\left(\frac{1}{\rho_n^2}\right)+\frac{1}{3}\frac{1}{\rho_n^2}\left(\varepsilon-\frac{2}{(n-1)\rho_n^2}\right)\right)}\right)}+\underbrace{P\Bigg(\underbrace{\max_{i,t}|2\varphi(\xi_i,\xi_t)|}_{\leq 2M_w^2}>\underbrace{4M_w^2-\varepsilon}_{>2M_w^2}\Bigg)}_{=0}\Bigg)\\
&\leq n^2\exp\left(-n\rho_n^2C_\varepsilon\right)
\end{aligned}
$$

We show that the second term in (35) goes to zero almost surely by applying the union bound and Bernstein's inequality. We also use that, by definition, $d_{ii} = \tilde{d}_{ii} = 0$.

$$P\left(\max_{i,j}\left|\tilde{d}_{ij}^2 - d_{ij}^2\right| > \frac{\varepsilon b_n r_n}{2}\right) \leq P\left(\max_{i,j,i\neq j}\left|\tilde{d}_{ij}^2 - d_{ij}^2\right| > \frac{\varepsilon b_n r_n}{2}\right) + \underbrace{P\Bigg(\underbrace{\max_{i}\left|\tilde{d}_{ii}^2 - d_{ii}^2\right|}_{=0} > \frac{\varepsilon b_n r_n}{2}\Bigg)}_{=0}$$

$$\leq n(n-1)E\left(P\left(\left|\tilde{d}_{ij}^2 - d_{ij}^2\right| > \frac{\varepsilon b_n r_n}{2}\middle|\xi_i,\xi_j\right)\right)$$

$$= n(n-1)E\left(P\left(\left|\frac{1}{n}\sum_{t=1}^{n}\left((\varphi(\xi_i,\xi_t)-\varphi(\xi_j,\xi_t))^2 - E\left((\varphi(\xi_i,\xi_t)-\varphi(\xi_j,\xi_t))^2\middle|\xi_i,\xi_j\right)\right)\right|\right.\right.$$
$$\left.\left. > \frac{\varepsilon b_n r_n}{2}\middle|\xi_i,\xi_j\right)\right)$$

$$\leq n(n-1)E\left(P\left(\left|\frac{1}{n-2}\sum_{\substack{t=1\\t\neq i,j}}^{n}\left((\varphi(\xi_i,\xi_t)-\varphi(\xi_j,\xi_t))^2\right.\right.\right.\right.$$
$$\left.\left.\left.\left. -E\left((\varphi(\xi_i,\xi_t)-\varphi(\xi_j,\xi_t))^2\middle|\xi_i,\xi_j\right)\right)\right| > \frac{\varepsilon b_n r_n}{2} - \frac{2}{(n-2)\rho_n^2}\middle|\xi_i,\xi_j\right)\right)$$

$$\leq 2n(n-1)\exp\left(\frac{-(n-2)\left(\frac{\varepsilon b_n r_n}{2} - \frac{2}{(n-2)\rho_n^2}\right)^2}{2 + \frac{2\left(\frac{\varepsilon b_n r_n}{2} - \frac{2}{(n-2)\rho_n^2}\right)}{3}}\right) \leq n^2\exp\left(-n b_n^2 r_n^2 C_\varepsilon\right).$$

The conclusion follows since

$$\sum_{n=3}^{\infty} P\left(\frac{1}{b_n r_n}\max_{i,j}\left|\hat{d}_{ij}^2 - d_{ij}^2\right| > \varepsilon\right) = O\left(\sum_{n=3}^{\infty} n^2\exp\left(-n b_n^2 r_n^2 \rho_n^2 C_\varepsilon\right)\right) = O(1)$$

The last term is bounded for any $C_\varepsilon > 0$ because under Assumption 1.4 $\frac{\log(n)}{n b_n^2 r_n^2 \rho_n^2} \to 0$. $\square$

A.2 Proofs of consistency of the bootstrap procedure

Proof of Theorem 2. We start by constructing a particular coupling in $\Gamma(A^*, H)$. Let $\tilde{\gamma}$ be a particular joint distribution over $\hat{F}_n$ and F_0, the details of which we specify later in the proof. We use $\tilde{\gamma}$ to construct a coupling between A^* and H: we draw pairs $\{(\xi_i^*, \xi_i)\}_{i=1}^n \overset{i.i.d.}{\sim} \tilde{\gamma}$. We also independently draw $\{\eta_{ij}\}_{i<j}^n \overset{i.i.d.}{\sim} \mathcal{U}[0,1]$ and set $\eta_{ij}^* = \eta_{ij}$. We denote $(A^*, H) \sim \tilde{\nu}$ and note that this construction gives correct marginal distributions of A^* and H, hence:

$$W_p^p(A^*, H) \leq \int d_{GM}^p(A^*, H)\, d\tilde{\nu} \leq \int\left(\binom{n}{2}^{-1}\frac{\|A^* - H\|_{1,1}}{2}\right)^p d\tilde{\nu}$$
$$\leq \int\binom{n}{2}^{-1}\sum_{i<j}\left|A_{ij}^* - H_{ij}\right|^p d\tilde{\nu} = \binom{n}{2}^{-1}\sum_{i<j}\int\left|A_{ij}^* - H_{ij}\right| d\tilde{\nu} = \int\left|A_{ij}^* - H_{ij}\right| d\tilde{\nu}$$

where the second inequality is due to the definition of d_{GM}, the third follows from the definition of $\frac{1}{2}\|A^*-H\|_{1,1}=\sum_{i<j}\left|A^*_{ij}-H_{ij}\right|$ and Jensen's inequality. The first equality is due to the fact that both adjacency matrices are binary ($1^p=1$, $0^p=0$) and the linearity of expectation. The final equality follows from the identity of distribution over all pairs (i,j). Expanding the final term:

$$
\begin{aligned}
&\int \left|A^*_{ij}-H_{ij}\right| d\tilde{\nu} = \tilde{\nu}\left(\left\{A^*_{ij}\neq H_{ij}\right\}\right) \\
&\quad = \tilde{\nu}\left(\left\{\mathbb{1}\left(\hat{h}_n\left(\xi^*_i,\xi^*_j\right)\geq \eta_{ij}\right)\neq \mathbb{1}\left(h_{0,n}\left(\xi_i,\xi_j\right)\geq \eta_{ij}\right)\right\}\right) \\
&\quad = \int_0^1\int\int\left|\mathbb{1}\left(\hat{h}_n\left(\xi^*_i,\xi^*_j\right)\geq \eta_{ij}\right)-\mathbb{1}\left(h_{0,n}\left(\xi_i,\xi_j\right)\geq \eta_{ij}\right)\right| d\tilde{\gamma}\left(\xi^*_i,\xi_i\right)d\tilde{\gamma}\left(\xi^*_j,\xi_j\right)d\eta_{ij} \\
&\quad = \int\int\left|\hat{h}_n\left(\xi^*_i,\xi^*_j\right)-h_{0,n}\left(\xi_i,\xi_j\right)\right| d\tilde{\gamma}\left(\xi^*_i,\xi_i\right)d\tilde{\gamma}\left(\xi^*_j,\xi_j\right) \\
&\quad \leq \int\int\left|\hat{h}_n\left(\xi^*_i,\xi^*_j\right)-h_{0,n}\left(\xi^*_i,\xi^*_j\right)\right| d\tilde{\gamma}\left(\xi^*_i,\xi_i\right)d\tilde{\gamma}\left(\xi^*_j,\xi_j\right) \\
&\qquad\quad + \int\int\left|h_{0,n}\left(\xi^*_i,\xi^*_j\right)-h_{0,n}\left(\xi_i,\xi_j\right)\right| d\tilde{\gamma}\left(\xi^*_i,\xi_i\right)d\tilde{\gamma}\left(\xi^*_j,\xi_j\right)
\end{aligned}
$$

The fourth equality follows from the fact that the two indicator functions differ in value only if η_{ij} falls into the interval between $h_{0,n}\left(\xi_i,\xi_j\right)$ and $\hat{h}_n\left(\xi^*_i,\xi^*_j\right)$, which happens with probability $\left|\hat{h}_n\left(\xi^*_i,\xi^*_j\right)-h_{0,n}\left(\xi_i,\xi_j\right)\right|$. In the last line we use triangle inequality.

We now look at the last two terms:

$$
\begin{aligned}
&\int\int\left|\hat{h}_n\left(\xi^*_i,\xi^*_j\right)-h_{0,n}\left(\xi^*_i,\xi^*_j\right)\right| d\tilde{\gamma}\left(\xi^*_i,\xi_i\right)d\tilde{\gamma}\left(\xi^*_j,\xi_j\right) \\
&\quad = \int\int\left|\hat{h}_n\left(\xi^*_i,\xi^*_j\right)-h_{0,n}\left(\xi^*_i,\xi^*_j\right)\right| d\hat{F}_n\left(\xi^*_i\right)d\hat{F}_n\left(\xi^*_j\right) \\
&\quad = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n\left|\hat{h}_n\left(\xi^A_i,\xi^A_j\right)-h_{0,n}\left(\xi^A_i,\xi^A_j\right)\right| \\
&\quad \leq \max_{i,j}\left|\hat{h}_n\left(\xi^A_i,\xi^A_j\right)-h_{0,n}\left(\xi^A_i,\xi^A_j\right)\right| = o_p(\rho_n)
\end{aligned}
$$

by Theorem 1, where $\xi^A_i\sim F_0$ refers to the unobserved characteristics used in the formation of the matrix A which A^* is bootstrapped from.

For the other term we use Assumption 1.2, which says:

$$
\inf_{\xi_j\in Supp(\xi_i)} P\left(\xi_k\in\left\{\sup_{\xi_t\in Supp(\xi_i)}\left|w_0(\xi_t,\xi_k)-w_0(\xi_t,\xi_j)\right|<\delta\right\}\right)\geq\left(\frac{\delta}{C}\right)^{\frac{1}{\alpha}}.
$$

We have:

$$\begin{aligned}
&\int\int \left|h_{0,n}\left(\xi_i^*,\xi_j^*\right)-h_{0,n}\left(\xi_i,\xi_j\right)\right| d\tilde{\gamma}\left(\xi_i,\xi_i^*\right) d\tilde{\gamma}\left(\xi_j,\xi_j^*\right)\\
&\quad\le \rho_n \int\int \left|w_0\left(\xi_i^*,\xi_j^*\right)-w_0\left(\xi_i,\xi_j^*\right)\right| d\tilde{\gamma}\left(\xi_i,\xi_i^*\right) d\hat{F}_n\left(\xi_j^*\right)\\
&\qquad + \rho_n \int\int \left|w_0\left(\xi_i,\xi_j^*\right)-w_0\left(\xi_i,\xi_j\right)\right| dF_0\left(\xi_i\right) d\tilde{\gamma}\left(\xi_j,\xi_j^*\right)\\
&\quad\le \rho_n \int \sup_{\xi_j^*\in Supp(\xi_i)} \left|w_0\left(\xi_i^*,\xi_j^*\right)-w_0\left(\xi_i,\xi_j^*\right)\right| d\tilde{\gamma}\left(\xi_i,\xi_i^*\right)\\
&\qquad + \rho_n \int \sup_{\xi_i\in Supp(\xi_i)} \left|w_0\left(\xi_i,\xi_j^*\right)-w_0\left(\xi_i,\xi_j\right)\right| d\tilde{\gamma}\left(\xi_j,\xi_j^*\right)\\
&\quad= 2\rho_n \int \sup_{\xi_i\in Supp(\xi_i)} \left|w_0\left(\xi_i,\xi_j^*\right)-w_0\left(\xi_i,\xi_j\right)\right| d\tilde{\gamma}\left(\xi_j,\xi_j^*\right)
\end{aligned}$$

The first inequality is due to the definition of $h_{0,n} = \rho_n w_0$ and triangle inequality. In the second inequality we take a supremum over the repeated index and note that the support of $\hat{F}_n$ is a subset of the support of F_0. As the terms no longer depend on ξ_j^* and ξ_i respectively, we integrate over their distributions. The resulting two terms are equal (note that w_0 is symmetric).

Fix $\varepsilon > 0$. For every $\xi_j^* \in Supp(\xi_i)$ there exists a neighbourhood $N(\xi_j^*, \varepsilon)$ of measure at least $\left(\frac{\varepsilon}{C}\right)^{\frac{1}{\alpha}}$ such that for all $\xi_j \in N(\xi_j^*, \varepsilon)$: $\sup_{\xi_i \in Supp(\xi_i)} \left|w_0\left(\xi_i,\xi_j^*\right)-w_0\left(\xi_i,\xi_j\right)\right| < \varepsilon$. Our task is to show that there exists a coupling $\tilde{\gamma}$ which aligns ξ_j^* with their corresponding neighbourhoods.

To that end, define

$$d_S(a,b) \equiv \sup_{\xi_i\in Supp(\xi_i)} \left|w_0\left(\xi_i,a\right)-w_0\left(\xi_i,b\right)\right|. \tag{36}$$

d_S is a pseudometric, i.e. it may fail positivity (the distance between two distinct points may be zero) but it satisfies all other properties of a distance (in particular the triangle inequality).

Take K points $\{a_1,\dots a_K\} \in Supp(\xi_i)$ which are at least ε apart: $\forall 1 \le i < j \le K$: $d_S(a_i,a_j) > \varepsilon$. Form a $\frac{\varepsilon}{2}$-neighbourhood around each a_k.

These neighbourhoods are non-overlapping: suppose there was a $b \in N\left(a_i, \frac{\varepsilon}{2}\right)$ and $b \in N\left(a_j, \frac{\varepsilon}{2}\right)$ for $i \ne j$. Then by triangle inequality: $d_S(a_i,a_j) \le d_S(a_i,b) + d_S(a_j,b) \le \varepsilon$. But we have assumed $d_S(a_i,a_j) > \varepsilon$, a contradiction.

By Assumption 1.2 we know that each of these neighbourhoods has a measure at least $\left(\frac{\varepsilon}{2C}\right)^{\frac{1}{\alpha}}$. It follows that:

$$1 \ge P_{b\sim F_0}\left(b \in \bigcup_{i=1}^{K} N\left(a_i, \frac{\varepsilon}{2}\right)\right) = \bigcup_{i=1}^{K} P_{b\sim F_0}\left(b \in N\left(a_i, \frac{\varepsilon}{2}\right)\right) \ge K\left(\frac{\varepsilon}{2C}\right)^{\frac{1}{\alpha}}$$

or $K \leq \left(\frac{\varepsilon}{2C}\right)^{-\frac{1}{\alpha}} < \infty$, so the set of $\{a_1, \dots a_K\}$ has finite cardinality.

Take the largest K possible. Then for all $b \in Supp(\xi_i) \exists k \leq K$ such that $d_S(b, a_k) \leq \varepsilon$, or in other words $\bigcup_{i=1}^{K} N(a_i, \varepsilon)$ is a finite cover of $Supp(\xi_i)$. Hence we can assign each $b \in Supp(\xi_i)$ to one of the $k \in \{1, \dots, K\}$: start with $N\left(a_k, \frac{\varepsilon}{2}\right)$ for all k, then for each point not yet assigned to a region add it to the region with (not necessarily unique) k which minimises the d_S distance from that point to a_k. This way we form K disjoint regions, say $\{N_k\}_{k=1}^{K}$, each of size at least $\left(\frac{\varepsilon}{2C}\right)^{\frac{1}{\alpha}}$ and such that whenever $b_1, b_2 \in N_k \subseteq N(a_k, \varepsilon)$ we have $d_S(b_1, b_2) \leq d_S(b_1, a_k) + d_S(b_2, a_k) \leq 2\varepsilon$.

Now instead of ξ_i report $k(\xi_i)$ such that $\xi_i \in N_{k(\xi_i)}$. This means we are replacing F_0 with an empirical distribution function G_ε which takes only K values, each with probability $P_{b\sim F_0}(b \in N_k) \geq \left(\frac{\varepsilon}{2C}\right)^{\frac{1}{\alpha}}$; and we replace $\hat{F}_n$ with an empirical distribution function $\hat{G}_{\varepsilon,n}$ from G_ε. We choose $\tilde{\gamma}$ to be any coupling of $F_0, \hat{F}_n$ consistent with the following: for $y \sim \mathcal{U}[0,1]$ set $k(\xi_j) = G_\varepsilon^{-1}(y)$, $k(\xi_j^*) = \hat{G}_{\varepsilon,n}^{-1}(y)$.

Then:

$$\int \sup_{\xi_i \in Supp(\xi_i)} \left| w_0(\xi_i, \xi_j^*) - w_0(\xi_i, \xi_j) \right| d\tilde{\gamma}(\xi_j, \xi_j^*) \leq 2\varepsilon + M_w \int \mathbb{1}\left(k(\xi_j) \neq k(\xi_j^*)\right) d\tilde{\gamma}(\xi_j, \xi_j^*)$$

For the first inequality we note that either ξ_j, ξ_j^* fall in the same N_k and hence $d_s(\xi_j^*, \xi_j) \leq 2\varepsilon$, or they come from different subsets of the domain, in which case their maximal possible distance is M_w. For the final term:

$$\begin{aligned}
\int \mathbb{1}\left(k(\xi_j) \neq k(\xi_j^*)\right) d\tilde{\gamma}(\xi_j, \xi_j^*) &\leq \int \left|k(\xi_j) - k(\xi_j^*)\right| d\tilde{\gamma}(\xi_j, \xi_j^*) \\
&= \int_0^1 \left| G_\varepsilon^{-1}(y) - \hat{G}_{\varepsilon,n}^{-1}(y) \right| dy \\
&= \int_1^K \left| G_\varepsilon(x) - \hat{G}_{\varepsilon,n}(x) \right| dx \\
&\leq K \sup_x \left| G_\varepsilon(x) - \hat{G}_{\varepsilon,n}(x) \right| \xrightarrow{a.s.} 0.
\end{aligned}$$

The first inequality is due to the fact that $k \in \mathbb{N}$ so if the terms are not equal their distance is at least 1. The next equality is by construction of $\tilde{\gamma}$, noting that $y \sim \mathcal{U}[0,1]$. We then do a change of variable (we switch from integrating the horizontal distance to the vertical distance between the plots of G_ε and $\hat{G}_{\varepsilon,n}$), noting that the plots can only differ on the domain $x \in [1, K]$. We use an upper bound in terms of a supremum over x and conclude that the final expression goes to zero almost surely by Glivenko-Cantelli Theorem. Hence for all n large enough $\int \mathbb{1}\left(k(\xi_j) \neq k(\xi_j^*)\right) d\tilde{\gamma}(\xi_j, \xi_j^*) \leq \frac{\varepsilon}{M_w}$ with probability one.

Since ε was arbitrary, the overall expression is $o_p(\rho_n)$, as required. □

For the proofs of the next section, in the appendix we split the argument into more steps and provide intermediate results which lead to the conclusions in Theorem 3, Lemma 2 and Corollary 1. The advantage of the additional steps is that they characterise moment conditions sufficient for bootstrap consistency which could be verified for other classes of functions or alternative estimators of the network-generating function. They have been left out of the main text to avoid introducing more complicated notation and improve the readability.

We begin with a general result, not specific to U-statistics. Because the many levels of randomness can get confusing very quickly, we have decided to tackle them one at a time: we firstly characterise a class of *non-random* estimators and distributions for which we get weak convergence of our statistic to the correct limit. We denote these generic non-random statistics and distribution as e.g. h_n, F_n and we can think of them as specific realisations of their random equivalents, e.g. F_n can be the empirical distribution $\hat{F}_n|\xi$ we get for a specific draw of ξ. In practice, the classes of h_n, F_n will often be wider and also contain elements which cannot be achieved as a specific realisation of our random procedure. Once we have characterised the class which ensures weak convergence to the desired limit, we show that, once we allow for randomness in ξ, the statistics based on the random $\hat{h}_n, \hat{F}_n$ belong that class with high probability, hence they converge weakly to the same limit either almost surely or in probability.

