Optimal prevention of unknown risks: A dynamic approach with learning

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Abstract

As illustrated by the recent terrorist attacks in the U.S., one must often take preventive actions to fight risks before having precise information on their probability distribution. In this paper, we explain how to adapt cost-benefit analysis to this situation. In an uncertain environment, the occurrence or the absence of loss allows the decision maker to update his belief about the frequency of losses in the future. Using a Bayesian expected utility model, we show that the uncertainty on the distribution of the loss raises the initial level of prevention if and only if the ratio of absolute prudence to absolute risk aversion is smaller than 2. In a numerical exercice, we show that the effect of this uncertainty may be large.

Keywords: learning, parameters uncertainty, predictability, precautionary principle, risk management, phantom risk.

Introduction

In traditional models of risk prevention,[1] the economy is static and the distribution of potential damages is known with certainty. This modeling is therefore not satisfactory to examine various relevant policy questions dealing with risk prevention. Consider for example the recent "mad cow crisis". One could reduce the potential damages to the health of the population by investing in various prevention activities, as for example by eliminating the use of animal-based feeds for cow, or even by excluding generations of cows borne before 1996 from the human diets. In France, the cost of this latter policy has been estimated to around 3 billions euros. The difficult challenge for economists is to determine how to use the standard cost-benefit analysis to determine the efficiency of the preventive policy when the distribution of potential damages to health is unknown. It is interesting to observe that economists have been particularly silent during the different stages of the crisis, leaving politicians facing the problem alone with agricultural lobbies, medical experts, sociologists and philosophers.

In this paper, we examine the problem of the optimal risk management of phantom risks, namely risks whose actual distribution is uncertain. It is a paradox that in spite of the ever improving scientific knowledge, the list of phantom risks is increasing over time. Here is a partial list of risks that have been phantom risks in the past, or are still considered as phantom: cancer from asbestos in the 50's, cancer from smoking in the 60's, contamination of AIDS by blood transfuses in the early 80's, environmental and health damages due to genetic manipulations of maize, global warming generated by the concentration of carbon dioxide in the atmosphere, health risks faced by individuals exposed to low nuclear radiations (or to electromagnetic fields, as those emitted by cellular phones), long term side effects of new drugs, effects due to the presence of various hazardous wastes, and the list of illustrations could go on by the hundreds.

What is the effect of the uncertainty affecting the distribution of a risk on the way we should perceive its prevention? The so-called precautionary principle,[2] which is now a legal rule in several European countries, states

[1]See Ehrlich and Becker (1972), Briys and Schlesinger (1990), Konrad and Skaperdas (1993), Lee (1998) and Jullien, Salanié and Salanié (1999).

[2]For a detailled discussion of the precautionary principle, see Godard (1997) and Gollier and Treich (2000).

that "where there are threats of serious or irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation" (Principle 15 of the 1992 Rio Declaration). It is still an open question to determine what this statement means for the cost-benefit analysis. In particular, it does not explain which distribution of the risk should be considered to measure costs and benefits when this distribution is uncertain.

In any static model, because expected utility is linear in the vector of probability p of the various states of nature, expected utility maximizers facing some uncertainty about p should do as if the distribution of the risk would be certain and fixed at the mean Ep. However, Ellsberg (1961) showed that several agents do not behave in the face of ambiguous probabilities like expected-utility maximizers. Gilboa and Schmeidler (1989) proposed a decision criteria in which agents would be averse to ambiguity. Under this criteria, agents compute their expected utility (EU) for the various possible distributions, and they do as if the distribution generating the lowest EU would hold with certainty. This min-max criteria raises several difficulties.[3] We will therefore assume in this paper that the population maximizes expected utility.

A common feature of the above-mentioned examples of phantom risks is that there is some uncertainty about the intensity of the risk, but this uncertainty will be reduced over time. This resolution of the uncertainty is expected to come either from the observation of actual damages by those