Definition A.1. *Set $C_{w,F,\rho}$. Let $\mathcal{h}$ denote a set of linking functions, let $\mathcal{F}$ denote a set of distributions, and let $(0,1]^{\mathbb{N}}$ denote a set of sequences of densities $\{\rho_n\}_{n=1}^{\infty}$, $0 < \rho_n \leq 1$. Let $(w, F, \rho) \in \mathcal{h} \times \mathcal{F} \times (0,1]^{\mathbb{N}}$ be a triple of a function w_0, a distribution F, and a sparsity sequence ρ. Let $\xi \sim F$ and $\xi^* \sim F_n$. For each $(w, F, \rho) \in \mathcal{h} \times \mathcal{F} \times (0,1]^{\mathbb{N}}$ let $C_{w,F,\rho}$ be the set of non-random sequences of pairs of functions and distributions $\{(h_n, F_n)\}_{n=1}^{\infty}$ characterised by a set of conditions on convergence of moments of the form $E_{F_n}\left(f\left(\frac{h_n}{\rho_n}(\xi^*), w(\xi^*)\right)\right) \to E_F\left(f\left(w(\xi), w(\xi)\right)\right)$ as $n \to \infty$ for some class of functions $f \in \mathcal{f}$. That is:*

$$C_{w,F,\rho} = \Big\{ \{(h_n, F_n)\}_{n=1}^{\infty} : \forall n \in \mathbb{N}, \forall f \in \mathcal{f} : \quad (h_n, F_n) \in \mathcal{h} \times \mathcal{F} \text{ and } \lim_{n \to \infty} E_{F_n}\left(f\left(\frac{h_n}{\rho_n}(\xi^*), w(\xi^*)\right)\right) = E_F\left(f\left(w(\xi), w(\xi)\right)\right) \Big\}. \tag{37}$$

We state the general version of the result[26]:

Theorem A.1. *Let $C_{w_0,F_0,\rho}$ be as defined in Definition A.1 and suppose that:*

(i) the set $C_{w_0,F_0,\rho}$ contains the sequence $\{(h_{0,n}, F_0)\}_{n=1}^{\infty}$;

(ii) for any sequence $\{(h_n, F_n)\}_{n=1}^{\infty}$ in $C_{w_0,F_0,\rho}$, $\tilde{J}_n(t, h_n, F_n)$ converges weakly to a common

26. The structure and the proof are strongly inspired by Theorem 1.2.1 of Politis et al. (1999).

distribution $J(t, w_0, F_0)$[27]*;*

(iii) for any sequence $\{(h_n, F_n)\}_{n=1}^{\infty}$ *in* $C_{w_0,F_0,\rho}$*:*

$$\lim_{n\longrightarrow\infty} E_{h_n,F_n}\left[\left(f_n\left(A^*\left(h_n\left(\xi^*\right), \eta^*\right), \rho_n, F_n\right) - \tilde{f}_n(h_n\left(\xi^*\right), \rho_n, F_n)\right)^2\right] = 0 \tag{38}$$

where $A^*\left(h_n\left(\xi^*\right), \eta^*\right)$ *denotes an adjacency matrix* A^* *based on a vector of observations of* $\xi^* \overset{i.i.d.}{\sim} F_n$*, with Bernoulli probabilities determined by* $h_n\left(\xi_i^*\right)$*.*

If the random sequence $\{(\hat{h}_n, \hat{F}_n)\}_{n=1}^{\infty}$ *belongs to* $C_{w_0,F_0,\rho}$ *with probability one, i.e.* $\forall n \in \mathbb{N}, \forall f \in \mathscr{f} : (\hat{h}_n, \hat{F}_n) \in \mathscr{h} \times \mathscr{F}$ *a.s. and* $E_{\hat{F}_n}\left(f\left(\frac{\hat{h}_n}{\rho_n}(\xi^*), w_0(\xi^*)\right)\right) \xrightarrow{a.s.} E_{F_0}\left(f\left(w_0(\xi), w_0(\xi)\right)\right)$*, then:*

1. $J_n(t, \hat{h}_n, \hat{F}_n) \overset{a.s.}{\Rightarrow} J(t, w_0, F_0)$*.*

2. *If* $J(t, w_0, F_0)$ *is continuous in* t *at* $t = 1 - \alpha$ *and strictly increasing at* $t = 1 - \alpha$*:*

$$J_n^{-1}(1 - \alpha, \hat{h}_n, \hat{F}_n) \xrightarrow{a.s.} J^{-1}(1 - \alpha, w_0, F_0). \tag{39}$$

3. *If* $J(t, w_0, F_0)$ *is continuous in* t *at* $t = 1 - \alpha$ *and is strictly increasing at* $t = 1 - \alpha$ *and if* F_0 *doesn't enter the function* f_n *directly but only through a parameter* θ[28]*:*
   $f_n(A(h_{0,n}(\xi), \eta), \rho_n, \theta)$*, then the* $(1 - \alpha)$ *confidence interval for* θ *constructed as:*

$$CI_n\left(1 - \alpha, A, \hat{h}_n, \hat{F}_n\right) = \left\{\theta : J_n^{-1}\left(\frac{\alpha}{2}, \hat{h}_n, \hat{F}_n\right) \leq f_n(A, \rho_n, \theta) \leq J_n^{-1}\left(1 - \frac{\alpha}{2}, \hat{h}_n, \hat{F}_n\right)\right\} \tag{40}$$

   is asymptotically valid:

$$P_{h_{0,n},F_0}\left(\theta \in CI_n\left(1 - \alpha, A, \hat{h}_n, \hat{F}_n\right)\right) \xrightarrow{a.s.} 1 - \alpha. \tag{41}$$

4. *If* $J\left(t, w_0, F_0\right)$ *is continuous in* t*, then*

$$\sup_t \left| J_n\left(t, \hat{h}_n, \hat{F}_n\right) - \tilde{J}_n\left(t, h_{0,n}, F_0\right)\right| \xrightarrow{a.s.} 0.$$

If the random sequence $\{(\hat{h}_n, \hat{F}_n)\}_{n=1}^{\infty}$ *satisfies the moment conditions for belonging to* $C_{w_0,F_0,\rho}$ *in probability:* $E_{\hat{F}_n}\left(f\left(\frac{\hat{h}_n}{\rho_n}(\xi^*), w_0(\xi^*)\right)\right) \xrightarrow{p} E_{F_0}\left(f\left(w_0(\xi), w_0(\xi)\right)\right)$*, then conclusions 1.-4. above hold with* $\overset{p}{\Rightarrow}$ *replacing* $\overset{a.s.}{\Rightarrow}$ *and* $\xrightarrow{p}$ *replacing* $\xrightarrow{a.s.}$*.*

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27. This is weaker than F_n converges weakly to F_0.
28. For example in equation (28) we have $\theta = E_{h_{0,n},F_0}(g(A(\iota)))$.

The above results follow straight from conveniently chosen assumptions, yet they are still useful because they provides a set of sufficient conditions for the convergence of the bootstrap distribution to the correct limit, the correctness of bootstrap confidence intervals, and the consistency of bootstrap.

Proof of Theorem A.1. We start from proving 1.

For any $\{(h_n, F_n)\}_{n=1}^{\infty}$ in $C_{w_0,F_0,\rho}$:

$$f_n\left(A\left(h_n\left(\xi^*\right), \eta^*\right), \rho_n, F_n\right) = \tilde{f}_n\left(h_n\left(\xi^*\right), \rho_n, F_n\right) + \Big(f_n\left(A\left(h_n\left(\xi^*\right), \eta^*\right), \rho_n, F_n\right) - \tilde{f}_n\left(h_n\left(\xi^*\right), \rho_n, F_n\right)\Big)$$

By assumption (ii), the distribution of $\tilde{f}_n\left(h_n\left(\xi^*\right), \rho_n, F_n\right)$ converges weakly to the desired limit: $\tilde{J}_n(t, h_n, F_n) \xrightarrow{weakly} J(t, w_0, F_0)$. By assumption (iii), the second term converges to 0 in second mean, hence it is $o_p(1)$ and does not affect the distribution limit[29]. For any sequence $\{(h_n, F_n)\}_{n=1}^{\infty} \in C_{w_0,F_0,\rho}$ we have:

$$J_n(t, h_n, F_n) \xrightarrow{weakly} J(t, w_0, F_0) \quad \text{i.e.} \quad d\left(J_n(t, h_n, F_n), J(t, w_0, F_0)\right) \to 0.$$

The random sequence $\left\{(\hat{h}_n, \hat{F}_n)\right\}_{n=1}^{\infty}$ belongs to $C_{w_0,F_0,\rho}$ with probability one, hence:

$$\begin{aligned}
&P\left(\lim_{n\to\infty} d\left(J_n(t, \hat{h}_n, \hat{F}_n), J(t, w_0, F_0)\right) = 0\right) \\
&\geq P\left(\{(\hat{h}_n, \hat{F}_n)\}_{n=1}^{\infty} \in C_{w_0,F_0,\rho} \text{ and } \lim_{n\to\infty} d\left(J_n(t, \hat{h}_n, \hat{F}_n), J(t, w_0, F_0)\right) = 0\right) \\
&= P\left(\{(\hat{h}_n, \hat{F}_n)\}_{n=1}^{\infty} \in C_{w_0,F_0,\rho}\right) \\
&= 1,
\end{aligned}$$

that is: $d\left(J_n(t, \hat{h}_n, \hat{F}_n), J(t, w_0, F_0)\right) \xrightarrow{a.s.} 0$.

For the case of convergence in probability, we use the following result:

Theorem (Billingsley (1995) Theorem 20.5 (ii))**.** *A necessary and sufficient condition for $X_n \xrightarrow{p} X$ is that each subsequence $\{X_{n'}\}$ has a further subsequence $\{X_{n''}\}$ such that $X_{n''} \xrightarrow{a.s.} X$.*

Given that $E_{\hat{F}_n}\left(f\left(\frac{\hat{h}_n}{\rho_n}(\xi^*), w_0(\xi^*)\right)\right) \xrightarrow{p} E_{F_0}\left(f\left(w_0(\xi), w_0(\xi)\right)\right)$, for any subsequence indexed by n' there is a further subsequence indexed by n'' which satisfies $E_{\hat{F}_{n''}}\left(f\left(\frac{\hat{h}_{n''}}{\rho_{n''}}(\xi^*), w_0(\xi^*)\right)\right) \xrightarrow{a.s.}$

29. By Theorem 25.4 in Billingsley (1995): $X_n \xrightarrow{d} X$ and $X_n - Y_n \xrightarrow{p} 0$, then $Y_n \xrightarrow{d} X$. Also, if F_{X_n} and F_X denote the distribution functions of random variables X_n and X, respectively, then $X_n \xrightarrow{d} X$ means $F_{X_n} \xrightarrow{weakly} F_X$.

$E_{F_0}\left(f\left(w_0(\xi), w_0(\xi)\right)\right)$. By what we have just shown, applied to $C_{w_0,F_0,\rho''}$, where ρ'' is the subsequence of ρ indexed by n'', $d\left(J_{n''}\left(t, \hat{h}_{n''}, \hat{F}_{n''}\right), J\left(t, w_0, F_0\right)\right) \xrightarrow{a.s.} 0$. Applying Theorem 20.5 (ii) from Billingsley (1995) in the other direction, this means that $d\left(J_n\left(t, \hat{h}_n, \hat{F}_n\right), J\left(t, w_0, F_0\right)\right) \xrightarrow{p} 0$. Hence 1. holds.

The remaining conclusions follow by arguments identical to those in the proof of Theorem 1.2.1 in Politis et al. (1999). For 2. we use the following Lemma:

Lemma (Lemma 1.2.1 of Politis et al. (1999))**.** *Let $\{G_n\}$ be a sequence of distribution functions on the real line converging weakly to a distribution function G (i.e. $G_n(x) \to G(x)$ for all continuity points of G). Assume G is continuous and strictly increasing at $y = G^{-1}(1-\alpha)$. Then,*

$$G_n^{-1}(1-\alpha) = \inf\{x : G_n(x) \geq 1-\alpha\} \to G^{-1}(1-\alpha). \tag{42}$$

Proof. See Politis et al. (1999) p.10. □

Together with the conclusion from 1. that $\tilde{J}_n(t, h_n, F_n) \xrightarrow{weakly} J(t, w_0, F_0)$ for all (h_n, F_n) in $C_{w_0,F_0,\rho}$, the lemma implies that $J_n^{-1}(1-\alpha, h_n, F_n) \to J^{-1}(1-\alpha, w_0, F_0)$ for all (h_n, F_n) in $C_{w_0,F_0,\rho}$. Arguments identical to those in the proof of 1. show that if $\{(\hat{h}_n, \hat{F}_n)\}_{n=1}^{\infty}$ belongs to $C_{w_0,F_0,\rho}$ with probability one, then $J_n^{-1}(1-\alpha, \hat{h}_n, \hat{F}_n) \xrightarrow{a.s.} J^{-1}(1-\alpha, w_0, F_0)$ and if $\{(\hat{h}_n, \hat{F}_n)\}_{n=1}^{\infty}$ satisfies the moment conditions for belonging to $C_{w_0,F_0,\rho}$ in probability: $E_{\hat{F}_n}\left(f\left(\frac{\hat{h}_n}{\rho_n}(\xi^*), w_0(\xi^*)\right)\right) \xrightarrow{p} E_{F_0}\left(f\left(w_0(\xi), w_0(\xi)\right)\right)$, then $J_n^{-1}(1-\alpha, \hat{h}_n, \hat{F}_n) \xrightarrow{p} J^{-1}(1-\alpha, w_0, F_0)$.

In order to show 3., we firstly prove the following Lemma:

Lemma A.6. *Let $\{G_n\}$ be a sequence of distribution functions on the real line converging weakly to a distribution function G (i.e. $G_n(x) \to G(x)$ for all continuity points of G). Let x_n be a real-valued sequence converging to x (i.e. $x_n \to x$). Assume that G is continuous and strictly increasing at x. Then,*

$$G_n(x_n) \to G(x). \tag{43}$$

Proof. Take any $\delta > 0$. Since G is continuous at x, there exists $\varepsilon > 0$ such that $x-\varepsilon$ and $x+\varepsilon$ are continuity points of G and

$$G(x-\varepsilon) - G(x) \geq -\frac{\delta}{2}$$
$$G(x+\varepsilon) - G(x) \leq \frac{\delta}{2}.$$

Since $x_n \to x$, $G_n(x-\varepsilon) \to G(x-\varepsilon)$, $G_n(x) \to G(x)$ and $G_n(x+\varepsilon) \to G(x+\varepsilon)$, there exists an

$N \in \mathbb{N}$ such that for all $n \geq N$:

$$\begin{aligned} |x_n - x| &\leq \varepsilon \\ |G_n(x-\varepsilon) - G(x-\varepsilon)| &\leq \frac{\delta}{2} \\ |G_n(x) - G(x)| &\leq \frac{\delta}{2} \\ |G_n(x+\varepsilon) - G(x+\varepsilon)| &\leq \frac{\delta}{2}. \end{aligned}$$

Since G_n are weakly increasing for all n:

$$G_n(x-\varepsilon) \leq G_n(x) \leq G_n(x+\varepsilon).$$

Hence for all $n \geq N$:

$$\begin{aligned} -\delta \leq G(x-\varepsilon) - G(x) - \frac{\delta}{2} &\leq G_n(x-\varepsilon) - G(x) \leq \\ &\leq G_n(x) - G(x) \leq \\ &\leq G_n(x+\varepsilon) - G(x) \leq G(x+\varepsilon) - G(x) + \frac{\delta}{2} \leq \delta. \end{aligned}$$

i.e. $|G_n(x_n) - G(x)| \leq \delta$. □

For 3, we start with any (h_n, F_n) in $C_{w_0,F_0,\rho}$. We have:

$$\begin{aligned} &P_{h_{0,n},F_0}\left(\theta \in CI_n\left(1-\alpha, A, h_n, F_n\right)\right) \\ &\quad= P_{h_{0,n},F_0}\left(J_n^{-1}\left(\frac{\alpha}{2}, h_n, F_n\right) \leq f_n(A, \rho_n, \theta) \leq J_n^{-1}\left(1-\frac{\alpha}{2}, h_n, F_n\right)\right) \\ &\quad= P_{h_{0,n},F_0}\left(f_n(A, \rho_n, \theta) \leq J_n^{-1}\left(1-\frac{\alpha}{2}, h_n, F_n\right)\right) \\ &\qquad\quad - P_{h_{0,n},F_0}\left(f_n(A, \rho_n, \theta) < J_n^{-1}\left(\frac{\alpha}{2}, h_n, F_n\right)\right) \\ &\quad= J_n\left(J_n^{-1}\left(1-\frac{\alpha}{2}, h_n, F_n\right), h_{0,n}, F_0\right) - J_n\left(J_n^{-1}\left(\frac{\alpha}{2}, h_n, F_n\right), h_{0,n}, F_0\right) \\ &\quad\to J\left(J^{-1}\left(1-\frac{\alpha}{2}, w_0, F_0\right), w_0, F_0\right) - J\left(J^{-1}\left(\frac{\alpha}{2}, w_0, F_0\right), w_0, F_0\right) = 1-\alpha. \end{aligned}$$

The convergence follows from Lemma A.6 used with 2. (for the convergence of the argument) and $J_n\left(t, h_{0,n}, F_0\right) \xrightarrow{weakly} J\left(t, w_0, F_0\right)$ (for the convergence in distribution). Arguments identical to those in the proof of 1. show that if $\left\{\left(\hat{h}_n, \hat{F}_n\right)\right\}_{n=1}^{\infty}$ belongs to $C_{w_0,F_0,\rho}$ with probability one, then $P_{h_{0,n},F_0}\left(\theta \in CI_n\left(1-\alpha, A, \hat{h}_n, \hat{F}_n\right)\right) \xrightarrow{a.s.} 1-\alpha$. If instead $\left\{\left(\hat{h}_n, \hat{F}_n\right)\right\}_{n=1}^{\infty}$ satisfies the moment conditions for belonging to $C_{w_0,F_0,\rho}$ only in probability, that is: $E_{\hat{F}_n}\left(f\left(\frac{\hat{h}_n}{\rho_n}(\xi^*), w_0(\xi^*)\right)\right) \xrightarrow{p}$

$E_{F_0}\left(f\left(w_0(\xi), w_0(\xi)\right)\right)$, then we have $P_{h_{0,n},F_0}\left(\theta \in CI_n\left(1-\alpha, A, \hat{h}_n, \hat{F}_n\right)\right) \xrightarrow{p} 1-\alpha$.

Finally, 4. follows from 1. and Polya's Theorem:

Theorem (Polya's Theorem, Satz I of Pólya (1920)). *Let X_n, X be random variables with distributions $F_n(x)$ and $F(x)$ respectively. If F is continuous*

$$X_n \xrightarrow{d} X \iff \sup_x |F_n(x) - F(x)| \to 0.$$

□

We can now provide more primitive conditions for the special class of f_n for which $\tilde{f}_n$ is a U-statistic.

Theorem A.2 (Consistency of bootstrap for U-statistics). *Let ι be a set of m nodes and denote the adjacency matrix on the subgraph with nodes in ι and linking probabilities $h_n(.,.)$ by $A(h_n(\xi(\iota)), \eta(\iota))$. Let $g : \{0,1\}^{\binom{m}{2}} \to \mathbb{R}$ be a symmetric function from a subgraph on $m < \infty$ nodes to the real line and let*

$$\begin{aligned} &f_n(A(h_n(\xi^*), \eta^*), \rho_n, F_n) \\ &\quad = \frac{\sqrt{n}}{\binom{n}{m}\rho_n^{\tau(g)}} \sum_{1 \le \iota_1 < \iota_2 < \cdots < \iota_m \le n} \left(g(A(h_n(\xi^*(\iota)), \eta^*(\iota))) - E_{h_n,F_n}(g(A(h_n(\xi^*(\iota)), \eta^*(\iota))))\right). \end{aligned}$$

and $\tilde{g}(h_{0,n}(\xi(\iota))) \equiv E(g(A(h_{0,n}(\xi(\iota)), \eta(\iota)))|\xi(\iota))$. There exists a normalisation $\tau(g)$[30] *and a function $\tilde{\tilde{g}} : Supp(\xi)^m \to \mathbb{R}$ such that:*

- $\frac{\tilde{g}(h_{0,n}(\xi(\iota)))}{\rho_n^{\tau(g)}} = \tilde{\tilde{g}}(w_0(\xi(\iota))) + O(\rho_n)$
- $E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi(j)))|\right) > 0$ *for some* $j \in \mathcal{M}_m$
- $E_{F_0}\left(\tilde{\tilde{g}}^2\left(w_0\left(\xi(j)\right)\right)\right) < \infty \quad \forall j \in \mathcal{M}_m$
- $Var_{F_0}(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi(\iota)))|\xi_{\iota_1}\right)) \equiv \sigma_1^2 < \infty$

Suppose that:

$$\sigma_1^2 > 0$$
$$\frac{n}{\binom{n}{m}\rho_n^{\tau(g)}} \to 0.$$

30. For $m = 2$, if $g(0) \neq 0$ we set $\rho_n^{-\tau(g)} = 1$, $\tilde{\tilde{g}}\left(w_0\left(\xi_i, \xi_j\right)\right) = g(0)$ and if $g(0) = 0$ but $g(1) \neq 0$ we set $\rho_n^{-\tau(g)} = \frac{1}{\rho_n}$ and $\tilde{\tilde{g}}\left(w_0\left(\xi_i, \xi_j\right)\right) = g(1) w_0\left(\xi_i, \xi_j\right)$. More generally, for $m \geq 2$, $\rho_n^{-\tau(g)} = \frac{1}{\rho_n^k}$ where k is the smallest number of ones such that $g(\cdot)$ evaluated at a vector of k ones and $\binom{m}{2} - k$ zeros is non-zero.

and let $C_{w_0,F_0,\rho}$ be a set of sequences $\{(h_n, F_n)\}_{n=1}^{\infty}$ which satisfy:

1. $E_{F_n}\left(\left(\frac{1}{\rho_n}\left(h_n(\xi_i^*, \xi_j^*) - h_{0,n}(\xi_i^*, \xi_j^*)\right)\right)^2\right) \to 0.$

2. $E_{F_n}\left(f\left(\xi^*(\iota)\right)\right) \to E_{F_0}\left(f\left(\xi(\iota)\right)\right)$ *for all* $f : Supp(\xi)^k \to \mathbb{R}$ *such that* $E_{F_0}\left(|f\left(\xi(\iota)\right)|\right) < \infty$ *for all* $\iota \in \mathcal{M}_k$*, for any* $k \leq 2m-1$.

Then $\{(\hat{h}_n, \hat{F})\}_{n=1}^{\infty}$ satisfies 1.-2. in probability and we get all conclusions of Theorem A.1 in probability with $J(t, w_0, F_0) = N(0, m^2\sigma_1^2)$.

Remark. *The advantage of stating our condition as in 2. instead of directly showing that it holds when $F_n = \hat{F}_n$ because of SLLN for U-statistics is that it characterises a wider class of distributions we could resample from. For example, when we adjust the resampling distribution for the purpose of GMM by adding weights to different observations in a way that ensures the moment conditions hold in the bootstrap world.*

Proof of Theorem A.2. The theorem was stated for a general $m < \infty$ but for simplicity of notation we present the proof for the case of $m = 2$. The structure of the argument remains identical if we use $m > 2$.

To show the existence of $\tilde{\tilde{g}}$ and $\tau(g)$ we start by analysing the form of $\tilde{g}$. Since g is a function from $\{0,1\}^{\binom{m}{2}}$ it takes at most $2^{\binom{m}{2}}$ distinct values. Each of those values is taken with probability that the input submatrix $A(\iota)$ matches a given pattern of 0s and 1s. Let $\Gamma(A(\iota))$ denote the set (of cardinality $2^{\binom{m}{2}}$) of all possible values $A(\iota)$ can take. Then:

$$\tilde{g}(h_{0,n}(\xi(\iota))) = \sum_{\gamma \in \Gamma(A(\iota))} g(\gamma) P(A(\iota) = \gamma | \xi(\iota)).$$

Conditional on $\xi(\iota)$, the elements of $A(\iota)$ are independent and $P(A_{ij} = 1|\xi) = h_{0,n}(\xi_i, \xi_j) = \rho_n w_0(\xi_i, \xi_j) \sim \rho_n$ while $P(A_{ij} = 0|\xi) = 1 - h_{0,n}(\xi_i, \xi_j) = 1 - \rho_n w_0(\xi_i, \xi_j) \sim 1$. The probability of the event that the upper triangle of $A(\iota)$ consists of k ones and $\binom{m}{2} - k$ zeros is proportional to ρ_n^k. The smallest k for which $g(\cdot)$ evaluated at an input γ with k ones and $\binom{m}{2} - k$ zeros in the upper triangle is non-zero is equal to the normalisation $\tau(g)$. By construction, all γs with fewer ones have a coefficient $g(\gamma) = 0$. All γs with more ones happen with probability proportional to ρ_n^l for $l > \tau(g)$, i.e. after a normalisation by $\rho_n^{-\tau(g)}$ are $O\left(\rho_n\right)$ and go to zero.

The terms in the sum proportional to $\rho_n^{\tau(g)}$ are of the form:

$$g(\gamma) \underbrace{h_{0,n}(\xi_{\iota_1}, \xi_{\iota_2}) \dots h_{0,n}(\xi_{\iota_3}, \xi_{\iota_4})}_{\tau(g) \text{ terms}} (1 - h_{0,n}(\xi_{\iota_5}, \xi_{\iota_6})) \dots (1 - h_{0,n}(\xi_{\iota_7}, \xi_{\iota_8})).$$

After a normalisation by $\rho_n^{-\tau(g)}$ we get:

$$g(\gamma)\underbrace{w_0(\xi_{\iota_1},\xi_{\iota_2})\dots w_0(\xi_{\iota_3},\xi_{\iota_4})}_{\tau(g)\text{ terms}}(1-h_{0,n}(\xi_{\iota_5},\xi_{\iota_6}))\dots(1-h_{0,n}(\xi_{\iota_7},\xi_{\iota_8}))$$

we keep the $g(\gamma)w_0(\xi_{\iota_1},\xi_{\iota_2})\dots w_0(\xi_{\iota_3},\xi_{\iota_4})$ part in $\tilde{\tilde{g}}$ and note that the remainder of the previous term is $O(\rho_n)$.