[3]First, it is hard to believe that people are as much pessimistic. In a world where extreme potential distributions of risk can rarely be excluded, the use of such a criteria leads to a dead end. In the case of the mad cow for example, we don't see a majority of consumers completely eliminating beef from their diet (see Gollier (2001b)), a strategy that would be optimal under ambiguity aversion. Second, such a model cannot be applied in a dynamic framework as long as we have no sensible rule to update ambiguous probabilities in a way similar to the Bayes rule for unambiguous probabilities. We believe that the dynamic aspect of risk management is an essential element of the problem. Third, on a more empirical ground, Viscusi and Chesson (1999), using a sample of 266 business owners facing risks from climate change, show evidence of both ambiguity-seeking behavior and ambiguity-averse behavior. More precisely, people seem to exhibit fears effect of ambiguity for small probabilities of suffering a loss, and hope effects for large probabilities. Fox and Tversky (1995) also showed in a series of experimental studies that ambiguity aversion, present in comparative contexts in which a person is confronted to both clear and ambiguous prospects, seems to disappear in noncomparative contexts in which a person evaluates only one of these prospects in isolation.

exposed to the risk, or by scientific progresses. In most cases, it seems that the first source of information dominates the second. It is the observation of recent extreme weather events that brings new knowledge about the risk of global warming, more than the improvement of climate modelling by meteorologists. It is the observation of the number of consumers hit by the human form of the mad cow disease in the U.K. that is the first source of information allowing for an update of our beliefs, as it happened that no apparent progress has been made about our understanding the biological mechanisms underlying the disease. Thus, our paper focuses on the optimal prevention of a recurrent risk whose distribution is updated from observing its past realizations by using Bayes rule.

How does this form of learning affect the initial risk-taking attitude of risk-averse expected-utility maximizers? The occurrence of a large loss in the first period has two effects. First, it reduces wealth and it raises the marginal utility of this wealth. Second, it induces agents to revise the probability of the worse scenario. The intuition suggests that this shift in the expected distribution of the risk should raise the marginal value of wealth. If this is true, then learning raises the marginal value of wealth where it is large, and reciprocally, it reduces the marginal value of wealth where it is low. In consequence, the process of learning plays a role on the initial risk attitude that is equivalent to an increase in risk aversion. In other words, the uncertainty about the distribution of damages makes agents more averse to the risk, i.e., they will invest more in prevention. This is compatible with the precautionary principle.

However, this result relies on the assumption that a bad news, i.e. an increase in the probability of the bad scenario, raises the marginal value of wealth. This is not true in general, however. Intuitively, an increase in the probability of damages has two effects on the marginal value of wealth. First, it makes agents poorer in expectation in the future. This wealth effect raises the marginal value of wealth. The intensity of this effect is increasing with the speed at which the marginal utility of consumption decreases with consumption, namely, it is increasing in the degree of absolute risk aversion A. But there is a second effect that goes the opposite direction. The increase in the probability of loss induces agents to reduce their exposure to the risk in the future. That yields a negative precautionary effect, in the sense that the reduced future risk makes wealth accumulation less valuable for a precautionary saving motive. This negative effect is increasing in the intensity of the precautionary saving motive which is proportional to absolute prudence P, an index introduced by Kimball (1990) to measure the degree of convexity of marginal utility. We are thus in a situation to understand that an increase in the loss probability raises the marginal value of wealth only if the wealth effect is sufficiently stronger than the negative precautionary effect. This is the case only if the ratio of absolute risk aversion to absolute prudence is larger than a critical level which we show to be equal to $1/2$. If, on the contrary, this ratio is less than $1/2$, the uncertainty surrounding the distribution of the risk should induce the social planner to reduce the prevention effort targeted to this risk in comparison to risks that are better known, contradicting the precautionary principle.

It is not a simple task to determine whether A/P is larger than $1/2$ because there is no estimation of absolute prudence in the literature. Following most researches in macroeconomic and finance, let us make the assumption that relative risk aversion is constant. Under this assumption it is easy to check that $A/P \geq 0.5$ if and only if relative risk aversion is larger than 1. This is a widely accepted assumption. Therefore, our model provides an argument in favor of the precautionary principle by claiming that more prevention efforts should be targeted to more uncertain risks.

This work is related to a reviving literature of learning in dynamic portfolio management. Our model is an example of a decision problem in which the future investment opportunity set is stochastic. Merton (1973) was the first to characterize rules for the optimal dynamic portfolio management. Detemple (1986), Gennotte (1986), Brennan (1998) and Brennan and Xia (1999)[4] examined the specific case where the opportunity set is stochastic due to the initial parameter uncertainty of the dynamic stochastic process. They solved various continuous-time infinite-horizon portfolio problems by assuming that relative risk aversion is constant. They showed that the sign of the effect of

[4]McCardle and Winkler (1992) examined a similar problem applied to gambling. In a casino, there is an urn of indistinguishable coins, half of which are "good" and half "bad". The good coins land heads with different probabilities that are known a priori. A single coin is picked at random from the urn that is used for n plays of the game. At each play of the game, you choose how much you want to bet. What is the optimal dynamic strategy in this game against nature? McCardle and Winkler (1992) raised this question to over 200 students and obtained that most people prefer not to bet at first, in order to gather information about the coin. This is fully compatible with our result under the assumption that $A/P \geq 0.5$.

learning on the initial optimal portfolio depends upon whether relative risk aversion is larger or smaller than unity. One of the achievements of our paper is to provide an intuition to this result by relying on the relative strength of the wealth effect and of the precautionary effect generated by a change in the distribution of the underlying risk.

Section 2 is devoted to the presentation of a simple two-period model. The link between learning and the marginal value of wealth is established in section 3. In section 4, we show that the effect of an increase in the loss probability in the second period raises the marginal value of wealth in the first period if and only if A/P is larger than 0.5. Our main result for the two-period prevention problem is established in section 5. We show that our main result can be extending to more than two periods in section 6, whereas section 7 provides a numerical illustration of the effect of the uncertainty surrounding the probability of loss.

2 The two-period model

We consider an economy with a representative agent leaving for two periods $t = 0, 1$. At the beginning of period t, the agent earns a sure income w_t. Consumption takes place at the end of each period. The von Neumann-Morgenstern utility function u on consumption is assumed to be increasing, concave and three time continuously differentiable. His lifetime utility is $u(c_0) + \beta u(c_1)$, where c_t is the consumption at the end of period t and β is the discount factor.

At each period t, the agent faces a risk $\widetilde{x}_t$ of loss L with probability p. We assume that $\widetilde{x}_0$ and $\widetilde{x}_1$ are i.i.d.. Whereas L is perfectly known, the probability p is subject to some uncertainty. Let $\widetilde{\pi}_0$ be the random variable representing the prior uncertainty about the probability of the loss. Let $p_0 = E\widetilde{\pi}_0$ be the expected prior probability of the loss.

At the end of the first period, the representative agent updates his beliefs about the probability of the loss. Let $\widetilde{\pi}_L$ and $\widetilde{\pi}_N$ be the posterior distribution of this probability respectively in the loss state and in the no loss state. Using Bayes' rule, it is straightforward to check that $p_L = E\widetilde{\pi}_L$ is larger than $p_N = E\widetilde{\pi}_N$: observing a loss in the first period raises the probability of a loss in the second period.

At the beginning of each period, the representative agent has to determine

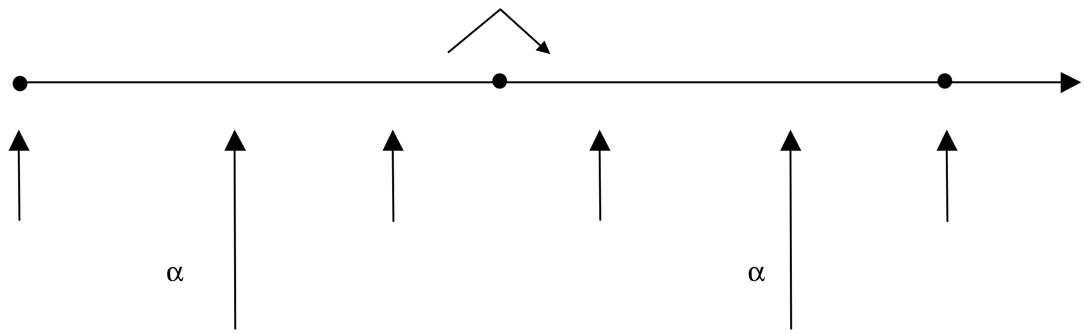

Figure 1: The timing of the decision process

how much to invest in a preventive activity. Each dollar invested in it reduces the loss by $k \geq 1$ dollars if it occurs. If k is smaller than the inverse of the probability of loss, the investment in prevention is actuarially unfair, and the myopic representative agent does not fully prevent the loss to occur, i.e., the investment is less than L/k. We hereafter assume that k is smaller than the inverse of the largest possible ex post expected probability of loss, i.e., $k \leq 1/p_L$. This assumption implies that the nonnegativity constraint on the investment on prevention is never binding. This assumption could be relaxed at the cost of more technicalities. Our results are robust to this relaxation.