To sum up, $\tilde{\tilde{g}}$ takes the form of a finite sum of non-zero constant (value of g at a specific realisation γ) times a product of $\tau(g)$ terms of the form $w_0(\xi_{\iota_i},\xi_{\iota_j})$.

The remaining terms in $\frac{\tilde{g}(h_{0,n}(\xi(\iota)))}{\rho_n^{\tau(g)}} - \tilde{\tilde{g}}(w_0(\xi(\iota)))$ vanish at the rate $O(\rho_n)$.

Since $w_0(\xi_i,\xi_j)$ is not identically equal to zero and there are non-zero coefficients $g(\gamma)$ multiplying products of $w_0(\xi_{\iota_i},\xi_{\iota_j})$ in $\tilde{\tilde{g}}$, there exists $j \in \mathcal{M}_m$ for which $E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi(j)))|\right) > 0$.

Since $w_0(\xi_i,\xi_j) < M_w$ for all ξ_i,ξ_j we have

$$E_{F_0}\left(\tilde{\tilde{g}}^2\left(w_0\left(\xi(j)\right)\right)\right) < \binom{\binom{m}{2}}{\tau(g)}^2 \left(\max_{\gamma\in\Gamma(A(\iota))} g^2(\gamma)\right) M_w^2 < \infty$$

for all $j \in \mathcal{M}_m$ (where the first constant says that there are $\binom{m}{2}$ ones and zeros that determine the value of $A(\iota)$, there are $\binom{\binom{m}{2}}{\tau(g)}$ ways to place $\tau(g)$ ones in them, and after squaring a sum of $\binom{\binom{m}{2}}{\tau(g)}$ terms we get $\binom{\binom{m}{2}}{\tau(g)}^2$ terms, each bounded above by the remaining part of the expression). Finally, since

$$Var(E(Y|X)) = Var(Y) - E(Var(Y|X)) \leq Var(Y) = E(Y^2) - E(Y)^2 \leq E(Y^2)$$

we also get that

$$\sigma_1^2 \equiv Var_{F_0}(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi(\iota)))|\xi_{\iota_1}\right)) < E_{F_0}\left(\tilde{\tilde{g}}^2\left(w_0\left(\xi(\iota)\right)\right)\right) < \infty.$$

Having established the existence of $\tilde{\tilde{g}}$ and $\tau(g)$, we now check that the elements of $C_{w_0,F_0,\rho}$ satisfy condition (i)-(iii) of Theorem A.1.

The sequence $\{(h_n,F_n)\}_{n=1}^{\infty} = \{(h_{0,n},F_0)\}_{n=1}^{\infty}$ satisfies the conditions and belongs to $C_{w_0,F_0,\rho}$ (sequences in 1. and 2. are constant and equal to the desired limit), hence (i) is satisfied.

To check condition (iii) we look at:

$$E_{h_n,F_n}\left(\left(f_n(A^*\left(h_n\left(\xi^*\right),\eta^*\right),\rho_n,F_n)-\tilde{f}_n(h_n\left(\xi^*\right),\rho_n,F_n)\right)^2\right)$$
$$=\frac{n}{\left(\binom{n}{2}\rho_n^{\tau(g)}\right)^2}\sum_{i^*<j^*}\sum_{k^*<l^*}E_{h_n,F_n}\left(\left(g(A_{i^*j^*})-E_{h_n}\left(g(A_{i^*j^*})|\xi^*\right)\right)\left(g(A_{k^*l^*})-E_{h_n}\left(g(A_{k^*l^*})|\xi^*\right)\right)\right)$$

To simplify the above expression notice that most terms in the summation are zero. In particular, consider different cases of overlap between the indices:

- if there is no overlap ($i^*\neq k^*$, $i^*\neq l^*$, $j^*\neq k^*$, $j^*\neq l^*$), by the independence assumption the term inside the sum is:

$$E_{h_n,F_n}\left(\left(g(A_{i^*j^*})-E_{h_n}\left(g(A_{i^*j^*})|\xi^*\right)\right)\right)^2=0^2=0.$$

- If there is partial overlap (e.g. $i^*=k^*$, $j^*\neq l^*$, or any symmetric situation):

$$\begin{aligned}&E_{h_n,F_n}\left(\left(g(A_{i^*j^*})-E_{h_n}\left(g(A_{i^*j^*})|\xi^*\right)\right)\left(g(A_{i^*l^*})-E_{h_n}\left(g(A_{i^*l^*})|\xi^*\right)\right)\right)\\ &\overset{LIE}{=}E_{h_n,F_n}\left(E_{h_n,F_n}\left(\left(g(A_{i^*j^*})-E_{h_n}\left(g(A_{i^*j^*})|\xi^*\right)\right)\left(g(A_{i^*l^*})-E_{h_n}\left(g(A_{i^*l^*})|\xi^*\right)\right)|\xi_i^*\right)\right)\\ &\overset{indep}{=}E_{h_n,F_n}\left(\left(E_{h_n,F_n}\left(\left(g(A_{i^*j^*})-E_{h_n}\left(g(A_{i^*j^*})|\xi^*\right)\right)|\xi_i^*\right)\right)^2\right)\\ &\overset{LIE}{=}E_{h_n,F_n}\left(\left(E_{h_n,F_n}\left(\left(E_{h_n}\left(g(A_{i^*j^*})|\xi^*\right)-E_{h_n}\left(g(A_{i^*j^*})|\xi^*\right)\right)|\xi_i^*\right)\right)^2\right)=0.\end{aligned}$$

- If there is full overlap ($i^*=k^*$ and $j^*=l^*$, or $i^*=l^*$ and $j^*=k^*$):

$$E_{h_n,F_n}\left(\left(g(A_{i^*j^*})-E_{h_n}\left(g(A_{i^*j^*})|\xi^*\right)\right)^2\right)\leq E_{h_n,F_n}\left(g^2(A_{i^*j^*})\right).$$

  There are $\binom{n}{2}$ terms of this final form in the sum.

Combining the three cases, we get:

$$E_{h_n,F_n}\left(\left(f_n\left(A^*\left(h_n\left(\xi^*\right),\eta^*\right),\rho_n,F_n\right)-\tilde{f}_n\left(h_n\left(\xi^*\right),\rho_n,F_n\right)\right)^2\right)\quad\leq\frac{2E_{h_n,F_n}\left(g^2(A_{i^*j^*})\right)}{(n-1)\rho_n^{2\tau(g)}}$$

In an analogous way to how we have defined $\tilde{g}$ and the corresponding $\tilde{\tilde{g}}$, we let $\tilde{g^2}\left(h_{0,n}\left(\xi_i,\xi_j\right)\right)\equiv E_{h_{0,n}}\left(g^2(A_{ij})|\xi\right)^{31}$, and we can find a function $\tilde{\tilde{g^2}}(w_0(\xi_i,\xi_j))$ with $\frac{\tilde{g^2}(h_{0,n}(\xi_i,\xi_j))}{\rho_n^{\tau(g)}}=\tilde{\tilde{g^2}}(w_0(\xi_i,\xi_j))+$

31. Comparing to the example given earlier:

$$\begin{aligned}E(g^2(A_{1,2},A_{2,3})|\xi)\equiv\tilde{g^2}(h_{0,n}(\xi_1,\xi_2),h_{0,n}(\xi_2,\xi_3))\\ &=g^2(0,0)(1-h_{0,n}(\xi_1,\xi_2))(1-h_{0,n}(\xi_2,\xi_3))+g^2(0,1)(1-h_{0,n}(\xi_1,\xi_2))h_{0,n}(\xi_2,\xi_3)\\ &+g^2(1,0)h_{0,n}(\xi_1,\xi_2)(1-h_{0,n}(\xi_2,\xi_3))+g^2(1,1)h_{0,n}(\xi_1,\xi_2)h_{0,n}(\xi_2,\xi_3).\end{aligned}$$

$O(\rho_n)$, $0 < E_{F_0}\left(\left|\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))\right|\right) < \infty$ and $0 < E_{F_0}\left(\left|\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_i))\right|\right) < \infty$. Then:

$$\begin{aligned}
&E_{h_n,F_n}\left(\rho_n^{-\tau(g)} g^2(A_{i^*j^*})\right)\\
&= E_{F_n}\left(E_{h_n}\left(\rho_n^{-\tau(g)} g^2(A_{i^*j^*})|\xi^*\right)\right)\\
&= E_{F_n}\left(\rho_n^{-\tau(g)}\tilde{g}^2\left(h_n\left(\xi_i^*,\xi_j^*\right)\right)\right)\\
&= E_{F_n}\left(\rho_n^{-\tau(g)}\tilde{g}^2\left(h_{0,n}\left(\xi_i^*,\xi_j^*\right)\right) + \rho_n^{-\tau(g)+1}\tilde{g}^{2'}\left(\bar{h}_n\left(\xi_i^*,\xi_j^*\right)\right)\frac{1}{\rho_n}\left(h_n\left(\xi_i^*,\xi_j^*\right) - h_{0,n}\left(\xi_i^*,\xi_j^*\right)\right)\right)\\
&\leq E_{F_n}\left(\rho_n^{-\tau(g)}\tilde{g}^2\left(h_{0,n}\left(\xi_i^*,\xi_j^*\right)\right)\right) + \underbrace{\rho_n^{-\tau(g)+1}\sup_h\left|\tilde{g}^{2'}(h)\right|}_{<\infty}\underbrace{E_{F_n}\left(\frac{1}{\rho_n}\left(h_n(\xi_i^*,\xi_j^*) - h_{0,n}(\xi_i^*,\xi_j^*)\right)\right)}_{=o(1)}\\
&= E_{F_n}\left(\tilde{\tilde{g}}^2\left(w_0\left(\xi_i^*,\xi_j^*\right)\right)\right) + O\left(\rho_n\right) + o(1)\\
&\xrightarrow{a.s.} E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))\right) < \infty.
\end{aligned}$$

Note that since the leading term of $\tilde{g}^2(h_{0,n})$ is proportional to the $\tau(g)$th power of $h_{0,n}$, the leading term of $\tilde{g}^{2'}(h_{0,n})$ has a $h_{0,n}$ to the power $\tau(g)-1$. Given the form of $\tilde{g}^{2'}(h_{0,n})$, which is a sum of finitely many terms of the form of a bounded constant times bounded powers of $h_{0,n}$, the whole derivative is bounded. It follows that:

$$E_{h_n,F_n}\left(\left(f_n\left(A^*\left(h_n\left(\xi^*\right),\eta^*\right),\rho_n,F_n\right) - \tilde{f}_n\left(h_n\left(\xi^*\right),\rho_n,F_n\right)\right)^2\right) \leq O\left(\frac{1}{n\rho_n^{\tau(g)}}\right) = o\left(1\right).$$

Hence (iii) holds.

Checking (ii) is a bit more involved. We start with a Hoeffding's (martingale) decomposition[32] of $\tilde{f}_n\left(h_n\left(\xi^*\right),\rho_n,F_n\right)$ for any $\{(h_n,F_n)\}_{n=1}^\infty$ in $C_{w_0,F_0,\rho}$:

$$\begin{aligned}
\tilde{f}_n\left(h_n\left(\xi^*\right),\rho_n,F_n\right) &= \frac{\sqrt{n}}{\binom{n}{2}\rho_n^{\tau(g)}}\sum_{i<j}\tilde{g}(h_n(\xi_i^*,\xi_j^*)) - E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*))\right)\\
&= \frac{2}{\sqrt{n}\rho_n^{\tau(g)}}\sum_{i=1}^n E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*))|\xi_i^*\right) - E_{F_n}\left(E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*))|\xi_i^*\right)\right)\\
&+ \frac{\sqrt{n}}{\binom{n}{2}\rho_n^{\tau(g)}}\sum_{i<j}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*)) - E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*))|\xi_i^*\right)\right.\\
&\qquad\left. - E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*))|\xi_j^*\right) + E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*))\right)\right)\\
&\equiv \tilde{U}_n(h_n,F_n) + \tilde{r}_n(h_n,F_n).
\end{aligned}$$

We firstly focus on $\tilde{U}_n(h_n,F_n)$, which is a (rescaled) average of i.i.d. terms. We add and subtract

This example illustrates why $\tilde{g}^2(h_{0,n}(\xi_1,\xi_2),h_{0,n}(\xi_2,\xi_3))$ is proportional to $\rho_n^{\tau(g)}$, not to $\rho_n^{2\tau(g)}$.

32. For more details see Chapter 5 of Serfling (2009), specifically section 5.1.5.

terms that swap h_n for $h_{0,n}$ and F_n for F_0:

$$\begin{aligned}
\tilde{U}_n(h_n, F_n) = & \frac{2}{\sqrt{n}\rho_n^{\tau(g)}} \sum_{i=1}^{n} \Big(E_{F_n}\left(\tilde{g}(h_n(\xi_i^*, \xi_j^*))|\xi_i^*\right) - E_{F_n}\left(\tilde{g}(h_n(\xi_i^*, \xi_j^*))\right) \\
& \qquad - E_{F_n}\left(\tilde{g}(h_{0,n}(\xi_i^*, \xi_j^*))|\xi_i^*\right) + E_{F_n}\left(\tilde{g}(h_{0,n}(\xi_i^*, \xi_j^*))\right) \Big) \\
& + \frac{2}{\sqrt{n}\rho_n^{\tau(g)}} \sum_{i=1}^{n} \Big(E_{F_n}\left(\tilde{g}(h_{0,n}(\xi_i^*, \xi_j^*))|\xi_i^*\right) - E_{F_n}\left(\tilde{g}(h_{0,n}(\xi_i^*, \xi_j^*))\right) \\
& \qquad - E_{F_0}\left(\tilde{g}(h_{0,n}(\xi_i^*, \xi_j))|\xi_i^*\right) + E_{F_n}\left(E_{F_0}\left(\tilde{g}(h_{0,n}(\xi_i^*, \xi_j))|\xi_i^*\right)\right) \Big) \\
& + \frac{2}{\sqrt{n}\rho_n^{\tau(g)}} \sum_{i=1}^{n} \left(E_{F_0}\left(\tilde{g}(h_{0,n}(\xi_i^*, \xi_j))|\xi_i^*\right) - E_{F_n}\left(E_{F_0}\left(\tilde{g}(h_{0,n}(\xi_i^*, \xi_j))|\xi_i^*\right)\right)\right) \\
& = T_1 + T_2 + T_3
\end{aligned}$$

We deal with these terms one by one.

For T_1, we do Taylor expansion of $\tilde{g}$ around $h_{0,n}$:

$$\tilde{g}(h_n(\xi_i^*, \xi_j^*)) - \tilde{g}(h_{0,n}(\xi_i^*, \xi_j^*)) = \tilde{g}^{'}\left(\tilde{h}_n(\xi_i^*, \xi_j^*)\right)\left(h_n(\xi_i^*, \xi_j^*) - h_{0,n}(\xi_i^*, \xi_j^*)\right)$$

where $\tilde{h}_n(\xi_i^*, \xi_j^*)$ is between $h_n(\xi_i^*, \xi_j^*)$ and $h_{0,n}(\xi_i^*, \xi_j^*)$. We can show that T_1 goes to zero in second mean, hence also in probability. Let:

$$\begin{aligned}
& T_1 \\
& = \frac{2}{\sqrt{n}\rho_n^{\tau(g)}} \sum_{i=1}^{n} \left(E_{F_n}\left(\tilde{g}(h_n(\xi_i^*, \xi_j^*)) - \tilde{g}(h_{0,n}(\xi_i^*, \xi_j^*))|\xi_i^*\right) - E_{F_n}\left(\tilde{g}(h_n(\xi_i^*, \xi_j^*)) - \tilde{g}(h_{0,n}(\xi_i^*, \xi_j^*))\right)\right) \\
& = \frac{2}{\sqrt{n}\rho_n^{\tau(g)}} \sum_{i=1}^{n} b_{i^*}.
\end{aligned}$$

Note that the terms inside the sum are independent and have zero expectation:

$$\begin{aligned}
E_{F_n}(b_{i^*}) &= E_{F_n}\left(E_{F_n}\left(\tilde{g}(h_n(\xi_i^*, \xi_k^*)) - \tilde{g}(h_{0,n}(\xi_i^*, \xi_k^*))|\xi_i^*\right) - E_{F_n}\left(\tilde{g}(h_n(\xi_k^*, \xi_l^*)) - \tilde{g}(h_{0,n}(\xi_k^*, \xi_l^*))\right)\right) \\
&\overset{LIE}{=} E_{F_n}\left(\tilde{g}(h_n(\xi_k^*, \xi_l^*)) - \tilde{g}(h_{0,n}(\xi_k^*, \xi_l^*))\right) - E_{F_n}\left(\tilde{g}(h_n(\xi_k^*, \xi_l^*)) - \tilde{g}(h_{0,n}(\xi_k^*, \xi_l^*))\right) = 0
\end{aligned}$$

Hence in the expansion of the square all terms with $i \neq j$ are zero:

$$
\begin{aligned}
E\left(T_1^2\right) &= E\left(\left(\frac{2}{\sqrt{n}\rho_n^{\tau(g)}}\sum_{i=1}^{n} b_{i^*}\right)^2\right) \\
&= 4\frac{1}{n\rho_n^{2\tau(g)}}\sum_{i=1}^{n} E_{F_n}\left(b_{i^*}^2\right) + 8\frac{1}{n\rho_n^{2\tau(g)}}\sum_{i<j} E_{F_n}\left(b_{i^*}b_{j^*}\right) \\
&\overset{i.i.d.}{=} 4\rho_n^{-2\tau(g)} E_{F_n}\left(b_{i^*}^2\right) + 8\frac{1}{n\rho_n^{2\tau(g)}}\sum_{i<j} E_{F_n}\left(b_{i^*}\right)E_{F_n}\left(b_{j^*}\right) \\
&= 4\rho_n^{-2\tau(g)} E_{F_n}\left(b_{i^*}^2\right) \\
&= 4\rho_n^{-2\tau(g)} E_{F_n}\Bigg(\bigg(E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*)) - \tilde{g}(h_{0,n}(\xi_i^*,\xi_j^*))|\xi_i^*\right) \\
&\qquad\qquad - E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*)) - \tilde{g}(h_{0,n}(\xi_i^*,\xi_j^*))\right)\bigg)^2\Bigg) \\
&= 4\rho_n^{-2\tau(g)} E_{F_n}\left(\left(E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*)) - \tilde{g}(h_{0,n}(\xi_i^*,\xi_j^*))|\xi_i^*\right)\right)^2\right) \\
&\quad - 4\rho_n^{-2\tau(g)}\left(E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*)) - \tilde{g}(h_{0,n}(\xi_i^*,\xi_j^*))\right)\right)^2 \\
&\leq 4\rho_n^{-2\tau(g)} E_{F_n}\left(\left(E_{F_n}\left(\tilde{g}^{'}\left(\tilde{h}_n(\xi_i^*,\xi_j^*)\right)\left(h_n(\xi_i^*,\xi_j^*) - h_{0,n}(\xi_i^*,\xi_j^*)\right)|\xi_i\right)\right)^2\right) \\
&\leq 4\underbrace{\left(\rho_n^{-\tau(g)+1}\sup_{h\in[0,M_w\rho_n]}\left|\tilde{g}^{'}(h)\right|\right)^2}_{<\infty}\underbrace{E_{F_n}\left(\left(\frac{1}{\rho_n}\left(h_n(\xi_i^*,\xi_j^*) - h_{0,n}(\xi_i^*,\xi_j^*)\right)\right)^2\right)}_{=o(1)} \to 0
\end{aligned}
$$

In the first inequality we use the fact that the second term is negative and smaller in magnitude than the first. We then pull the supremum over derivatives of $\tilde{g}$ out of the expectation, use Jensen's inequality to put the square inside the inner expectation, apply the law of iterated expectations, and use the assumption 1. to get the conclusion.

As mentioned before, the derivative of $\tilde{g}$ is bounded for any choice of g, and as we take a derivative with respect to h the leading term of the $\tilde{g}^{'}$ becomes proportional to power one lower than $\tilde{g}$, i.e. $\rho_n^{-\tau(g)+1}\tilde{g}^{'} = O_p(1)$.

For the middle term, T_2, we show that it goes to zero in mean squared. To simplify notation, let $T_2 = \frac{2}{\sqrt{n}\rho_n^{\tau(g)}}\sum_{i=1}^{n} a_{i^*}$ and notice that:

$$
\begin{aligned}
E_{F_n}\left(a_{i^*}\right) = &\underbrace{E_{F_n}\left(E_{F_n}\left(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j^*))|\xi_i^*\right)\right) - E_{F_n}\left(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j^*))\right)}_{=0} \\
&\underbrace{-E_{F_n}\left(E_{F_0}\left(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*\right)\right) + E_{F_n}\left(E_{F_0}\left(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*\right)\right)}_{=0} = 0.
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
E\left(T_2^2\right) &= E\left(\left(\frac{2}{\sqrt{n}\rho_n^{\tau(g)}}\sum_{i=1}^{n} a_{i^*}\right)^2\right)\\
&= 4\frac{1}{n\rho_n^{2\tau(g)}}\sum_{i=1}^{n} E_{F_n}\left(a_{i^*}^2\right) + 8\frac{1}{n\rho_n^{2\tau(g)}}\sum_{i<j} E_{F_n}\left(a_{i^*}a_{j^*}\right)\\
&\overset{i.i.d.}{=} 4\rho_n^{-2\tau(g)}E_{F_n}\left(a_{i^*}^2\right) + 8\frac{1}{n\rho_n^{2\tau(g)}}\sum_{i<j} E_{F_n}\left(a_{i^*}\right)E_{F_n}\left(a_{j^*}\right)\\
&= 4\rho_n^{-2\tau(g)}E_{F_n}\left(a_{i^*}^2\right)\\
&= 4E_{F_n}\Big(\big(E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))|\xi_i^*\right) + E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))\right) - E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*\right)\\
&\quad + E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))\right) - E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))\right) + E_{F_n}\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*\right)\right)\big)^2\Big) + O\left(\rho_n\right)\\
&\leq 8\underbrace{E_{F_n}\left(\left(E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))|\xi_i^*\right)\right)^2\right)}_{\to E_{F_0}\left(\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)\right)^2\right)} + 8\underbrace{E_{F_n}\left(\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*\right)\right)^2\right)}_{\to E_{F_0}\left(\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)\right)^2\right)}\\
&\quad - 16\underbrace{E_{F_n}\left(E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))|\xi_i^*\right)E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*\right)\right)}_{\to E_{F_0}\left(\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)\right)^2\right)}\\
&\quad + 8\underbrace{\left(E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))\right) - E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))\right)\right)^2}_{\to 0}\\
&\quad + 8\underbrace{\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))\right) - E_{F_n}\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*\right)\right)\right)^2}_{\to 0} + O\left(\rho_n\right)\\
&\to 0
\end{aligned}
$$

In the 5th equality we plug in the definition of a_{i^*}, we add and subtract the term $E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))\right)$, we bring the normalisation by $\rho_n^{-2\tau(g)}$ inside the expectation and use $\frac{\tilde{g}(\xi_i,\xi_j)}{\rho_n^{\tau(g)}} = \tilde{\tilde{g}}\left(w_0\left(\xi_i,\xi_j\right)\right) + O\left(\rho_n\right)$. In the next step, we apply $(a+b)^2 \leq 2a^2 + 2b^2$, where a corresponds to the first four terms in the previous summation, for which we expand the square, and b corresponds to the last two terms. We now verify that we can apply property 2. to all resulting terms:

- By the independence between ξ_j^* and ξ_k^* when $j \neq k$ and the law of iterated expectations we can rewrite the first term as:

$$
\begin{aligned}
E_{F_n}\left(\left(E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))|\xi_i^*\right)\right)^2\right) &= E_{F_n}\left(E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))|\xi_i^*\right)E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_k^*))|\xi_i^*\right)\right)\\
&= E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_k^*))\right)
\end{aligned}
$$

We now check the conditions for 2. when all indices are unique:

$$\begin{aligned}
E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))\tilde{\tilde{g}}(w_0(\xi_i,\xi_k))|\right) &\overset{LIE}{=} E_{F_0}\left(E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))\tilde{\tilde{g}}(w_0(\xi_i,\xi_k))|\,|\,\xi_i\right)\right) \\
&\leq E_{F_0}\left(E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\,|\,\xi_i\right) E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_k))|\,|\,\xi_i\right)\right) \\
&= E_{F_0}\left(E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\,|\,\xi_i\right)^2\right) \\
&\leq E_{F_0}\left(E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))\big|\,\xi_i\right)\right) \\
&\overset{LIE}{=} E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))\right) < \infty.
\end{aligned}$$

When two indices are repeated we use Cauchy-Schwarz inequality:

$$E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))\tilde{\tilde{g}}(w_0(\xi_i,\xi_i))|\right) \leq \sqrt{E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))\right) E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_i))\right)} < \infty.$$

And when all indices are equal the condition $E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_i))\right) < \infty$ follows straight from the assumptions. Hence we have

$$\begin{aligned}
E_{F_n}\left(\left(E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))|\xi_i^*\right)\right)^2\right) &\to E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))\tilde{\tilde{g}}(w_0(\xi_i,\xi_k))|\right) \\
&= E_{F_0}\left(\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)\right)^2\right).
\end{aligned}$$

- For the second term, we can verify the condition for 2. when the indices are unique:

$$\begin{aligned}
E_{F_0}\left(\left|E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)^2\right|\right) &= E_{F_0}\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)^2\right) \\
&\leq E_{F_0}\left(E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))|\xi_i\right)\right) \\
&\overset{LIE}{=} E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))\right) < \infty,
\end{aligned}$$

where the inequality follows from Jensen's inequality. When the indices are repeated:

$$E_{F_0}\left(\left|E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_i))|\xi_i\right)^2\right|\right) = E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_i))\right) < \infty.$$

hence $E_{F_n}\left(\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*\right)\right)^2\right) \to E_{F_0}\left(\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)\right)^2\right)$.

- The third term can be rewritten as:

$$\begin{aligned}
&E_{F_n}\left(E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))|\xi_i^*\right) E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*\right)\right) \\
&\quad = E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*\right)\right).
\end{aligned}$$

Using Jensen's inequality, we verify the condition for 2. when the indices are unique:

$$
\begin{aligned}
&E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_k))|\xi_i\right)|\right)\\
&\quad\le E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\,E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_k))|\,|\xi_i\right)\right)\\
&\quad\overset{LIE}{=} E_{F_0}\left(E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\,|\xi_i\right)E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_k))|\,|\xi_i\right)\right)\\
&\quad= E_{F_0}\left(E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\,|\xi_i\right)^2\right)\\
&\quad\le E_{F_0}\left(E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))|\xi_i\right)\right)\\
&\quad\overset{LIE}{=} E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))\right)<\infty
\end{aligned}
$$

and using Jensen's and Cauchy-Schwarz inequalities we verify it when the indices are equal:

$$
\begin{aligned}
E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_i))E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)|\right) &\le E_{F_0}\left(E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_i))\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\,|\xi_i\right)\right)\\
&\overset{LIE}{=} E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_i))\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\right)\\
&\le \sqrt{E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))\right)E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_i))\right)}<\infty.
\end{aligned}
$$

hence

$$
\begin{aligned}
&E_{F_n}\left(E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))|\xi_i^*\right)E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*\right)\right)\\
&\quad\to E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)\right)=E_{F_0}\left(\left(E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)\right)^2\right)
\end{aligned}
$$

- For the fourth term we can verify that

$$
\begin{aligned}
E_{F_0}\left(|E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i\right)|\right) &\le \sqrt{E_{F_0}\left(E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))|\xi_i\right)\right)}\\
&\overset{LIE}{=} \sqrt{E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))\right)}<\infty,\\
E_{F_0}\left(|E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_i))|\xi_i\right)|\right) = E_{F_0}\left(|\tilde{\tilde{g}}(w_0(\xi_i,\xi_i))|\right) &\le \sqrt{E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_i))\right)}<\infty.
\end{aligned}
$$

hence $E_{F_n}\left(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j^*))\right)\to E_{F_0}\left(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))\right)$.

We combine all terms using continuous mapping theorem and see that they all cancel out and the limit is zero.

For T_3, we can write:

$$T_3 = 2\rho_n^{-\tau(g)}\sqrt{Var_{F_n}(E_{F_0}(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*))}$$
$$\times \sum_{i=1}^{n} \frac{E_{F_0}(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*) - E_{F_n}(E_{F_0}(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*))}{\sqrt{n}\sqrt{Var_{F_n}(E_{F_0}(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*))}}.$$

Denote the terms inside the sum by X_{in}. They have zero expectation:

$$E_{F_n}(X_{in}) = \frac{E_{F_n}(E_{F_0}(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*) - E_{F_n}(E_{F_0}(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*)))}{\sqrt{n}\sqrt{Var_{F_n}(E_{F_0}(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*))}} = 0.$$

Their variances sum to 1 for each n:

$$\sum_{i=1}^{n} Var_{F_n}(X_{in}) = n\frac{Var_{F_n}(E_{F_0}(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*))}{nVar_{F_n}(E_{F_0}(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*))} = 1.$$

And for all n when $i \neq j$ the terms X_{in} and X_{jn} are independent and identically distributed. Hence by Lindeberg-Levy CLT for triangular arrays their sum converges in distribution to a standard normal random variable.

For the multiplier term we have:

$$\begin{aligned}
\rho_n^{-2\tau(g)}Var_{F_n}(E_{F_0}(\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*)) &= Var_{F_n}\left(E_{F_0}\left(\rho_n^{-\tau(g)}\tilde{g}(h_{0,n}(\xi_i^*,\xi_j))|\xi_i^*\right)\right) \\
&= Var_{F_n}(E_{F_0}(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*)) + O(\rho_n) \\
&= E_{F_n}\left(E_{F_0}(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*)^2\right) - (E_{F_n}(E_{F_0}(\tilde{\tilde{g}}(w_0(\xi_i^*,\xi_j))|\xi_i^*)))^2 + O(\rho_n) \\
&\to E_{F_0}\left(E_{F_0}(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i)^2\right) - (E_{F_0}(E_{F_0}(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i)))^2 \\
&= Var_{F_0}(E_{F_0}(\tilde{\tilde{g}}(w_0(\xi_i,\xi_j))|\xi_i)) \equiv \sigma_1^2 < \infty.
\end{aligned}$$

The first equality is pulling the normalisation inside the variance. The second equality applies the definition og $\tilde{\tilde{g}}$. The third equality is rewriting variance in terms of expectations. The limit follows from 2. (we have already checked that the relevant absolute moments are finite when we were checking conditions for convergence of T_2, terms two and four) and the continuous mapping theorem. The final line is by definition. Hence $T_3 \xrightarrow{d} N(0, 4\sigma_1^2)$.

It remains to show that $\tilde{r}_n(h_n, F_n) = o_p(1)$. We can check that the expression is $\sqrt{n}$ times a U-statistic with a kernel function $G(\xi_i^*,\xi_j^*) = \rho_n^{-\tau(g)}\big(\tilde{g}(h_n(\xi_i^*,\xi_j^*)) - E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*))|\xi_i^*\right) - E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*))|\xi_j^*\right) + E_{F_n}\left(\tilde{g}(h_n(\xi_i^*,\xi_j^*))\right)\big)$. Note that $E(G(\xi_i^*,\xi_j^*)) = E(G(\xi_i^*,\xi_j^*)|\xi_i^*) = 0$, i.e. it is a degenerate U-statistic with $Var(E(G(\xi_i^*,\xi_j^*)|\xi_i^*)) = 0$. We could show that the whole term is negligible by convergence in second mean from definition, or rely on a Theorem from section 5.3.2

in Serfling (2009) which, in the present setting, can be stated as:

Lemma (Theorem 5.3.2 in Serfling (2009))**.** *If* $E_{F_n}\left(\left(G(\xi_i^*,\xi_j^*)\right)^2\right)<\infty$ *then*

$$E_{F_n}\left(\left(\tilde{r}_n(h_n,F_n)\right)^2\right)=O\left(\frac{1}{n}\right)=o(1).$$

By Jensen's inequality and the law of large numbers, $\rho_n^{-2\tau(g)}E_{F_n}\left(\tilde{g}^2\left(h_n\left(\xi_i^*,\xi_j^*\right)\right)\right)$ is an upper bound for all terms in the expansion of $E_{F_n}\left(\left(G(\xi_i^*,\xi_j^*)\right)^2\right)$. Hence the sufficient condition is implied by:

$$\begin{aligned}
\rho_n^{-2\tau(g)}E_{F_n}\left(\tilde{g}^2\left(h_n\left(\xi_i^*,\xi_j^*\right)\right)\right) &\leq 2\underbrace{E_{F_n}\left(\tilde{\tilde{g}}^2\left(w_0\left(\xi_i^*,\xi_j^*\right)\right)\right)}_{\to E_{F_0}\left(\tilde{\tilde{g}}^2(w_0(\xi_i,\xi_j))\right)<\infty}+O(\rho_n) \\
&+2\underbrace{\left(\sup_{h\in[0,M_w\rho_n]}\left|\frac{\tilde{g}'(h)}{\rho_n^{\tau(g)-1}}\right|\right)^2}_{<\infty}\underbrace{E_{F_n}\left(\left(\frac{1}{\rho_n}\left(h_n\left(\xi_i^*,\xi_j^*\right)-\hat{h}_n\left(\xi_i^*,\xi_j^*\right)\right)\right)^2\right)}_{\to 0} \\
&\leq 2E_{F_0}\left(\tilde{\tilde{g}}^2\left(w_0\left(\xi_i,\xi_j\right)\right)\right)+o(1).
\end{aligned}$$

Hence for any $\varepsilon>0$ we can find an N sufficiently large so that the condition is satisfied: $E_{F_n}\left(\left(G(\xi_i^*,\xi_j^*)\right)^2\right)<8E_{F_0}\left(\tilde{\tilde{g}}^2\left(w_0\left(\xi_i,\xi_j\right)\right)\right)+\varepsilon<\infty$ for all $n>N$.

Moving on to the second part of the proof, we check that the sequence $\left\{\hat{h}_n,\hat{F}_n\right\}_{n=1}^{\infty}$ satisfies assumptions 1. and 2. in probability:

1. Follows from Theorem 1:

$$\begin{aligned}
E_{\hat{F}_n}\left(\left(\frac{1}{\rho_n}\left(\hat{h}_n(\xi_i^*,\xi_j^*)-h_{0,n}(\xi_i^*,\xi_j^*)\right)\right)^2\right) &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{\rho_n}\left(\hat{h}_n(\xi_i,\xi_j)-h_{0,n}(\xi_i,\xi_j)\right)\right)^2 \\
&\leq\left(\max_{i,j}\left|\frac{\hat{h}_n(\xi_i,\xi_j)-h_{0,n}(\xi_i,\xi_j)}{\rho_n}\right|\right)^2 \\
&= o_p(1)^2=o_p(1).
\end{aligned}$$

2. Let $f:Supp(\xi)^3\to\mathbb{R}$ be any symmetric function for which $E_{F_0}\left(\left|f\left(\xi_i,\xi_j,\xi_k\right)\right|\right)<\infty$,

$E_{F_0}\left(\left|f\left(\xi_i,\xi_i,\xi_j\right)\right|\right)<\infty$ and $E_{F_0}\left(\left|f\left(\xi_i,\xi_i,\xi_i\right)\right|\right)<\infty$. We have:

$$\begin{aligned}
E_{\hat{F}_n}\left(f(\xi_i^*,\xi_j^*,\xi_k^*)\right) &= \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(\xi_i,\xi_j,\xi_k) \\
&= \underbrace{\frac{(n-1)(n-2)}{n^2}}_{\to 1}\underbrace{\frac{1}{\binom{n}{3}}\sum_{i<j<k} f(\xi_i,\xi_j,\xi_k)}_{\xrightarrow{a.s.} E_{F_0}(f(\xi_i,\xi_j,\xi_k))<\infty} + \underbrace{\frac{3(n-1)}{n^2}}_{\to 0}\underbrace{\frac{1}{\binom{n}{2}}\sum_{i<j} f(\xi_i,\xi_i,\xi_j)}_{\xrightarrow{a.s.} E_{F_0}(f(\xi_i,\xi_i,\xi_j))<\infty} \\
&\quad + \underbrace{\frac{1}{n^2}}_{\to 0}\underbrace{\frac{1}{n}\sum_{i=1}^{n} f(\xi_i,\xi_i,\xi_i)}_{\xrightarrow{a.s} E_{F_0}(f(\xi_i,\xi_i,\xi_i))<\infty} \\
&\xrightarrow{a.s.} E_{F_0}\left(f(\xi_i,\xi_j,\xi_k)\right)
\end{aligned}$$

The first equality follows from the definition of the empirical distribution function $\hat{F}_n$. The convergence of the two terms in the second line follows from the SLLN for U-statistics (see e.g. Theorem A. in section 5.4 of Serfling (2009), p.190) given that $E_{F_0}\left(\left|f\left(\xi_i,\xi_j,\xi_k\right)\right|\right)<\infty$ and $E_{F_0}\left(\left|f\left(\xi_i,\xi_i,\xi_j\right)\right|\right)<\infty$. The convergence of the term in the third line follows from Kolmogorov's SLLN for i.i.d. random variables which applies under the assumption that $E_{F_0}\left(\left|f\left(\xi_i,\xi_i,\xi_i\right)\right|\right)<\infty$. The final line is by continuous mapping theorem for almost sure convergence. Condition 2. holds almost surely (hence also in probability), but because we only get condition 1. in probability the overall result is for convergence weakly in probability.

□

The above result was stated for a normalisation using the unknown ρ_n. We now show that the conclusions remain true when we replace it with an estimate.

Corollary A.1. *Under the assumptions of Theorem A.2*

$$\hat{\rho}_n-\rho_n=o_p(\rho_n),$$

hence

$$f_n\left(A\left(\hat{h}_n(\xi^*),\eta^*\right),\hat{\rho}_n,\hat{F}_n\right)=f_n\left(A\left(\hat{h}_n(\xi^*),\eta^*\right),\rho_n,\hat{F}_n\right)+o_p(1)$$

and we get all conclusions of Theorem A.1 in probability with $J(t,w_0,F_0)=N(0,m^2\sigma_1^2)$ *for*

$$\begin{aligned}
&f_n\left(A\left(\hat{h}_n\left(\xi^*\right),\eta^*\right),\hat{\rho}_n,\hat{F}_n\right)\\
&\quad=\frac{\sqrt{n}}{\binom{n}{m}\hat{\rho}_n^{\tau(g)}}\sum_{1\le\iota_1<\cdots<\iota_m\le n}\left(g\left(A\left(\hat{h}_n\left(\xi^*\left(\iota\right)\right),\eta^*\left(\iota\right)\right)\right)-E_{\hat{h}_n,\hat{F}_n}\left(g\left(A\left(\hat{h}_n\left(\xi^*\left(\iota\right)\right),\eta^*\left(\iota\right)\right)\right)\right)\right).
\end{aligned}$$

Proof of Corollary A.1. Let ρ_n^* denote the density of a bootstrap adjacency matrix formed by $\xi^*\sim F_n$ with linking probabilities h_n. We can write

$$f_n\left(A\left(h_n\left(\xi^*\right),\eta^*\right),\rho_n^*,F_n\right)=\left(\frac{\rho_n}{\hat{\rho}_n^*}\right)^{\tau(g)}f_n\left(A\left(h_n\left(\xi^*\right),\eta^*\right),\rho_n,F_n\right)$$

hence it is sufficient to show $\rho_n^*-\rho_n=o_p(\rho_n)$ which, by Slutsky's theorem, implies $\left(\frac{\rho_n}{\rho_n^*}\right)^{\tau(g)}\xrightarrow{p}1$.

Applying Theorem 3 to $g(A_{ij})=A_{ij}$, for which $m=2$ and $\tau(g)=1$, we have:

$$\frac{\sqrt{n}}{\binom{n}{2}\rho_n}\sum_{1\le i^*<j^*\le n}\left(A_{i^*j^*}-E_{h_n,F_n}(A_{i^*j^*})\right)=O_p(1)$$

where the expression is bounded in probability because it has a well-defined limiting distribution.

Hence

$$\begin{aligned}
\rho_n^*&=\frac{1}{\binom{n}{2}}\sum_{1\le i^*<j^*\le n}A_{i^*j^*}\\
&=E_{h_n,F_n}(A_{i^*j^*})+\underbrace{\frac{\rho_n}{\sqrt{n}}O_p(1)}_{=\frac{1}{\sqrt{n}}O_p(\rho_n)=o_p(\rho_n)}\\
&=E_{F_n}\left(E_{h_n}\left(A_{i^*j^*}\middle|\,\xi^*\right)\right)+o_p(\rho_n)\\
&=E_{F_n}\left(E_{h_n}\left(h_n(\xi_i^*,\xi_j^*)\middle|\,\xi^*\right)\right)+o_p(\rho_n)\\
&=\rho_n\left(E_{F_n}\left(\frac{1}{\rho_n}\left(h_n(\xi_i^*,\xi_j^*)-h_{0,n}(\xi_i^*,\xi_j^*)\right)\right)+E_{F_n}\left(\frac{1}{\rho_n}h_{0,n}(\xi_i^*,\xi_j^*)\right)\right)+o_p(\rho_n)\\
&\le\rho_n\left(\sqrt{E_{F_n}\left(\left(\frac{1}{\rho_n}\left(h_n(\xi_i^*,\xi_j^*)-h_{0,n}(\xi_i^*,\xi_j^*)\right)\right)^2\right)}+E_{F_n}\left(w_0(\xi_i^*,\xi_j^*)\right)\right)+o_p(\rho_n)\\
&=\rho_n\left(o(1)+E_{F_0}\left(w_0\left(\xi_i,\xi_j\right)+o(1)\right)\right)+o_p(\rho_n)\\
&=\rho_n(o(1)+1+o(1))+o_p(\rho_n)\\
&=\rho_n+o_p(\rho_n)
\end{aligned}$$

The first equality is by definition, the second follows from the above expression and result from Theorem 3. The third equality uses the law of iterated expectations. The fourth equality is by definition of A. For the fifth equality we add and subtract $E_{F_n}\left(h_{0,n}(\xi_i^*,\xi_j^*)\right)$ and pull ρ_n out of the bracket. The inequality is due to Jensen's inequality where the final term is transformed according to the definition of w_0. The sixth equality uses assumptions 1. and 2.. The seventh equality is due to the definition of w_0 which is assumed to integrate to 1.

The above derivation applies to the case of $\rho_n^* = \hat{\rho}_n$ (for $h_n = h_{0,n}$ and $f_n = F_0$), proving that $f_n\left(A\left(h_{0,n}\left(\xi\right),\eta\right),\hat{\rho}_n,F_0\right)$ and $f_n\left(A\left(h_{0,n}\left(\xi\right),\eta\right),\rho_n,F_0\right)$ have the same asymptotic limit.

If we replaces ρ_n^* with $\hat{\rho}_n^*$ the $o(1)$ terms in the derivation are replaced by $o_p(1)$, which does not affect the overall result. Hence we also get the same limit of $f_n\left(A\left(\hat{h}_n\left(\xi^*\right),\eta^*\right),\hat{\rho}_n^*,\hat{F}_n\right)$. □

We note that all conclusions of Theorem 3, Lemma 2 and Corollary 1 follow from Theorem A.2 and Corollary A.1, hence they have also been proven.

A.3 Useful results

For reference, we list some results which we use in our proofs:

Theorem (Bernstein's inequality for bounded random variables[33]). *Let $Z_1,\ldots,Z_n$ be independent random variables. Assume that there exist some positive constant M such that $|Z_i| \le M$ with probability one for each i. Let also $\sigma^2 = \frac{1}{n}\sum_{i=1}^n V(Z_i)$. Then, for all $\varepsilon > 0$:*

$$P\left(\left|\frac{1}{n}\sum_{i=1}^n (Z_i - E(Z_i))\right| \ge \varepsilon\right) \le 2\exp\left(-\frac{n\varepsilon^2}{2\left(\sigma^2 + \frac{1}{3}M\varepsilon\right)}\right). \tag{44}$$

B Appendix: additional tables, plots, codes

B.1 Codes

In this section we present some of the codes used for simulations. A full package should eventually become available online.

We start with the definitions of different distances and estimators of the linking functions:

```
def D2(A):
    #a function which maps A into a matrix of distances D2
    n= len(A)
    V=(1/n)*np.matmul(A,A)
    C=(1/n)*np.matmul(V,V)
    B=np.matmul(np.diag(np.diag(C)),np.ones((n,n)))
    D = B+B.T-2*C
    return D
```

33. Copied after Zeleneev (2020).

```
def Dmax(A):
    #a function which maps A into a matrix of distances Dmax
    n = len(A)
    V = (1/n)*np.matmul(A,A)
    F = torch.tensor(np.tensordot(np.ones(n),V,0))
    G = torch.transpose(F, 1,0)
    H = F-G
    J=1-torch.transpose(np.fmax(torch.eye(n).repeat(n, 1, 1),np.tensordot(np.eye(n)
                                                ,np.ones(n),0)),2,1)
    D = np.array(torch.amax(np.fmin(abs(H),J), dim=2))
    return D
```

```
def HK1h(D,A,h):
    # gives a kernel approximation to the linking function based on a one-way
                                                normal kernel, with bandwidth h,
                                                based on distance D
    n= len(A)
    K = np.exp(-0.5*(D/h)**2)
    K[np.isnan(K)] = 0
    T=np.matmul(K,A)
    B=np.matmul(K,np.ones((n,n)))-K
    H = T/B
    H=(H+np.transpose(H))/2
    return H
```

```
def HK2h(D,A,h):
    # gives a kernel approximation to the linking function based on a two-way
                                                normal kernel, with bandwidth h,
                                                based on distance D
    n= len(A)
    K = np.exp(-0.5*(D/h)**2)
    K[np.isnan(K)] = 0
    H=np.matmul(K,np.matmul(A,K))/np.matmul(K,np.matmul(1-np.eye(n),K))
    return H
```

```
def HNN1(D,A):
    #gives a kernel approximation to h based on one-way nearest neighbours, with
                                                bandwidth h, based on distance D
    #uses the optimal neighbourhood size (n log(n))^{1/2}
    n= len(A)
    N_size = round(np.sqrt(n*np.log(n)))
    N=np.argpartition(D, N_size+1)[:,:N_size+1]
```

```
    mask = np.ones((n,N_size+1), dtype=bool)
    mask[range(n), np.argmax(N==np.array(range(n)).reshape(n,1), axis=1)] = False
    N = N[mask].reshape(n, N_size)
    mask2 = np.zeros((n,n), dtype=bool)
    mask2[np.tile(np.array(range(n)).reshape(n,1),N_size), N] = True
    mask_long=np.tile(mask2,(n,1))
    A_long=np.tile(A,(1,n)).reshape(n*n,n)
    Amlong = A_long[mask_long].reshape(n,n, N_size)
    H=np.sum(Amlong,2)/N_size
    H=(H+np.transpose(H))/2
    return H
```

In simulations we generate the true matrices using one of the following functions:

```
def high_rho_generate(n, r, rep):
    #generates a rep number of true n by n matrices from the high rho function with
                                                density r/1.35, outputs only the
                                               adjacency matrices
    A_true = []
    for s in range(rep):
        w = np.random.uniform(0,1,(n))
        u = np.random.uniform(0,1,(n,n))
        eta = np.tril(u) + np.tril(u, -1).T
        Wi = np.tensordot(w,np.ones(n),0)
        Wj = np.tensordot(np.ones(n),w,0)
        A = (eta < r*(1-((abs(0.5-Wi)<0.05) & (abs(0.5-Wj)<0.05)))*(1-0.5*(abs(0.5-
                                                    Wi)+abs(0.5-Wj)))/(0.975))*1
        np.fill_diagonal(A, 0)
        A_true.append(A)
    return A_true
```

```
def horse_generate(n, r, rep):
    #generates a rep number of true n by n matrices from the horseshoe function
                                               with density r/4.44, outputs only the
                                                adjacency matrices
    A_true = []
    for s in range(rep):
        w = np.random.uniform(0,1,(n))
        u = np.random.uniform(0,1,(n,n))
        eta = np.tril(u) + np.tril(u, -1).T
        Wi = np.tensordot(w,np.ones(n),0)
        Wj = np.tensordot(np.ones(n),w,0)
        A = (eta < r*((np.exp(-200*(Wi-Wj**2)**2)+np.exp(-200*(Wj-Wi**2)**2))/2))*1
        np.fill_diagonal(A, 0)
```

```
        A_true.append(A)
    return A_true
```

```
def product_generate_A_h_xi(n, r, rep):
    #generates a rep number of true n by n matrices from the product function with
                                                density r/4, outputs the adjacency
                                                matrices, the true linking function,
                                                and the true values of the underlying
                                                 characteristics $\xi_i$
    A_true = []
    xi_true = []
    h_true = []
    for s in range(rep):
        w = np.random.uniform(0,1,(n))
        u = np.random.uniform(0,1,(n,n))
        eta = np.tril(u) + np.tril(u, -1).T
        Wi = np.tensordot(w,np.ones(n),0)
        Wj = np.tensordot(np.ones(n),w,0)
        h = r*Wi*Wj
        A = (eta < h)*1
        np.fill_diagonal(A, 0)
        np.fill_diagonal(h, 0)
        A_true.append(A)
        h_true.append(h)
        xi_true.append(list(w))
    return (A_true, h_true, np.array(xi_true))
```

Code for running bootstrap and finding the optimal bandwidth:

```
    def boot_HK1h(A,h,B):
    #outputs B bootstrapped adjacency matrices based on matrix A with bandwidth h
                                                using linking function estimate HK1
    n=len(A)
    H_true = HK1h(D2(A),A,h)
    #choose nodes for bootstrap villages:
    v = np.random.randint(0, n, size=(B,n))
    #generate new adjacency matrices
    row = np.tensordot(v,np.ones(n),0).astype(int)
    column = np.tile(np.array(v),n).reshape(B,n,n)
    G = H_true[row,column]
    u = np.random.rand(B, n, n)
    m = np.tril(u) + np.transpose(np.tril(u, -1),[0,2,1])
    A_boot = (m < G)*1
    [np.fill_diagonal(A_boot[i], 0) for i in range(B)]
```

```
    return A_boot
```

```
def HK1h_loo(D,A,h):
    #gives a leave-obe-out kernel approximation to h based on a one-way normal
                                                    kernel, with bandwidth h, based on
                                                    distance D
    n= len(A)
    K = np.exp(-0.5*(D/h)**2)
    K[np.isnan(K)] = 0
    T = np.matmul(K,A) - np.matmul(np.diag(np.diag(K)),np.ones((n,n)))*A
    B = np.matmul(K,np.ones((n,n)))-(K-np.diag(np.diag(K)))-np.matmul(np.diag(np.
                                                    diag(K)),np.ones((n,n)))
    H = T/B
    H=(H+np.transpose(H))/2
    return H
```

```
def log_likelihood(A,H):
    #the log-likelihood estimation for an adjacency matrix A under the assumption
                                                    it comes from a distribution with
                                                    linking probabilities in H
    log_likelihood = np.sum(A*np.log(H)+(1-A)*np.log(1-H))
    return log_likelihood
```

```
def ll(h, A):
    #the leave-one-out log-likelihood objective function for use in minimising
                                                    procedures
    return -log_likelihood(A,HK1h_loo(D2(A),A,h))
```

A sample Monte Carlo simulation code using the above definitions and the WARP procedure from Giacomini, Politis, and White (2013) to obtain the confidence interval coverage:

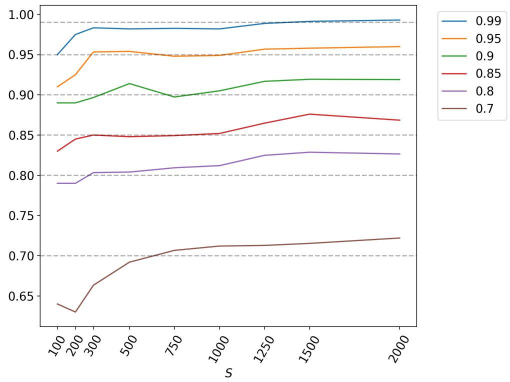
```
#define the output data frame:
df_loo = pd.DataFrame(columns=['S', 'B', 'n', 'rho','average for true graphs', '
                                                true value', 'alpha', 'proportion of
                                                bootstrap CI that cover truth', 'average
                                                length of bootstrap CI', 'statistic','h']
                                                )


#run the simulations:
ALPHA = [0.01, 0.05, 0.1, 0.15, 0.2, 0.3] #sizes of confidence intervals
NN = [25, 50, 100, 150, 200, 300] #sample sizes
SS = [1000] #number of true graphs
RR = [1, 0.75,0.5,0.25,0.1] #sparsity level
```

```
for n in NN:
    for r in RR:
        S_max = max(SS)
        (A_true, h_true, xi_true) = product_generate_A_h_xi(n, r, S_max)
        A_boot = []
        h_list = []
        B=1 #because of WARP we only need one bootstrap replication

        true_density = r*0.25

        for A in A_true[0:min(len(A_true),S_max)]:
            #find the optimal bandwidth by minimising the leave-one-out log-
                                                    likelihood
            h_guess = 0.2090189845643738*true_density**1.38258532*n**(-1.55268817)*
                                                    np.log(n)**1.82661653
            res = minimize(ll, h_guess, args=A, method = 'Nelder-Mead', tol=1e-7,
                                                    bounds=((0,1.1),))
            h = res.x[0]
            #do bootstrap for matrix A using the optimal bandwidth
            Ab = boot_HK1h(A,h,B)
            #save the bootstrapped adjacency matrices and the bandwidth
            A_boot.append(Ab)
            h_list.append(h)

        #estimate the statistic of interest for true and bootstrapped graphs
        true_density_all = [nx.density(nx.from_numpy_array(A_true[s])) for s in
                                                range(S_max)]]
        true_density_mean = np.mean(true_density_all)
        boot_density_all = [nx.density(nx.from_numpy_array(A_boot[s][0])) for s in
                                                range(S_max)]

        #find the confidence interval coverage using WARP
        for S in SS:
            if (S<=len(A_true)):
                true_density_vec = true_density_all[0:S]
                boot_density = boot_density_all[0:S]
                stat = 'density'
                density_minus_true = np.array(boot_density) - np.array(
                                                        true_density_vec)
                for alpha in ALPHA:
                    qu= np.percentile(density_minus_true, 100*(1-alpha/2))
```

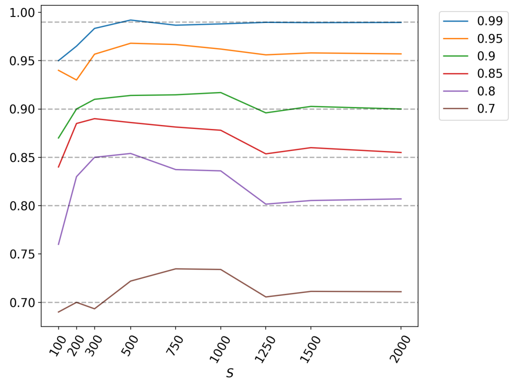
```
            ql= np.percentile(density_minus_true, 100*(alpha/2))
            bl = true_density_vec - qu
            bu = true_density_vec - ql
            co= np.mean([bl[i] <= true_density <= bu[i] for i in range(S)])
            me = np.mean(bu-bl)
            df_loo = df_loo._append({'S': S, "B": B, 'n': n, "rho": r, '
                                                    average for true
                                                    graphs':
                                                    true_density_mean, '
                                                    true value':
                                                    true_density, 'alpha'
                                                    : alpha, 'proportion
                                                    of bootstrap CI that
                                                    cover truth': co, '
                                                    average length of
                                                    bootstrap CI': me, '
                                                    statistic': stat, 'h'
                                                    : np.mean(h_list)},
                                                    ignore_index = True)

    #save the output
    df_loo.to_csv('df_loo_product_n_25_300_true_dens.csv')
```

B.2 Monte Carlo simulations: tables and a sensitivity check

Since we are using the WARP procedure instead of traditional Monte Carlo simulations, we test its sensitivity by checking the effect of varying the number of simulated true graphs S rather than the number of bootstrap replications B, which is always kept at $B = 1$. Fig. 11a shows that the predictions for different statistics stabilise above S around 750 or higher. This is true in most simulations (see Table 4, Table 5), with the exception of networks with high density such as that in Fig. 11b (and Table 6) in which the predictions don't stabilise until $S = 1250$ or even $S = 1500$. In all other sections we use $S = 1000$. Running more repetitions is computationally expensive and provides little advantage in terms of accuracy in the majority of cases.



(a) Confidence interval coverage for density using the product generating function at $n = 500$ and $\rho_n = 0.1875$.



(b) Confidence interval coverage for transitivity using the high density generating function at $n = 500$ and $\rho_n = 0.759$.

Figure 11: Confidence interval coverage for different number of simulated true graphs S based on Monte Carlo simulations.

			average for	Proportion of bootstrap CI that cover truth for					
n	statistic	ρ_n	true graphs	α =0.01	α =0.05	α =0.1	α =0.15	α =0.2	α =0.3
500	λ_1	0.024	17.116	1.000	0.971	0.925	0.858	0.804	0.660
		0.078	52.502	0.998	0.961	0.900	0.854	0.811	0.706
		0.139	93.282	0.997	0.956	0.884	0.844	0.799	0.685
		0.250	166.453	0.995	0.954	0.887	0.825	0.778	0.692
	λ_3	0.024	7.823	0.005	0.000	0.000	0.000	0.000	0.000
		0.078	12.825	0.000	0.000	0.000	0.000	0.000	0.000
		0.139	15.582	0.000	0.000	0.000	0.000	0.000	0.000
		0.250	17.042	0.000	0.000	0.000	0.000	0.000	0.000
	λ_{10}	0.024	7.022	0.997	0.985	0.943	0.876	0.803	0.596
		0.078	11.627	0.391	0.058	0.018	0.008	0.004	0.002
		0.139	14.197	0.082	0.000	0.000	0.000	0.000	0.000
		0.250	15.667	0.000	0.000	0.000	0.000	0.000	0.000
	Louvain CDA modularity	0.024	0.243	0.000	0.000	0.000	0.000	0.000	0.000
		0.078	0.119	0.001	0.000	0.000	0.000	0.000	0.000
		0.139	0.082	0.015	0.001	0.001	0.001	0.000	0.000
		0.250	0.052	0.198	0.061	0.021	0.008	0.006	0.002
	density	0.024	0.024	0.405	0.177	0.096	0.059	0.044	0.024
		0.078	0.078	0.989	0.909	0.811	0.752	0.699	0.620
		0.139	0.139	0.993	0.955	0.889	0.827	0.771	0.672
		0.250	0.250	0.992	0.953	0.897	0.839	0.788	0.659
	max betweenness centrality	0.024	0.019	0.304	0.058	0.033	0.023	0.013	0.010
		0.078	0.011	1.000	0.965	0.809	0.692	0.591	0.439
		0.139	0.009	1.000	0.995	0.981	0.923	0.847	0.668
		0.250	0.008	1.000	0.997	0.969	0.901	0.835	0.713
	transitivity	0.024	0.043	1.000	1.000	1.000	0.997	0.993	0.952
		0.078	0.138	1.000	0.967	0.928	0.899	0.864	0.774
		0.139	0.247	0.985	0.954	0.903	0.859	0.818	0.719
		0.250	0.443	0.991	0.947	0.890	0.835	0.779	0.672
	triangle density	0.024	0.000	0.990	0.852	0.706	0.563	0.456	0.309
		0.078	0.001	0.998	0.965	0.905	0.845	0.798	0.701
		0.139	0.006	0.991	0.959	0.902	0.840	0.794	0.692
		0.250	0.037	0.992	0.956	0.882	0.828	0.779	0.678
1000	λ_1	0.013	18.974	0.894	0.677	0.470	0.379	0.270	0.157
		0.058	77.998	0.991	0.951	0.886	0.847	0.812	0.715
		0.120	160.817	0.988	0.959	0.911	0.859	0.814	0.749
		0.250	332.967	0.987	0.959	0.904	0.852	0.810	0.747
	λ_3	0.013	8.573	0.000	0.000	0.000	0.000	0.000	0.000
		0.058	16.482	0.000	0.000	0.000	0.000	0.000	0.000
		0.120	21.588	0.000	0.000	0.000	0.000	0.000	0.000
		0.250	24.642	0.000	0.000	0.000	0.000	0.000	0.000
	λ_{10}	0.013	8.002	0.044	0.000	0.000	0.000	0.000	0.000
		0.058	15.493	0.000	0.000	0.000	0.000	0.000	0.000
		0.120	20.377	0.000	0.000	0.000	0.000	0.000	0.000
		0.250	23.414	0.000	0.000	0.000	0.000	0.000	0.000
	Louvain CDA modularity	0.013	0.234	0.000	0.000	0.000	0.000	0.000	0.000
		0.058	0.100	0.000	0.000	0.000	0.000	0.000	0.000
		0.120	0.064	0.000	0.000	0.000	0.000	0.000	0.000
		0.250	0.037	0.000	0.000	0.000	0.000	0.000	0.000
	density	0.013	0.013	0.000	0.000	0.000	0.000	0.000	0.000
		0.058	0.058	0.937	0.849	0.777	0.705	0.669	0.576
		0.120	0.120	0.984	0.943	0.892	0.836	0.800	0.725
		0.250	0.250	0.992	0.953	0.896	0.846	0.800	0.732
	max betweenness centrality	0.013	0.011	0.101	0.010	0.006	0.004	0.003	0.001
		0.058	0.006	1.000	0.960	0.872	0.776	0.690	0.531
		0.120	0.005	0.998	0.994	0.975	0.957	0.916	0.775
		0.250	0.004	1.000	0.993	0.982	0.947	0.899	0.791
	transitivity	0.013	0.024	1.000	1.000	0.998	0.982	0.955	0.880
		0.058	0.103	0.997	0.980	0.942	0.906	0.864	0.778
		0.120	0.214	0.983	0.952	0.913	0.868	0.835	0.733
		0.250	0.444	0.988	0.959	0.922	0.880	0.827	0.716
	triangle density	0.013	0.000	0.385	0.144	0.064	0.040	0.027	0.011
		0.058	0.000	0.991	0.946	0.884	0.846	0.804	0.714
		0.120	0.004	0.988	0.967	0.921	0.857	0.809	0.745
		0.250	0.037	0.984	0.966	0.910	0.856	0.807	0.746

Table 1: Confidence interval coverage for different densities based on Monte Carlo simulations using the product generating function when $S = 1000$.

n	statistic	ρ_n	average for true graphs	Proportion of bootstrap CI that cover truth for α =0.01	α =0.05	α =0.1	α =0.15	α =0.2	α =0.3
250	λ_1	0.127	33.515	0.993	0.961	0.885	0.805	0.744	0.614
		0.307	79.477	0.995	0.965	0.932	0.892	0.851	0.754
		0.476	122.926	0.992	0.976	0.938	0.901	0.845	0.748
		0.740	190.475	0.989	0.970	0.932	0.886	0.838	0.732
	λ_3	0.127	10.044	0.000	0.000	0.000	0.000	0.000	0.000
		0.307	13.573	0.000	0.000	0.000	0.000	0.000	0.000
		0.476	14.350	0.000	0.000	0.000	0.000	0.000	0.000
		0.740	12.038	0.608	0.107	0.020	0.004	0.000	0.000
	λ_{10}	0.127	8.891	0.000	0.000	0.000	0.000	0.000	0.000
		0.307	12.043	0.000	0.000	0.000	0.000	0.000	0.000
		0.476	12.728	0.000	0.000	0.000	0.000	0.000	0.000
		0.740	10.505	0.996	0.972	0.947	0.886	0.839	0.722
	Louvain CDA modularity	0.127	0.137	0.000	0.000	0.000	0.000	0.000	0.000
		0.307	0.074	0.000	0.000	0.000	0.000	0.000	0.000
		0.476	0.049	0.000	0.000	0.000	0.000	0.000	0.000
		0.740	0.024	0.981	0.900	0.787	0.716	0.610	0.440
	density	0.130	0.130	1.000	1.000	0.991	0.977	0.959	0.885
		0.314	0.314	0.998	0.986	0.964	0.930	0.886	0.802
		0.488	0.488	0.996	0.980	0.946	0.902	0.858	0.744
		0.759	0.759	0.993	0.968	0.927	0.887	0.840	0.741
	max betweenness centrality	0.127	0.009	0.750	0.271	0.119	0.079	0.067	0.040
		0.307	0.005	1.000	0.813	0.605	0.466	0.390	0.249
		0.476	0.003	0.997	0.852	0.761	0.639	0.533	0.379
		0.740	0.001	0.994	0.930	0.882	0.821	0.766	0.651
	transitivity	0.127	0.132	0.999	0.990	0.971	0.923	0.886	0.771
		0.307	0.318	0.995	0.970	0.931	0.893	0.856	0.749
		0.476	0.494	0.992	0.976	0.944	0.907	0.857	0.745
		0.740	0.768	0.993	0.970	0.943	0.884	0.824	0.728
	triangle density	0.127	0.002	0.999	0.991	0.968	0.910	0.844	0.769
		0.307	0.032	0.995	0.975	0.948	0.917	0.871	0.783
		0.476	0.119	0.994	0.981	0.944	0.903	0.854	0.755
		0.740	0.446	0.989	0.971	0.934	0.893	0.836	0.731
500	λ_1	0.071	37.783	0.966	0.846	0.712	0.602	0.504	0.325
		0.230	119.330	0.998	0.977	0.933	0.907	0.859	0.787
		0.413	213.258	0.999	0.980	0.947	0.904	0.864	0.781
		0.740	381.727	0.999	0.960	0.926	0.879	0.833	0.752
	λ_3	0.071	11.362	0.000	0.000	0.000	0.000	0.000	0.000
		0.230	18.167	0.000	0.000	0.000	0.000	0.000	0.000
		0.413	20.806	0.000	0.000	0.000	0.000	0.000	0.000
		0.740	17.709	0.000	0.000	0.000	0.000	0.000	0.000
	λ_{10}	0.071	10.551	0.000	0.000	0.000	0.000	0.000	0.000
		0.230	16.902	0.000	0.000	0.000	0.000	0.000	0.000
		0.413	19.377	0.000	0.000	0.000	0.000	0.000	0.000
		0.740	16.336	0.000	0.000	0.000	0.000	0.000	0.000
	Louvain CDA modularity	0.071	0.135	0.000	0.000	0.000	0.000	0.000	0.000
		0.230	0.066	0.000	0.000	0.000	0.000	0.000	0.000
		0.413	0.041	0.000	0.000	0.000	0.000	0.000	0.000
		0.740	0.018	0.671	0.153	0.038	0.019	0.005	0.002
	density	0.073	0.073	1.000	0.998	0.990	0.975	0.946	0.882
		0.236	0.236	0.999	0.992	0.977	0.938	0.905	0.839
		0.423	0.423	0.999	0.984	0.953	0.915	0.885	0.791
		0.759	0.759	0.998	0.965	0.921	0.883	0.845	0.755
	max betweenness centrality	0.071	0.005	0.097	0.031	0.012	0.009	0.006	0.002
		0.230	0.003	0.959	0.643	0.462	0.352	0.269	0.190
		0.413	0.002	0.995	0.902	0.764	0.630	0.517	0.344
		0.740	0.001	0.988	0.943	0.891	0.846	0.791	0.671
	transitivity	0.071	0.074	1.000	0.999	0.979	0.953	0.887	0.772
		0.230	0.239	0.998	0.987	0.952	0.906	0.872	0.792
		0.413	0.428	0.999	0.986	0.959	0.921	0.882	0.808
		0.740	0.769	0.997	0.962	0.930	0.897	0.835	0.754
	triangle density	0.071	0.000	0.998	0.977	0.911	0.849	0.809	0.668
		0.230	0.013	0.998	0.988	0.957	0.927	0.890	0.811
		0.413	0.077	0.999	0.983	0.956	0.920	0.862	0.797
		0.740	0.446	0.999	0.957	0.928	0.885	0.833	0.754

Table 2: Confidence interval coverage for different densities based on Monte Carlo simulations using the high density generating function when $S = 1000$.

n	statistic	ρ_n	average for true graphs	Proportion of bootstrap CI that cover truth for $\alpha=0.01$	$\alpha=0.05$	$\alpha=0.1$	$\alpha=0.15$	$\alpha=0.2$	$\alpha=0.3$
750	λ_1	0.008	7.696	0.981	0.899	0.700	0.563	0.446	0.311
		0.029	25.866	0.991	0.936	0.890	0.841	0.786	0.673
		0.058	49.481	0.985	0.944	0.880	0.828	0.794	0.696
		0.113	95.648	0.983	0.946	0.897	0.829	0.779	0.681
	λ_3	0.008	5.338	0.000	0.000	0.000	0.000	0.000	0.000
		0.029	12.664	0.758	0.567	0.431	0.333	0.279	0.209
		0.058	22.937	0.979	0.898	0.825	0.764	0.709	0.619
		0.113	43.090	0.984	0.902	0.833	0.759	0.705	0.610
	λ_{10}	0.008	4.773	0.698	0.171	0.052	0.017	0.012	0.004
		0.029	8.859	0.000	0.000	0.000	0.000	0.000	0.000
		0.058	11.656	0.000	0.000	0.000	0.000	0.000	0.000
		0.113	14.127	0.989	0.937	0.876	0.831	0.793	0.701
	Louvain CDA modularity	0.008	0.436	0.000	0.000	0.000	0.000	0.000	0.000
		0.029	0.323	0.891	0.806	0.712	0.646	0.585	0.494
		0.058	0.307	0.974	0.915	0.836	0.756	0.689	0.589
		0.113	0.294	0.988	0.948	0.903	0.842	0.754	0.673
	density	0.008	0.008	0.007	0.000	0.000	0.000	0.000	0.000
		0.029	0.029	0.952	0.719	0.569	0.413	0.317	0.203
		0.058	0.058	0.980	0.932	0.876	0.821	0.758	0.678
		0.113	0.113	0.993	0.948	0.905	0.855	0.802	0.710
	max betweenness centrality	0.008	0.023	0.806	0.334	0.176	0.125	0.094	0.062
		0.029	0.008	1.000	0.990	0.885	0.797	0.692	0.508
		0.058	0.006	1.000	0.969	0.869	0.713	0.586	0.464
		0.113	0.005	1.000	0.996	0.960	0.890	0.811	0.665
	transitivity	0.008	0.010	1.000	1.000	1.000	0.998	0.995	0.973
		0.029	0.039	0.987	0.946	0.901	0.848	0.794	0.714
		0.058	0.076	0.983	0.945	0.884	0.840	0.809	0.699
		0.113	0.148	0.976	0.926	0.870	0.824	0.782	0.668
	triangle density	0.008	0.000	1.000	0.998	0.992	0.972	0.935	0.869
		0.029	0.000	0.998	0.961	0.920	0.883	0.835	0.743
		0.058	0.000	0.992	0.930	0.894	0.846	0.800	0.709
		0.113	0.002	0.976	0.930	0.876	0.824	0.771	0.682
1000	λ_1	0.006	8.002	0.977	0.788	0.611	0.472	0.332	0.204
		0.026	30.381	0.998	0.941	0.894	0.844	0.787	0.689
		0.054	61.907	0.998	0.942	0.886	0.831	0.792	0.694
		0.113	127.414	0.996	0.950	0.900	0.844	0.781	0.672
	λ_3	0.006	5.528	0.000	0.000	0.000	0.000	0.000	0.000
		0.026	14.699	0.903	0.682	0.563	0.461	0.416	0.333
		0.054	28.486	0.986	0.947	0.891	0.846	0.789	0.678
		0.113	57.194	0.989	0.934	0.887	0.836	0.776	0.694
	λ_{10}	0.006	5.000	0.141	0.001	0.000	0.000	0.000	0.000
		0.026	9.885	0.000	0.000	0.000	0.000	0.000	0.000
		0.054	13.380	0.000	0.000	0.000	0.000	0.000	0.000
		0.113	16.554	0.995	0.972	0.914	0.848	0.801	0.697
	Louvain CDA modularity	0.006	0.425	0.000	0.000	0.000	0.000	0.000	0.000
		0.026	0.319	0.958	0.898	0.834	0.786	0.731	0.604
		0.054	0.302	0.971	0.881	0.804	0.747	0.643	0.557
		0.113	0.291	0.998	0.969	0.924	0.875	0.834	0.707
	density	0.006	0.006	0.000	0.000	0.000	0.000	0.000	0.000
		0.026	0.026	0.810	0.537	0.366	0.285	0.173	0.101
		0.054	0.054	0.980	0.895	0.828	0.776	0.735	0.605
		0.113	0.113	0.988	0.966	0.909	0.869	0.793	0.690
	max betweenness centrality	0.006	0.018	0.658	0.210	0.104	0.066	0.045	0.028
		0.026	0.006	1.000	0.995	0.897	0.776	0.697	0.507
		0.054	0.004	1.000	0.936	0.765	0.643	0.522	0.370
		0.113	0.003	1.000	0.997	0.918	0.874	0.843	0.706
	transitivity	0.006	0.008	1.000	1.000	1.000	0.998	0.993	0.968
		0.026	0.034	0.993	0.957	0.898	0.846	0.794	0.685
		0.054	0.071	0.991	0.945	0.888	0.815	0.763	0.654
		0.113	0.147	0.992	0.944	0.868	0.800	0.747	0.625
	triangle density	0.006	0.000	1.000	0.996	0.974	0.946	0.906	0.800
		0.026	0.000	0.998	0.980	0.937	0.872	0.818	0.716
		0.054	0.000	0.993	0.957	0.875	0.832	0.765	0.649
		0.113	0.002	0.994	0.943	0.886	0.805	0.737	0.606

Table 3: Confidence interval coverage for different densities based on Monte Carlo simulations using the horseshoe generating function when $S = 1000$.

n	average $a^{(opt)}$	statistic	average for true graphs	S	Proportion of bootstrap CI that cover truth for α =0.01	α =0.05	α =0.1	α =0.15	α =0.2	α =0.3
300	0.000105	density	0.187	100	1.000	0.990	0.930	0.880	0.880	0.760
				200	0.995	0.950	0.910	0.870	0.840	0.770
				300	0.993	0.947	0.910	0.850	0.803	0.740
				500	0.996	0.950	0.916	0.854	0.806	0.726
				750	0.996	0.943	0.897	0.835	0.795	0.709
				1000	0.996	0.939	0.897	0.847	0.812	0.712
				1250	0.993	0.941	0.902	0.845	0.811	0.714
				1500	0.995	0.945	0.901	0.846	0.807	0.713
				2000	0.995	0.949	0.900	0.849	0.809	0.719
		λ_1	75.206	100	1.000	0.990	0.920	0.890	0.840	0.700
				200	0.995	0.955	0.905	0.870	0.815	0.730
				300	0.993	0.950	0.890	0.853	0.820	0.733
				500	0.996	0.944	0.894	0.860	0.824	0.722
				750	0.996	0.941	0.892	0.843	0.819	0.719
				1000	0.994	0.942	0.898	0.843	0.813	0.725
				1250	0.993	0.941	0.897	0.850	0.818	0.721
				1500	0.993	0.949	0.899	0.853	0.815	0.724
				2000	0.995	0.952	0.899	0.851	0.815	0.719
		transitivity	0.332	100	1.000	0.980	0.940	0.890	0.810	0.700
				200	1.000	0.965	0.925	0.890	0.825	0.680
				300	0.997	0.947	0.917	0.870	0.817	0.700
				500	0.992	0.946	0.896	0.836	0.796	0.702
				750	0.991	0.949	0.904	0.841	0.815	0.715
				1000	0.990	0.946	0.888	0.847	0.812	0.712
				1250	0.990	0.946	0.890	0.839	0.806	0.712
				1500	0.989	0.946	0.893	0.845	0.809	0.718
				2000	0.990	0.949	0.905	0.848	0.805	0.712
500	0.000059	density	0.188	100	0.950	0.910	0.890	0.830	0.790	0.640
				200	0.975	0.925	0.890	0.845	0.790	0.630
				300	0.983	0.953	0.897	0.850	0.803	0.663
				500	0.982	0.954	0.914	0.848	0.804	0.692
				750	0.983	0.948	0.897	0.849	0.809	0.707
				1000	0.982	0.949	0.905	0.852	0.812	0.712
				1250	0.989	0.957	0.917	0.865	0.825	0.713
				1500	0.991	0.958	0.919	0.876	0.829	0.715
				2000	0.993	0.960	0.919	0.869	0.827	0.722
		λ_1	125.416	100	0.940	0.910	0.860	0.790	0.780	0.660
				200	0.990	0.920	0.890	0.840	0.790	0.670
				300	0.990	0.947	0.910	0.867	0.817	0.670
				500	0.986	0.954	0.928	0.858	0.828	0.668
				750	0.991	0.951	0.913	0.863	0.815	0.696
				1000	0.988	0.951	0.915	0.866	0.811	0.709
				1250	0.990	0.954	0.914	0.872	0.820	0.716
				1500	0.991	0.955	0.919	0.875	0.817	0.719
				2000	0.995	0.959	0.920	0.881	0.816	0.716
		transitivity	0.333	100	0.990	0.960	0.800	0.770	0.710	0.630
				200	0.990	0.950	0.935	0.820	0.775	0.660
				300	0.987	0.953	0.930	0.863	0.803	0.717
				500	0.986	0.964	0.926	0.874	0.814	0.706
				750	0.988	0.969	0.931	0.875	0.825	0.727
				1000	0.982	0.957	0.922	0.865	0.812	0.722
				1250	0.993	0.962	0.921	0.862	0.815	0.720
				1500	0.990	0.961	0.917	0.862	0.815	0.713
				2000	0.997	0.963	0.923	0.874	0.824	0.713

Table 4: Confidence interval coverage for different number of simulated true graphs S based on Monte Carlo simulations using the product generating function when $n = 300$ or $n = 500$ and $\rho_n = 0.1874$.

n	average $a^{(opt)}$	statistic	average for true graphs	S	Proportion of bootstrap CI that cover truth for α =0.01	α =0.05	α =0.1	α =0.15	α =0.2	α =0.3
300	0.000046	density	0.112	100	0.990	0.990	0.960	0.960	0.950	0.770
				200	0.990	0.980	0.955	0.945	0.925	0.850
				300	0.990	0.983	0.963	0.940	0.903	0.827
				500	0.992	0.976	0.952	0.922	0.880	0.802
				750	0.992	0.972	0.948	0.905	0.860	0.780
				1000	0.994	0.974	0.945	0.901	0.866	0.781
				1250	0.995	0.975	0.943	0.898	0.858	0.775
				1500	0.995	0.975	0.937	0.899	0.859	0.774
				2000	0.998	0.973	0.940	0.901	0.855	0.761
		λ_2	25.306	100	0.970	0.960	0.810	0.720	0.700	0.630
				200	0.980	0.965	0.910	0.875	0.800	0.665
				300	0.990	0.963	0.930	0.900	0.830	0.680
				500	0.994	0.954	0.910	0.876	0.818	0.662
				750	0.991	0.953	0.905	0.861	0.800	0.669
				1000	0.995	0.959	0.911	0.859	0.805	0.669
				1250	0.995	0.957	0.912	0.866	0.819	0.674
				1500	0.995	0.965	0.917	0.871	0.821	0.682
				2000	0.997	0.963	0.919	0.869	0.819	0.693
		transitivity	0.146	100	0.980	0.850	0.810	0.780	0.750	0.670
				200	1.000	0.910	0.845	0.790	0.755	0.685
				300	0.987	0.873	0.820	0.793	0.743	0.663
				500	0.986	0.912	0.846	0.802	0.772	0.690
				750	0.983	0.920	0.856	0.809	0.772	0.667
				1000	0.995	0.932	0.881	0.813	0.777	0.674
				1250	0.994	0.934	0.885	0.818	0.779	0.674
				1500	0.995	0.943	0.889	0.830	0.791	0.685
				2000	0.994	0.945	0.885	0.822	0.776	0.681
500	0.000031	density	0.112	100	0.970	0.910	0.860	0.830	0.810	0.750
				200	0.985	0.955	0.890	0.855	0.815	0.750
				300	0.990	0.953	0.937	0.873	0.820	0.773
				500	1.000	0.962	0.946	0.892	0.836	0.746
				750	0.999	0.961	0.949	0.896	0.848	0.761
				1000	0.998	0.961	0.936	0.878	0.832	0.748
				1250	0.997	0.966	0.940	0.890	0.843	0.746
				1500	0.996	0.966	0.937	0.886	0.831	0.731
				2000	0.998	0.964	0.931	0.883	0.834	0.743
		λ_2	41.732	100	0.990	0.940	0.900	0.830	0.770	0.710
				200	1.000	0.970	0.905	0.795	0.765	0.695
				300	0.997	0.943	0.887	0.817	0.783	0.717
				500	0.998	0.948	0.910	0.844	0.802	0.734
				750	0.999	0.955	0.897	0.829	0.797	0.727
				1000	0.998	0.960	0.905	0.834	0.801	0.725
				1250	0.998	0.962	0.900	0.842	0.806	0.731
				1500	0.998	0.958	0.896	0.834	0.797	0.715
				2000	0.998	0.956	0.905	0.848	0.804	0.726
		transitivity	0.147	100	0.990	0.950	0.890	0.860	0.750	0.690
				200	0.985	0.960	0.940	0.880	0.845	0.720
				300	0.983	0.947	0.930	0.850	0.807	0.700
				500	0.988	0.958	0.908	0.838	0.790	0.670
				750	0.989	0.953	0.896	0.843	0.795	0.676
				1000	0.989	0.955	0.898	0.855	0.812	0.697
				1250	0.990	0.958	0.904	0.859	0.813	0.701
				1500	0.991	0.951	0.901	0.859	0.806	0.703
				2000	0.991	0.956	0.890	0.846	0.792	0.684

Table 5: Confidence interval coverage for different number of simulated true graphs S based on Monte Carlo simulations using the horseshoe generating function when $\rho_n = 0.1125$.

ρ_n	statistic	average for true graphs	S	Proportion of bootstrap CI that cover truth for α =0.01	α =0.05	α =0.1	α =0.15	α =0.2	α =0.3
0.569	density	0.570	100	1.000	0.990	0.970	0.950	0.910	0.810
			200	1.000	0.980	0.955	0.930	0.855	0.760
			300	1.000	0.980	0.947	0.913	0.843	0.757
			500	1.000	0.980	0.946	0.890	0.832	0.754
			750	1.000	0.975	0.925	0.855	0.815	0.715
			1000	0.996	0.975	0.928	0.858	0.821	0.733
			1250	0.998	0.970	0.918	0.851	0.815	0.718
			1500	0.997	0.969	0.924	0.863	0.820	0.723
			2000	0.994	0.970	0.924	0.862	0.820	0.722
	λ_1	286.543	100	1.000	0.940	0.800	0.740	0.730	0.590
			200	0.985	0.960	0.905	0.840	0.770	0.650
			300	0.983	0.940	0.910	0.840	0.750	0.623
			500	0.984	0.940	0.898	0.822	0.742	0.638
			750	0.992	0.948	0.907	0.841	0.792	0.687
			1000	0.993	0.944	0.903	0.842	0.784	0.701
			1250	0.996	0.962	0.908	0.848	0.796	0.700
			1500	0.997	0.963	0.907	0.862	0.797	0.699
			2000	0.998	0.966	0.910	0.868	0.814	0.708
	transitivity	0.576	100	1.000	0.920	0.870	0.730	0.650	0.580
			200	0.985	0.955	0.900	0.840	0.795	0.670
			300	0.983	0.927	0.903	0.803	0.753	0.630
			500	0.986	0.940	0.886	0.834	0.760	0.652
			750	0.999	0.949	0.908	0.856	0.795	0.688
			1000	0.998	0.948	0.901	0.855	0.800	0.699
			1250	0.998	0.967	0.909	0.866	0.806	0.698
			1500	0.997	0.967	0.911	0.869	0.807	0.691
			2000	0.998	0.969	0.912	0.869	0.821	0.703
0.759	density	0.759	100	0.990	0.990	0.920	0.920	0.910	0.770
			200	0.995	0.970	0.925	0.900	0.840	0.690
			300	0.990	0.977	0.917	0.870	0.840	0.723
			500	0.988	0.974	0.918	0.882	0.842	0.712
			750	0.996	0.977	0.928	0.893	0.849	0.747
			1000	0.990	0.970	0.926	0.895	0.861	0.748
			1250	0.992	0.965	0.928	0.890	0.855	0.739
			1500	0.990	0.958	0.915	0.873	0.827	0.707
			2000	0.990	0.957	0.906	0.861	0.818	0.700
	λ_1	381.690	100	0.960	0.940	0.860	0.850	0.800	0.680
			200	0.970	0.945	0.900	0.895	0.870	0.675
			300	0.987	0.957	0.910	0.897	0.813	0.710
			500	0.992	0.964	0.918	0.882	0.834	0.732
			750	0.987	0.967	0.907	0.863	0.832	0.737
			1000	0.989	0.962	0.912	0.874	0.833	0.739
			1250	0.990	0.958	0.903	0.845	0.809	0.710
			1500	0.990	0.961	0.908	0.850	0.813	0.711
			2000	0.989	0.957	0.902	0.849	0.814	0.708
	transitivity	0.768	100	0.950	0.940	0.870	0.840	0.760	0.690
			200	0.965	0.930	0.900	0.885	0.830	0.700
			300	0.983	0.957	0.910	0.890	0.850	0.693
			500	0.992	0.968	0.914	0.886	0.854	0.722
			750	0.987	0.967	0.915	0.881	0.837	0.735
			1000	0.988	0.962	0.917	0.878	0.836	0.734
			1250	0.990	0.956	0.896	0.854	0.802	0.706
			1500	0.989	0.958	0.903	0.860	0.805	0.711
			2000	0.990	0.957	0.900	0.855	0.807	0.711

Table 6: Confidence interval coverage for different number of simulated true graphs S based on Monte Carlo simulations using the high density generating function when $n = 500$.

n	statistic	average for true graphs	method	Proportion of bootstrap CI that cover truth for α =0.01	α =0.05	α =0.1	α =0.15	α =0.2	α =0.3
100	λ_1	17.222	HK1	0.998	0.982	0.940	0.895	0.854	0.775
			HK2	0.998	0.986	0.939	0.894	0.861	0.769
			HNN1	0.947	0.777	0.648	0.590	0.507	0.397
			emp	0.999	0.953	0.912	0.877	0.842	0.742
	λ_2	6.418	HK1	0.974	0.858	0.704	0.599	0.504	0.341
			HK2	1.000	0.947	0.879	0.796	0.679	0.514
			HNN1	0.966	0.896	0.771	0.675	0.612	0.537
			emp	0.343	0.057	0.012	0.004	0.002	0.000
	transitivity	0.220	HK1	0.999	0.992	0.971	0.938	0.907	0.847
			HK2	0.998	0.984	0.963	0.933	0.901	0.831
			HNN1	0.986	0.937	0.838	0.785	0.739	0.665
			emp	0.992	0.966	0.897	0.839	0.800	0.703
300	λ_1	50.535	HK1	0.995	0.959	0.918	0.876	0.836	0.739
			HK2	0.994	0.958	0.923	0.889	0.859	0.765
			HNN1	0.936	0.847	0.724	0.647	0.597	0.509
			emp	0.990	0.953	0.900	0.859	0.824	0.733
	λ_2	11.778	HK1	0.000	0.000	0.000	0.000	0.000	0.000
			HK2	0.001	0.000	0.000	0.000	0.000	0.000
			HNN1	0.978	0.915	0.857	0.814	0.754	0.651
			emp	0.000	0.000	0.000	0.000	0.000	0.000
	transitivity	0.221	HK1	0.997	0.982	0.940	0.897	0.859	0.759
			HK2	0.996	0.984	0.954	0.904	0.860	0.782
			HNN1	0.981	0.919	0.872	0.823	0.747	0.629
			emp	0.988	0.947	0.897	0.847	0.781	0.678
500	λ_1	83.925	HK1	0.983	0.955	0.904	0.849	0.782	0.680
			HK2	0.986	0.944	0.902	0.858	0.791	0.701
			HNN1	0.950	0.860	0.773	0.682	0.633	0.534
			emp	0.981	0.937	0.888	0.851	0.817	0.708
	λ_2	15.448	HK1	0.000	0.000	0.000	0.000	0.000	0.000
			HK2	0.000	0.000	0.000	0.000	0.000	0.000
			HNN1	0.982	0.915	0.854	0.814	0.753	0.658
			emp	0.000	0.000	0.000	0.000	0.000	0.000
	transitivity	0.222	HK1	0.986	0.945	0.899	0.867	0.814	0.708
			HK2	0.987	0.956	0.911	0.851	0.812	0.734
			HNN1	0.982	0.920	0.838	0.780	0.734	0.630
			emp	0.982	0.968	0.905	0.837	0.772	0.692

Table 7: Confidence interval coverage based on Monte Carlo simulations for different bootstrap methods for the true graphs from the product generating function with density $\rho_n = 0.125$, $S = 1000$ true graphs, with sample size n ranging from 100 to 500. The methods are: HK1 (our main method based on $\hat{h} \equiv \hat{h}^{(K1)}$ with $a^{(opt)}$), HK2 (our bootstrap method but using the linking function estimator $\hat{h}^{(K2)}$ with $a^{(optK2)}$ based on $\hat{h}^{(K2)}$), HNN1 (our bootstrap method but but using the linking function estimator $\hat{h}^{(NN1)}$ from Zhang, Levina, and Zhu (2017) with their optimal choice of neighbourhood size), emp (empirical bootstrap from Green and Shalizi (2022)), dot_prod_k (the bootstrap method from Levin and Levina (2019) based on assuming a k-dimensional ξ_i).

statistic	n	average for true graphs	method	Proportion of bootstrap CI that cover truth for α =0.01	α =0.05	α =0.1	α =0.15	α =0.2	α =0.3
density	250	0.160767	HK1	0.991	0.959	0.914	0.874	0.823	0.736
			HK2	0.993	0.956	0.908	0.868	0.826	0.721
			HNN1	0.981	0.914	0.873	0.818	0.762	0.636
			LLS_L	1.000	1.000	1.000	0.999	0.999	0.990
			asymp estimated var	0.993	0.973	0.948	0.909	0.864	0.781
			asymp infeasible var	0.990	0.943	0.895	0.846	0.795	0.701
			dot_prod_1	1.000	0.999	0.998	0.990	0.973	0.913
			dot_prod_3	1.000	0.999	0.998	0.991	0.967	0.906
			emp	0.990	0.961	0.910	0.871	0.828	0.742
	500	0.139211	HK1	0.993	0.955	0.889	0.827	0.771	0.672
			HK2	0.993	0.954	0.872	0.816	0.753	0.667
			HNN1	0.987	0.896	0.795	0.730	0.675	0.590
			asymp estimated var	0.998	0.978	0.948	0.907	0.868	0.770
			asymp infeasible var	0.993	0.952	0.900	0.855	0.797	0.676
			dot_prod_1	1.000	1.000	0.996	0.988	0.979	0.920
			dot_prod_3	1.000	1.000	0.997	0.991	0.985	0.936
			emp	0.991	0.942	0.891	0.841	0.784	0.677
	750	0.127979	HK1	0.979	0.937	0.877	0.842	0.788	0.684
			HK2	0.973	0.928	0.873	0.824	0.774	0.665
			HNN1	0.944	0.866	0.775	0.721	0.659	0.556
			asymp estimated var	0.994	0.980	0.953	0.914	0.883	0.799
			asymp infeasible var	0.989	0.948	0.901	0.852	0.803	0.704
			dot_prod_1	1.000	0.999	0.993	0.982	0.964	0.920
			dot_prod_3	1.000	0.999	0.993	0.982	0.965	0.923
			emp	0.978	0.933	0.887	0.835	0.795	0.691
triangle density	250	0.009854	HK1	0.990	0.969	0.942	0.886	0.852	0.745
			HK2	0.990	0.974	0.941	0.902	0.869	0.749
			HNN1	0.990	0.946	0.902	0.839	0.788	0.675
			LLS_L	1.000	0.999	0.995	0.983	0.968	0.925
			LLS_Q	1.000	0.999	0.995	0.983	0.966	0.923
			asymp infeasible var	0.987	0.953	0.905	0.852	0.794	0.697
			dot_prod_1	1.000	0.999	0.997	0.983	0.975	0.930
			dot_prod_3	1.000	0.999	0.999	0.983	0.976	0.933
			emp	0.994	0.966	0.935	0.871	0.821	0.732
	500	0.006402	HK1	0.991	0.959	0.902	0.840	0.794	0.692
			HK2	0.997	0.960	0.896	0.847	0.790	0.700
			HNN1	0.986	0.929	0.826	0.760	0.704	0.606
			asymp infeasible var	0.993	0.951	0.889	0.852	0.797	0.695
			dot_prod_1	1.000	1.000	0.997	0.987	0.970	0.911
			dot_prod_3	1.000	1.000	0.998	0.993	0.973	0.921
			emp	0.996	0.951	0.882	0.839	0.789	0.692
	750	0.004970	HK1	0.987	0.946	0.898	0.854	0.805	0.695
			HK2	0.987	0.943	0.892	0.852	0.793	0.697
			HNN1	0.962	0.904	0.820	0.752	0.703	0.598
			asymp infeasible var	0.987	0.940	0.900	0.852	0.803	0.703
			dot_prod_1	1.000	1.000	0.995	0.985	0.965	0.919
			dot_prod_3	1.000	1.000	0.995	0.985	0.967	0.922
			emp	0.977	0.938	0.900	0.841	0.804	0.709

Table 8: Confidence interval coverage based on Monte Carlo simulations for different bootstrap methods for the true graphs from the product generating function with density $\rho_n \sim \sqrt[4]{\frac{\log(n)}{n}}$, $S = 1000$ true graphs. The methods are: HK1 (our main method based on $\hat{h} \equiv \hat{h}^{(K1)}$ with $a^{(opt)}$), HK2 (our bootstrap method but using the linking function estimator $\hat{h}^{(K2)}$ with $a^{(optK2)}$ based on $\hat{h}^{(K2)}$), HNN1 (our bootstrap method but but using the linking function estimator $\hat{h}^{(NN1)}$ from Zhang, Levina, and Zhu (2017) with their optimal choice of neighbourhood size), emp (empirical bootstrap from Green and Shalizi (2022)), dot_prod_k (the bootstrap method from Levin and Levina (2019) based on assuming a k-dimensional ξ_i), asymp estimated var (the asymptotic distribution from Bickel, Chen, and Levina (2011) with variance estimated according to the formula in Green and Shalizi (2022)), asymp infeasible var (the asymptotic distribution from Bickel, Chen, and Levina (2011) with the true theoretical variance), LLS_L and LLS_Q (the linear and quadratic methods from Lin, Lunde, and Sarkar (2020)).

ρ_n	statistic	average for true graphs	method	Proportion of bootstrap CI that cover truth for α =0.01	α =0.05	α =0.1	α =0.15	α =0.2	α =0.3
0.056	density	0.056	HK1	0.989	0.927	0.864	0.786	0.721	0.634
			HK2	0.968	0.825	0.654	0.514	0.419	0.346
			HNN1	0.253	0.066	0.034	0.017	0.005	0.001
			dot_prod_1	0.005	0.000	0.000	0.000	0.000	0.000
			dot_prod_3	1.000	0.998	0.985	0.973	0.955	0.901
			emp	0.999	0.985	0.960	0.920	0.875	0.800
	λ_1	32.530	HK1	0.988	0.968	0.931	0.866	0.820	0.722
			HK2	0.996	0.977	0.945	0.898	0.849	0.767
			HNN1	0.925	0.751	0.631	0.522	0.440	0.332
			emp	0.979	0.925	0.871	0.813	0.763	0.669
	λ_2	21.481	HK1	0.997	0.957	0.903	0.860	0.827	0.729
			HK2	0.991	0.911	0.807	0.748	0.686	0.589
			HNN1	0.970	0.845	0.696	0.585	0.507	0.409
			emp	0.935	0.811	0.694	0.631	0.560	0.480
	transitivity	0.073	HK1	0.997	0.938	0.890	0.848	0.800	0.672
			HK2	0.970	0.930	0.867	0.826	0.781	0.688
			HNN1	0.966	0.901	0.818	0.767	0.702	0.590
			emp	0.993	0.956	0.904	0.851	0.786	0.692
0.084	density	0.084	HK1	0.990	0.953	0.914	0.874	0.833	0.722
			HK2	0.990	0.950	0.909	0.861	0.794	0.650
			HNN1	0.910	0.854	0.732	0.619	0.563	0.436
			dot_prod_1	0.000	0.000	0.000	0.000	0.000	0.000
			dot_prod_3	0.999	0.996	0.987	0.970	0.948	0.903
			emp	0.996	0.986	0.957	0.914	0.875	0.797
	λ_1	48.240	HK1	0.997	0.971	0.912	0.863	0.817	0.705
			HK2	0.998	0.976	0.934	0.875	0.821	0.721
			HNN1	0.973	0.919	0.843	0.809	0.760	0.638
			emp	0.997	0.967	0.900	0.837	0.786	0.681
	λ_2	31.542	HK1	0.996	0.971	0.931	0.876	0.827	0.730
			HK2	0.993	0.951	0.896	0.854	0.797	0.672
			HNN1	0.989	0.959	0.902	0.834	0.805	0.694
			emp	0.991	0.941	0.876	0.824	0.780	0.663
	transitivity	0.110	HK1	0.997	0.949	0.886	0.825	0.779	0.669
			HK2	0.996	0.950	0.896	0.845	0.796	0.702
			HNN1	0.955	0.885	0.814	0.761	0.688	0.578
			emp	0.998	0.965	0.922	0.874	0.819	0.738
0.113	density	0.113	HK1	0.993	0.975	0.922	0.866	0.823	0.732
			HK2	0.989	0.970	0.914	0.869	0.829	0.718
			HNN1	0.990	0.937	0.875	0.827	0.777	0.661
			dot_prod_1	0.000	0.000	0.000	0.000	0.000	0.000
			dot_prod_3	1.000	0.993	0.985	0.975	0.958	0.898
			emp	0.993	0.982	0.955	0.918	0.870	0.792
	λ_1	63.879	HK1	0.997	0.968	0.908	0.856	0.781	0.662
			HK2	1.000	0.973	0.916	0.874	0.807	0.690
			HNN1	0.989	0.937	0.879	0.789	0.738	0.654
			emp	0.993	0.958	0.899	0.839	0.798	0.693
	λ_2	41.713	HK1	0.994	0.961	0.909	0.855	0.810	0.694
			HK2	0.994	0.941	0.898	0.845	0.793	0.697
			HNN1	0.988	0.955	0.901	0.857	0.799	0.686
			emp	0.994	0.945	0.883	0.814	0.749	0.643
	transitivity	0.147	HK1	0.996	0.956	0.895	0.820	0.766	0.672
			HK2	1.000	0.957	0.902	0.837	0.784	0.661
			HNN1	0.956	0.846	0.771	0.694	0.655	0.550
			emp	0.989	0.944	0.891	0.846	0.792	0.682

Table 9: Confidence interval coverage based on Monte Carlo simulations for different bootstrap methods for the true graphs from the horseshoe generating function with sample size $n = 500$, $S = 1000$ true graphs, and different values of density ρ_n ranging from 0.056 to 0.1125. The methods are: HK1 (our main method based on $\hat{h} \equiv \hat{h}^{(K1)}$ with $a^{(opt)}$), HK2 (our bootstrap method but using the linking function estimator $\hat{h}^{(K2)}$ with $a^{(optK2)}$ based on $\hat{h}^{(K2)}$), HNN1 (our bootstrap method but but using the linking function estimator $\hat{h}^{(NN1)}$ from Zhang, Levina, and Zhu (2017) with their optimal choice of neighbourhood size), emp (empirical bootstrap from Green and Shalizi (2022)), dot_prod_k (the bootstrap method from Levin and Levina (2019) based on assuming a k-dimensional ξ_i).

ρ_n	statistic	average for true graphs	method	Proportion of bootstrap CI that cover truth for α =0.01	α =0.05	α =0.1	α =0.15	α =0.2	α =0.3
0.380	density	0.380	HK1	0.998	0.984	0.964	0.928	0.879	0.800
			HK2	0.995	0.978	0.954	0.931	0.888	0.795
			HNN1	0.982	0.921	0.859	0.783	0.735	0.643
			dot_prod_1	1.000	0.996	0.987	0.975	0.948	0.873
			dot_prod_3	1.000	0.996	0.986	0.973	0.944	0.873
			emp	0.998	0.983	0.948	0.932	0.885	0.772
	λ_1	114.861	HK1	0.994	0.964	0.937	0.892	0.851	0.777
			HK2	0.999	0.970	0.939	0.907	0.869	0.793
			HNN1	0.990	0.973	0.926	0.878	0.834	0.728
			emp	1.000	0.984	0.957	0.935	0.902	0.817
	λ_2	15.962	HK1	0.000	0.000	0.000	0.000	0.000	0.000
			HK2	0.000	0.000	0.000	0.000	0.000	0.000
			HNN1	1.000	0.992	0.980	0.952	0.929	0.847
			emp	0.000	0.000	0.000	0.000	0.000	0.000
	transitivity	0.384	HK1	0.994	0.973	0.937	0.902	0.868	0.786
			HK2	1.000	0.981	0.937	0.897	0.859	0.775
			HNN1	0.997	0.973	0.948	0.915	0.880	0.774
			emp	0.998	0.973	0.888	0.839	0.778	0.694
0.570	density	0.570	HK1	0.994	0.969	0.945	0.904	0.849	0.756
			HK2	0.993	0.973	0.923	0.882	0.827	0.713
			HNN1	0.994	0.964	0.926	0.889	0.836	0.731
			dot_prod_1	1.000	0.996	0.991	0.973	0.947	0.886
			dot_prod_3	1.000	0.996	0.987	0.963	0.939	0.857
			emp	0.991	0.945	0.915	0.863	0.814	0.723
	λ_1	171.787	HK1	0.999	0.978	0.941	0.905	0.852	0.767
			HK2	0.999	0.978	0.941	0.900	0.857	0.779
			HNN1	0.990	0.972	0.939	0.884	0.838	0.740
			emp	0.990	0.971	0.937	0.887	0.850	0.757
	λ_2	15.963	HK1	0.000	0.000	0.000	0.000	0.000	0.000
			HK2	0.000	0.000	0.000	0.000	0.000	0.000
			HNN1	1.000	0.993	0.977	0.949	0.895	0.832
			emp	0.000	0.000	0.000	0.000	0.000	0.000
	transitivity	0.576	HK1	0.998	0.979	0.934	0.908	0.864	0.770
			HK2	0.997	0.978	0.940	0.902	0.867	0.776
			HNN1	0.987	0.969	0.920	0.877	0.830	0.730
			emp	0.985	0.938	0.891	0.819	0.787	0.652
0.759	density	0.759	HK1	0.994	0.963	0.911	0.872	0.841	0.737
			HK2	0.992	0.961	0.919	0.870	0.825	0.747
			HNN1	0.986	0.942	0.902	0.859	0.809	0.694
			dot_prod_1	1.000	0.998	0.990	0.975	0.945	0.881
			dot_prod_3	1.000	0.997	0.986	0.959	0.922	0.850
			emp	0.985	0.947	0.890	0.835	0.810	0.706
	λ_1	228.714	HK1	0.996	0.965	0.926	0.884	0.842	0.731
			HK2	0.993	0.959	0.926	0.877	0.841	0.734
			HNN1	0.991	0.962	0.931	0.865	0.819	0.702
			emp	0.983	0.953	0.914	0.864	0.813	0.702
	λ_2	13.730	HK1	0.020	0.000	0.000	0.000	0.000	0.000
			HK2	0.978	0.817	0.615	0.498	0.379	0.217
			HNN1	0.999	0.988	0.965	0.932	0.892	0.784
			emp	0.000	0.000	0.000	0.000	0.000	0.000
	transitivity	0.768	HK1	0.994	0.961	0.916	0.885	0.833	0.720
			HK2	0.993	0.961	0.916	0.886	0.829	0.732
			HNN1	0.992	0.965	0.932	0.864	0.815	0.712
			emp	0.970	0.920	0.872	0.814	0.776	0.643

Table 10: Confidence interval coverage based on Monte Carlo simulations for different bootstrap methods for the true graphs from the high density generating function with sample size $n = 300$, $S = 1000$ true graphs, and different values of density ρ_n ranging from 0.38 to 0.79. The methods are: HK1 (our main method based on $\hat{h} \equiv \hat{h}^{(K1)}$ with $a^{(opt)}$), HK2 (our bootstrap method but using the linking function estimator $\hat{h}^{(K2)}$ with $a^{(optK2)}$ based on $\hat{h}^{(K2)}$), HNN1 (our bootstrap method but but using the linking function estimator $\hat{h}^{(NN1)}$ from Zhang, Levina, and Zhu (2017) with their optimal choice of neighbourhood size), emp (empirical bootstrap from Green and Shalizi (2022)), dot_prod_k (the bootstrap method from Levin and Levina (2019) based on assuming a k-dimensional ξ_i).

true value	c	average $a^{(opt)}$	Proportion of bootstrap CI that cover truth for α=0.01	α=0.05	α=0.1	α=0.15	α=0.2	α=0.3
0.1109	0.10	1.045577e-06	0.990	0.962	0.936	0.857	0.818	0.736
	0.25	2.613943e-06	0.997	0.987	0.949	0.913	0.862	0.783
	0.50	5.227885e-06	0.999	0.984	0.956	0.927	0.887	0.790
	0.75	7.841828e-06	0.995	0.976	0.950	0.912	0.865	0.767
	0.90	9.410194e-06	0.999	0.987	0.961	0.926	0.880	0.778
	1.00	1.045577e-05	0.996	0.989	0.961	0.947	0.912	0.834
	1.10	1.150135e-05	0.999	0.985	0.971	0.948	0.912	0.829
	1.25	1.306971e-05	0.998	0.979	0.947	0.913	0.842	0.743
	2	2.091154e-05	0.477	0.223	0.117	0.071	0.051	0.032
	4	4.182308e-05	0.000	0.000	0.000	0.000	0.000	0.000
	10	1.045577e-04	0.000	0.000	0.000	0.000	0.000	0.000
0.2218	0.10	3.524835e-06	0.987	0.941	0.886	0.849	0.791	0.704
	0.25	8.812088e-06	0.997	0.956	0.908	0.862	0.826	0.715
	0.50	1.762418e-05	0.996	0.973	0.926	0.903	0.853	0.771
	0.75	2.643626e-05	0.993	0.953	0.913	0.878	0.806	0.713
	0.90	3.172352e-05	0.984	0.955	0.930	0.903	0.869	0.769
	1.00	3.524835e-05	0.995	0.969	0.947	0.905	0.856	0.745
	1.10	3.877319e-05	0.991	0.959	0.941	0.891	0.850	0.743
	1.25	4.406044e-05	0.995	0.968	0.940	0.900	0.860	0.763
	2	7.049670e-05	0.992	0.939	0.883	0.819	0.768	0.691
	4	1.409934e-04	0.785	0.473	0.390	0.317	0.258	0.171
	10	3.524835e-04	0.012	0.003	0.002	0.000	0.000	0.000
0.3327	0.10	5.842118e-06	0.990	0.935	0.883	0.837	0.784	0.676
	0.25	1.460529e-05	0.974	0.935	0.881	0.816	0.769	0.667
	0.50	2.921059e-05	0.991	0.968	0.909	0.865	0.802	0.674
	0.75	4.381588e-05	0.992	0.966	0.927	0.893	0.834	0.722
	0.90	5.257906e-05	0.990	0.956	0.907	0.873	0.830	0.700
	1.00	5.842118e-05	0.991	0.958	0.921	0.874	0.829	0.716
	1.10	6.426330e-05	0.993	0.965	0.929	0.881	0.835	0.720
	1.25	7.302647e-05	0.993	0.955	0.899	0.868	0.818	0.690
	2	1.168424e-04	0.990	0.955	0.923	0.876	0.824	0.710
	4	2.336847e-04	0.969	0.917	0.854	0.808	0.739	0.647
	10	5.842118e-04	0.732	0.509	0.384	0.312	0.277	0.195
0.4436	0.10	7.561251e-06	0.995	0.955	0.903	0.852	0.788	0.708
	0.25	1.890313e-05	0.994	0.954	0.903	0.852	0.805	0.719
	0.50	3.780625e-05	0.990	0.949	0.894	0.857	0.823	0.722
	0.75	5.670938e-05	0.994	0.969	0.921	0.853	0.811	0.712
	0.90	6.805126e-05	0.993	0.962	0.900	0.862	0.839	0.763
	1.00	7.561251e-05	0.991	0.954	0.887	0.848	0.803	0.691
	1.10	8.317376e-05	0.986	0.948	0.899	0.861	0.820	0.715
	1.25	9.451564e-05	0.990	0.952	0.906	0.868	0.830	0.749
	2	1.512250e-04	0.989	0.937	0.884	0.849	0.806	0.715
	4	3.024500e-04	0.989	0.955	0.911	0.851	0.796	0.707
	10	7.561251e-04	0.951	0.875	0.792	0.731	0.673	0.573

Table 11: Confidence interval coverage for transitivity at different bandwidths $c \times a^{(opt)}$ and at different densities ρ_n, based on Monte Carlo simulations using the product generating function when $n = 500$ and $S = 1000$.

ρ_n	c	average $a^{(opt)}$	Proportion of bootstrap CI that cover truth for α=0.01	α=0.05	α=0.1	α=0.15	α=0.2	α=0.3
0.028125	0.01	1.402691e-08	1.000	0.991	0.979	0.962	0.935	0.854
	0.10	1.402691e-07	1.000	0.988	0.976	0.960	0.931	0.843
	0.25	3.506728e-07	1.000	0.992	0.978	0.965	0.944	0.870
	0.50	7.013456e-07	0.999	0.989	0.970	0.949	0.918	0.842
	0.75	1.052018e-06	0.999	0.945	0.884	0.831	0.778	0.666
	0.90	1.262422e-06	0.986	0.831	0.733	0.570	0.450	0.317
	1.00	1.402691e-06	0.853	0.538	0.370	0.256	0.194	0.102
	1.10	1.542960e-06	0.687	0.325	0.174	0.105	0.064	0.025
	1.25	1.753364e-06	0.355	0.092	0.026	0.008	0.005	0.001
	2	2.805383e-06	0.000	0.000	0.000	0.000	0.000	0.000
	4	5.610765e-06	0.020	0.005	0.001	0.000	0.000	0.000
	10	1.402691e-05	0.853	0.706	0.593	0.519	0.457	0.379
	20	2.805383e-05	0.979	0.931	0.848	0.796	0.746	0.634
	100	1.402691e-04	0.981	0.949	0.887	0.829	0.755	0.658
0.056250	0.01	6.201768e-08	0.997	0.985	0.964	0.921	0.887	0.807
	0.10	6.201768e-07	0.997	0.984	0.957	0.929	0.877	0.799
	0.25	1.550442e-06	0.998	0.979	0.956	0.926	0.883	0.789
	0.50	3.100884e-06	0.997	0.988	0.966	0.931	0.897	0.808
	0.75	4.651326e-06	0.997	0.983	0.958	0.916	0.880	0.800
	0.90	5.581592e-06	0.996	0.969	0.930	0.856	0.810	0.704
	1.00	6.201768e-06	0.985	0.930	0.864	0.806	0.751	0.647
	1.10	6.821945e-06	0.986	0.876	0.799	0.720	0.654	0.543
	1.25	7.752210e-06	0.939	0.814	0.720	0.610	0.502	0.313
	2	1.240354e-05	0.835	0.465	0.268	0.185	0.126	0.067
	4	2.480707e-05	0.641	0.363	0.202	0.139	0.111	0.067
	10	6.201768e-05	0.767	0.638	0.525	0.466	0.428	0.343
	20	1.240354e-04	0.930	0.825	0.771	0.702	0.657	0.577
	100	6.201768e-04	0.969	0.871	0.802	0.745	0.698	0.599
0.084375	0.01	1.469518e-07	0.997	0.982	0.947	0.917	0.867	0.774
	0.10	1.469518e-06	0.992	0.977	0.947	0.910	0.867	0.764
	0.25	3.673796e-06	0.992	0.977	0.950	0.922	0.886	0.792
	0.50	7.347591e-06	0.995	0.984	0.945	0.909	0.861	0.787
	0.75	1.102139e-05	0.992	0.980	0.953	0.918	0.882	0.774
	0.90	1.322566e-05	0.988	0.959	0.933	0.888	0.835	0.723
	1.00	1.469518e-05	0.992	0.965	0.933	0.900	0.849	0.750
	1.10	1.616470e-05	0.982	0.943	0.906	0.857	0.786	0.695
	1.25	1.836898e-05	0.990	0.953	0.903	0.813	0.772	0.653
	2	2.939036e-05	0.982	0.920	0.819	0.766	0.705	0.593
	4	5.878073e-05	0.965	0.862	0.741	0.666	0.610	0.439
	10	1.469518e-04	0.829	0.668	0.581	0.485	0.412	0.323
	20	2.939036e-04	0.902	0.783	0.722	0.655	0.607	0.520
	100	1.469518e-03	0.944	0.852	0.784	0.718	0.666	0.550
0.112500	0.01	3.076687e-07	0.997	0.978	0.926	0.885	0.850	0.744
	0.10	3.076687e-06	0.997	0.961	0.926	0.885	0.845	0.732
	0.25	7.691717e-06	1.000	0.967	0.923	0.882	0.847	0.754
	0.50	1.538343e-05	0.997	0.971	0.927	0.891	0.849	0.731
	0.75	2.307515e-05	0.997	0.964	0.925	0.884	0.844	0.740
	0.90	2.769018e-05	0.994	0.966	0.921	0.884	0.849	0.741
	1.00	3.076687e-05	0.996	0.954	0.911	0.871	0.823	0.738
	1.10	3.384355e-05	0.997	0.960	0.911	0.878	0.809	0.725
	1.25	3.845858e-05	0.996	0.949	0.904	0.861	0.792	0.715
	2	6.153373e-05	0.992	0.951	0.894	0.850	0.806	0.714
	4	1.230675e-04	0.987	0.933	0.866	0.805	0.748	0.657
	10	3.076687e-04	0.884	0.729	0.634	0.555	0.499	0.418
	20	6.153373e-04	0.843	0.773	0.679	0.604	0.529	0.462
	100	3.076687e-03	0.909	0.823	0.760	0.673	0.626	0.525

Table 12: Confidence interval coverage for density at different bandwidths $c \times a^{(opt)}$ and at different densities ρ_n, based on Monte Carlo simulations using the horseshoe generating function when $n = 500$ and $S = 1000$.

ρ_n	c	average $a^{(opt)}$	Proportion of bootstrap CI that cover truth for α=0.01	α=0.05	α=0.1	α=0.15	α=0.2	α=0.3
0.37950	0.001	9.252546e-08	0.000	0.000	0.000	0.000	0.000	0.000
	0.01	9.252546e-07	0.000	0.000	0.000	0.000	0.000	0.000
	0.1	9.257237e-06	0.000	0.000	0.000	0.000	0.000	0.000
	0.25	2.314309e-05	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	4.628619e-05	0.000	0.000	0.000	0.000	0.000	0.000
	0.75	6.942928e-05	0.000	0.000	0.000	0.000	0.000	0.000
	0.9	8.331514e-05	0.000	0.000	0.000	0.000	0.000	0.000
	1	9.257237e-05	0.000	0.000	0.000	0.000	0.000	0.000
	1.1	1.018296e-04	0.000	0.000	0.000	0.000	0.000	0.000
	1.25	1.157155e-04	0.000	0.000	0.000	0.000	0.000	0.000
	2	1.851447e-04	1.000	0.990	0.983	0.956	0.920	0.852
	4	3.702895e-04	0.999	0.993	0.981	0.953	0.897	0.820
	10	9.257237e-04	0.999	0.987	0.941	0.896	0.845	0.765
	100	9.252546e-03	0.997	0.959	0.899	0.828	0.773	0.661
	1000	9.252546e-02	0.997	0.955	0.899	0.833	0.791	0.668
0.56925	0.001	1.108253e-07	0.000	0.000	0.000	0.000	0.000	0.000
	0.01	1.108253e-06	0.000	0.000	0.000	0.000	0.000	0.000
	0.1	1.107591e-05	0.000	0.000	0.000	0.000	0.000	0.000
	0.25	2.768978e-05	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	5.537956e-05	0.000	0.000	0.000	0.000	0.000	0.000
	0.75	8.306935e-05	0.000	0.000	0.000	0.000	0.000	0.000
	0.9	9.968322e-05	0.000	0.000	0.000	0.000	0.000	0.000
	1	1.107591e-04	0.000	0.000	0.000	0.000	0.000	0.000
	1.1	1.218350e-04	0.000	0.000	0.000	0.000	0.000	0.000
	1.25	1.384489e-04	0.000	0.000	0.000	0.000	0.000	0.000
	2	2.215183e-04	1.000	0.990	0.974	0.956	0.936	0.853
	4	4.430365e-04	0.998	0.976	0.938	0.889	0.837	0.744
	10	1.107591e-03	0.983	0.931	0.842	0.771	0.705	0.585
	100	1.108253e-02	0.907	0.738	0.505	0.435	0.346	0.259
	1000	1.108253e-01	0.885	0.640	0.495	0.437	0.380	0.259
0.75900	0.001	9.286943e-08	0.000	0.000	0.000	0.000	0.000	0.000
	0.01	9.286943e-07	0.000	0.000	0.000	0.000	0.000	0.000
	0.1	9.277168e-06	0.000	0.000	0.000	0.000	0.000	0.000
	0.25	2.319292e-05	0.000	0.000	0.000	0.000	0.000	0.000
	0.5	4.638584e-05	0.000	0.000	0.000	0.000	0.000	0.000
	0.75	6.957876e-05	0.000	0.000	0.000	0.000	0.000	0.000
	0.9	8.349452e-05	0.000	0.000	0.000	0.000	0.000	0.000
	1	9.277168e-05	0.000	0.000	0.000	0.000	0.000	0.000
	1.1	1.020489e-04	0.030	0.001	0.000	0.000	0.000	0.000
	1.25	1.159646e-04	0.860	0.650	0.449	0.303	0.221	0.133
	2	1.855434e-04	0.999	0.982	0.954	0.926	0.879	0.787
	4	3.710867e-04	1.000	0.981	0.952	0.910	0.856	0.755
	10	9.277168e-04	0.982	0.914	0.867	0.793	0.735	0.650
	100	9.286943e-03	0.929	0.865	0.738	0.648	0.595	0.479
	1000	9.286943e-02	0.906	0.772	0.659	0.605	0.535	0.444

Table 13: Confidence interval coverage for λ_{10} at different bandwidths $c \times a^{(opt)}$ and at different densities ρ_n, based on Monte Carlo simulations using the high density generating function when $n = 500$ and $S = 1000$.

B.3 Plots

The linking functions we consider are:

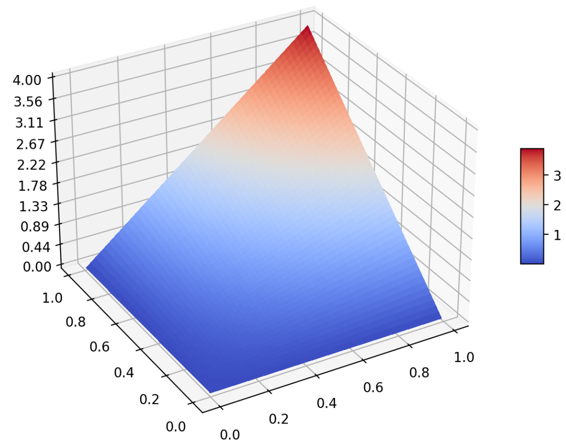


Figure 12: Dot product linking function: $h(\xi_i, \xi_j) = \rho_n \times 4\xi_i\xi_j$.

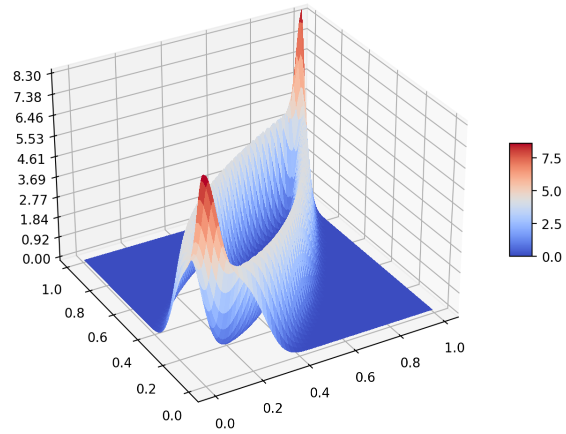


Figure 13: Horseshoe linking function: $h(\xi_i, \xi_j) = \rho_n \times 4.44286 \left(e^{-200\left(\xi_i - \xi_j^2\right)^2} + e^{-200\left(\xi_j - \xi_i^2\right)^2} \right)$.

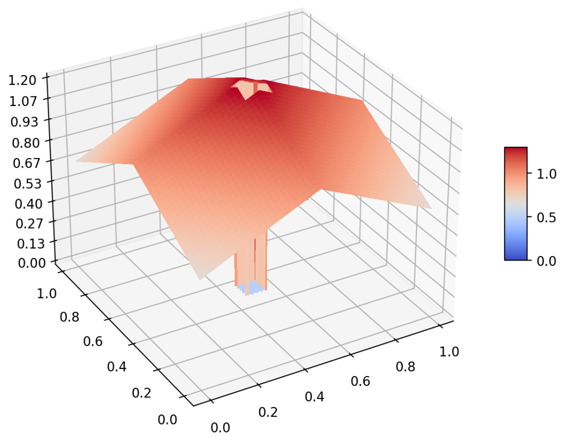


Figure 14: High-density linking function:
$h(\xi_i, \xi_j) = \rho_n \times 1.35 \left(1 - \mathbb{1}\left(\left|\frac{1}{2} - \xi_i\right| \leq 0.05\right) \mathbb{1}\left(\left|\frac{1}{2} - \xi_j\right| \leq 0.05\right)\right) \left(1 - \frac{1}{2}\left(\left|\frac{1}{2} - \xi_i\right| + \left|\frac{1}{2} - \xi_j\right|\right)\right).$

C Extensions and alternative specifications

C.1 Possible extensions of application

On a technical level, one possible extension would be analysing different ways of aggregating the twelve observed types of household interactions into the adjacency matrix. Like the original paper, we have used the union of the twelve characteristic-specific adjacency matrices, but there are many other possible choices, e.g. taking an intersection (this may be less desirable as it leads to a significantly sparser network) or an average (which gives a weighted adjacency matrix). Our method allows for comparison of the adjacency matrices achieved though different aggregating functions and checking if they lead to different structures. For example, we can compare the largest eigenvalues λ_1 obtained for villages using different aggregating functions and check if they have overlapping confidence intervals. If they don't, this shows that the choice of the aggregating function is not without loss of generality.

Another possible modification of our procedure would be estimating the linking function under the assumption that each of the twelve characteristics is a separate draw from the Bernoulli distribution. We could redefine the distance function to depend directly on the twelve characteristics instead of a single aggregate adjacency matrix. Let $\mathbf{A}_{ij}$ denote a 12×1 vector of indicators whether households i and j are related according to the twelve characteristics. Let $\|.\|$ be some vector norm (e.g. max norm, min norm, Euclidean norm, or a weighted norm[34]). Then we can define

$$d_{ij}^{(\|\cdot\|,2)} = \left(\frac{1}{n}\sum_{t=1}^{n}\left(\frac{1}{n}\sum_{s=1}^{n}\|\mathrm{diag}\left(\mathbf{A}_{ts}\right)\left(\mathbf{A}_{is}-\mathbf{A}_{js}\right)\|\right)^{2}\right)^{\frac{1}{2}}$$

where $\mathrm{diag}(v)$ is a diagonal matrix with diagonal entries from a vector v. To obtain the bootstrap version of adjacency matrices $\mathbf{A}$ we could draw from a joint Bernoulli distribution with probabilities estimated using $\hat{h}_n$ based on the distance $d_{ij}^{(\|\cdot\|,2)}$ and the adjacency matrices for individual characteristics, and a covariance matrix equal to the sample covariance between different characteristics.

C.2 Sensitivity checks

We rerun the estimation for a subset of villages using 300 repetitions of the simulated information spreading through the network to estimate the simulated moments instead of the original 75. The outcomes (middle panel in Fig. 15) for that subsample were very similar to the outcomes based on the original specification (left panel in Fig. 15). We conclude that 75 simulations are sufficient for the estimation of simulated moments.

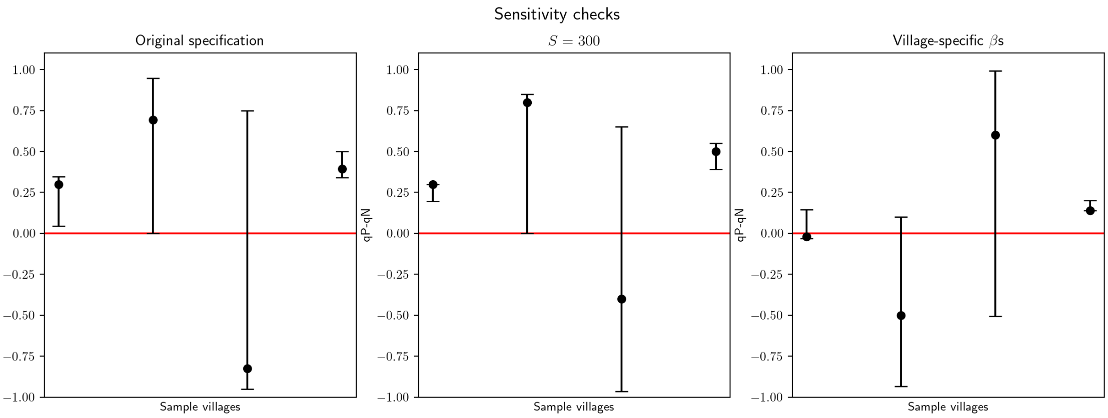


Figure 15: A comparison of the estimates of $q^P - q^N$ for a subset of villages with 95% confidence intervals based simulated moments estimated using the original specification with 75 simulations and β estimated using all villages (left), 300 simulations and β estimated using all villages (middle), and 75 simulations and β estimated using village-specific data only (right).

We also tried to estimate the β coefficients using village-specific data (rather than aggregating over all villages). This did make a difference for the estimates and confidence intervals (right panel of Fig. 15), though the conclusions remain similar. However, since the regression used to identify β is run using only the information about the leaders, we found the sample sizes for individual villages

34. A weighted norm could be of the form $\|x\| = \sqrt{x'W_x x}$, with a weight matrix W_x that may depend on the adjacency matrices, e.g. $W_x = \left(\frac{1}{n}\sum_{v=1}^{n} x_v x_v'\right)^{-1}$.

too small to give reliable estimates. Hence we chose to use aggregate β in our main simulations.

C.3 Alternative bootstrap procedure: links only

Instead of the procedure we use in the main paper, we could skip the step 2. of resampling nodes from the original graph and go straight into resampling links according to $\hat{h}_n$ for the original set of nodes. The motivation for this procedure is similar to the one we use: the original sample comes from the true data generating distribution, we have a good estimate for the distribution of the adjacency matrix, hence the networks simulated this way should preserve the structure of the original network. Skipping one step in the simulation simplifies the procedure, improves computational time and would also simplify the proofs. We run some simulations using this method and found that, unfortunately, it does not perform as well as our main approach. Table Table 14 shows the results of some of our simulations. We see that the confidence interval coverage is very poor, other than for a few special cases where the sample size is small ($n = 25$, in which case the bias is small relative to variance and the true value may still be included in the confidence interval) or we are estimating a statistic which is relatively tricky to estimate (e.g. λ_2) and hence measured with more variation than e.g. λ_1 or density.

Our hypothesis for why the performance is so poor is that if the bandwidth a_n is small, or if for some individuals there are no close neighbours, our procedure becomes similar to the empirical bootstrap of Green and Shalizi (2022): we draw a link between two individuals if and only if they were linked in the original graph. Hence if we don't resample individuals, the bootstrapped graph may become too similar to the original graph, at least on the subgraphs consisting of individuals with few neighbours. This can lead to insufficient variation in the bootstrapped graphs and worse performance of the bootstrap procedure. Fig. 16 and Fig. 17 shows a comparison of our original method (in blue) and the version which only resamples links while keeping the original nodes (in orange). While both do a good job of replicating the statistic values in the bootstrapped graph (dashed black line), the version which only resamples links is too concentrated around the value in the bootstrapped graph and often misses the population value of the statistic (red solid line), leading to poor confidence interval coverage of the links-only procedure.

This indicates that if one is interested in uncovering the population values, our main procedure is more reliable. However, in applications where we are only interested in confidence intervals for a specific sample, bootstrapping links only does provide narrower confidence intervals and would be preferred.

generating function	n	ρ_n	95% CI coverage for density	transitivity	λ_1	λ_2
high density	25	0.379500	0.803	0.848	0.791	0.978
		0.569250	0.803	0.788	0.776	0.979
		0.759000	0.755	0.730	0.752	0.973
	100	0.379500	0.541	0.537	0.495	0.824
		0.569250	0.509	0.554	0.528	0.925
		0.759000	0.532	0.518	0.529	0.963
	300	0.379500	0.168	0.144	0.156	0.662
		0.569250	0.211	0.222	0.227	0.824
		0.759000	0.340	0.337	0.339	0.947
	500	0.379500	0.068	0.059	0.058	0.410
		0.569250	0.087	0.110	0.092	0.673
		0.759000	0.251	0.282	0.260	0.950
horseshoe	25	0.056250	0.727	0.920	0.838	0.956
		0.084375	0.776	0.910	0.770	0.936
		0.112500	0.745	0.757	0.715	0.916
	100	0.056250	0.530	0.520	0.406	0.819
		0.084375	0.616	0.419	0.387	0.717
		0.112500	0.686	0.365	0.359	0.679
	200	0.056250	0.430	0.344	0.243	0.659
		0.084375	0.585	0.308	0.302	0.606
		0.112500	0.592	0.270	0.287	0.501
	300	0.056250	0.433	0.285	0.215	0.575
		0.084375	0.589	0.266	0.227	0.504
		0.112500	0.568	0.237	0.240	0.460
	500	0.056250	0.407	0.213	0.158	0.498
		0.084375	0.530	0.215	0.221	0.413
		0.112500	0.490	0.217	0.218	0.326
product	25	0.125000	0.562	0.907	0.652	0.911
		0.250000	0.484	0.785	0.513	0.963
	100	0.125000	0.331	0.567	0.366	0.887
		0.250000	0.263	0.423	0.298	0.957
	300	0.125000	0.203	0.253	0.164	0.867
		0.250000	0.124	0.177	0.136	0.914
	500	0.125000	0.156	0.165	0.140	0.836
		0.250000	0.089	0.126	0.090	0.865

Table 14: 95% confidence interval coverage for density, transitivity, λ_1 and λ_2 for different generating functions, different sample sizes n from 25 ot 500 and at different densities ρ_n, based on Monte Carlo simulations using a version of the algorithm which keeps the original set of nodes and only resamples the links between them.

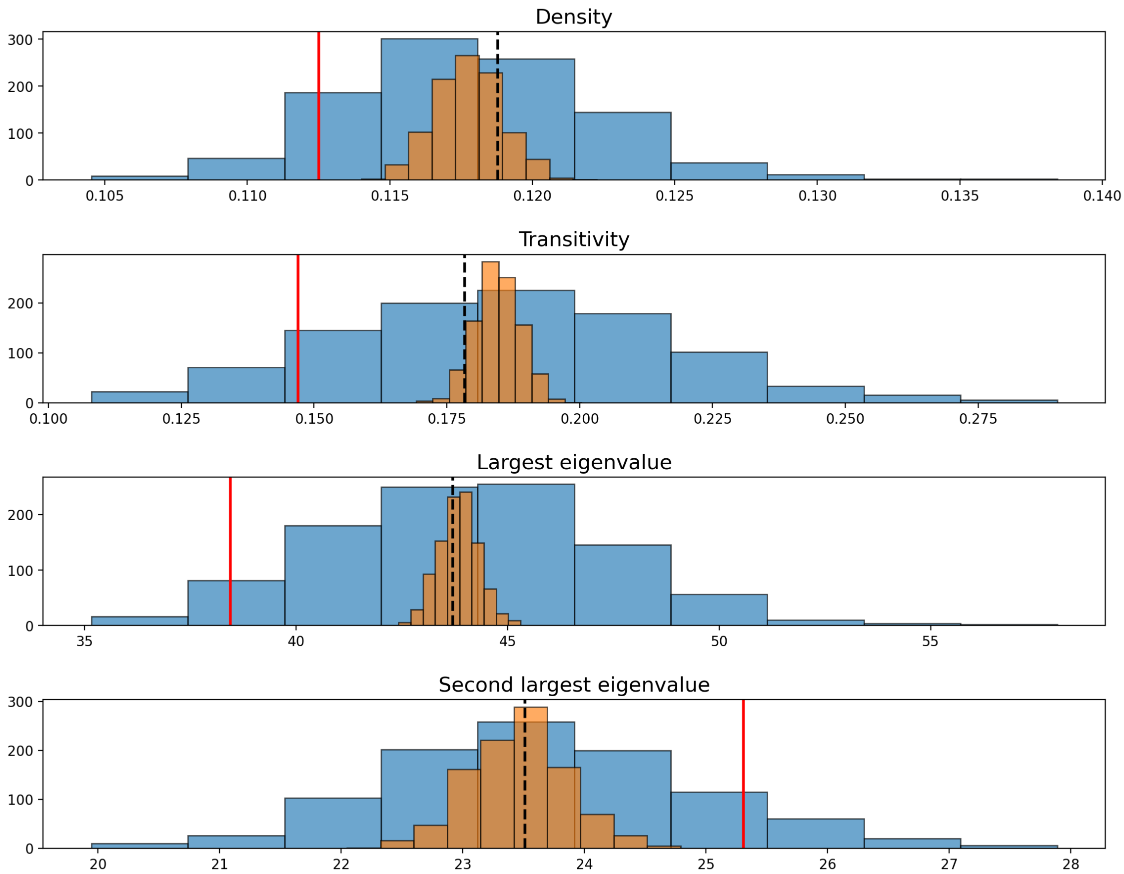


Figure 16: Comparison of histograms for our main bootstrap approach (in blue) and for the links-only version (in orange) from estimation of one specific network from the horseshoe generating function with $n = 300$ and $\rho_n = 0.1125$. The estimated statistics are, from top to bottom: density, transitivity, λ_1 and λ_2. The red solid line denotes the population true value of the statistic while the black dashed line denotes the value of the statistic in the bootstrapped graph.

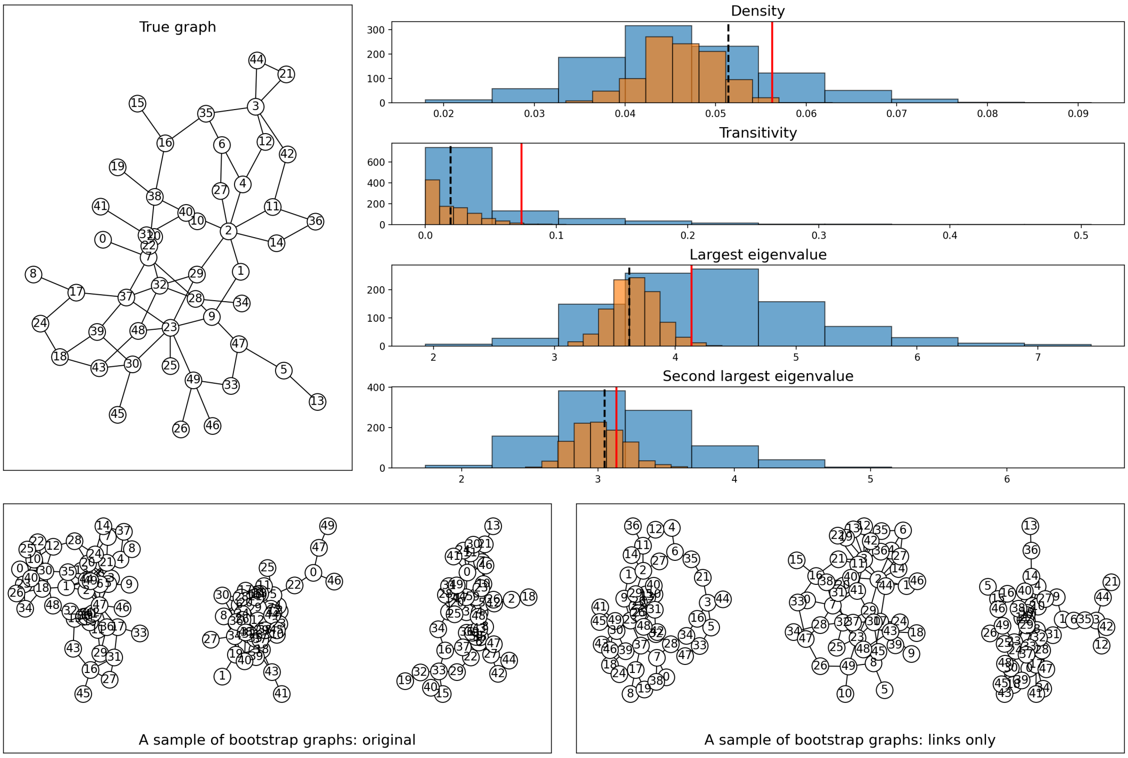


Figure 17: Comparison of histograms for our main bootstrap approach (in blue) and for the links-only version (in orange) from estimation of one specific network from the horseshoe generating function with $n = 50$ and $\rho_n = 0.05625$. The estimated statistics are, from top to bottom: density, transitivity, λ_1 and λ_2. The red solid line denotes the population true value of the statistic while the black dashed line denotes the value of the statistic in the bootstrapped graph.