A crucial element of the model is the opportunity for the agent to save some of his initial income for consumption in the second period. Let $R = 1+r$ be the gross interest rate in the economy. We assume that it is not affected by first period state. The saving decision takes place after observing the first period state. The timing of the decision process is described in Figure 1.

We solve the problem of the decision maker by backward induction. We start with the second period prevention problem, given saving s and the first-period state x. It is written as

$$\max_{\alpha} \quad E\left[\widetilde{\pi}_x u(w_1 + Rs - \alpha - (L - k\alpha)) + (1 - \widetilde{\pi}_x) u(w_1 + Rs - \alpha)\right], \quad (1)$$

where α is the amount invested in prevention in the second period. Because of the linearity of the objective function with respect to probabilities, program (1) can be rewritten as

$$v(s,p_x) = \max_{\alpha} \quad p_x u(w_1 + Rs - \alpha - (L - k\alpha)) + (1 - p_x) u(w_1 + Rs - \alpha). \quad (2)$$

The decision maker just need to know the expected probability of loss to solve his problem. The ambiguity about $\widetilde{\pi}$ does not matter, as is standard under expected utility. This principle was also recognized in Gennotte (1996) who stated that the agent solves his dynamic decision problem in two stages. First, the agent updates his beliefs using Bayes' rule. Second, there is a choice stage in which the agent maximizes his expected utility by using the estimated probabilities obtained in the first stage. In general, the optimal prevention $\alpha^*(s,p_x)$ depends upon the level of savings s and the expected probability p_x.

One can use the maximal expected utility $v(s,p_x)$ of the second period consumption to solve the saving problem that arises at the end of the first period. Let z denote the cash-on-hand of the agent at the end of the first period, that is after incurring the first period loss if any, but before first period consumption. The consumption-saving problem is written as follows:

$$V(z,p_x) = \max_{s} \quad u(z - s) + \beta v(s,p_x). \quad (3)$$

It can be verified that V is increasing and concave with respect to its first argument.

Finally, the agent solves the first period prevention problem:

$$\max_{\alpha} \quad p_0 V(w_0 - \alpha - (L - k\alpha), p_L) + (1 - p_0) V(w_0 - \alpha, p_N). \quad (4)$$

The optimal level of prevention in the first period is denoted α_0^*.We again used the linearity of the objective function with respect to probabilities to replace the uncertainty on $\widetilde{\pi}_0$ by its expectation p_0. The difficulty of this decision problem comes from the state dependency of the indirect utility function V. However, because the objective function is a weighted sum of concave functions of the decision variable, the first-order condition is necessary and sufficient for an optimum.

Our objective in this paper is to compare α_0^* to the optimal level of initial prevention when the probability of loss is known with certainty and is equal to p_0. Notice that there is no learning in this alternative model. The decision problem of the representative agent is written as

$$\max_{\alpha} \quad p_0 V(w_0 - \alpha - (L - k\alpha), p_0) + (1 - p_0) V(w_0 - \alpha, p_0). \tag{5}$$

We want to determine the condition under which the optimal solution $\widehat{\alpha}_0$ of this program is smaller than α_0^*, the optimal level of prevention when the probability of loss is uncertain, but with the same prior expected loss.

3 The effect of the state dependent utility on the optimal prevention

The first-order condition for the first period level of prevention when there is no uncertainty about the probability of loss yields

$$(k-1) p_0 \frac{\partial V}{\partial z}(z_L, p_0) = (1 - p_0) \frac{\partial V}{\partial z}(z_N, p_0), \tag{6}$$

where $z_L = w_0 - \widehat{\alpha}_0 - (L - k\widehat{\alpha}_0)$ is the cash-on-hand when a loss occurs, and $z_N = w_0 - \widehat{\alpha}_0$ is the cash-on-hand when there is no accident. Because the objective function of program (4) is concave in the decision variable α, its optimal solution will be larger than $\widehat{\alpha}_0$ if and only if its derivative with respect to α is positive when evaluated at $\alpha = \widehat{\alpha}_0$. This condition is equivalent to

$$(k-1) p_0 \frac{\partial V}{\partial z}(z_L, p_L) - (1 - p_0) \frac{\partial V}{\partial z}(z_N, p_N) \geq 0. \tag{7}$$

Eliminating k from this condition by using equation (6) makes it equivalent to

$$\frac{\frac{\partial V}{\partial z}(z_L, p_L)}{\frac{\partial V}{\partial z}(z_L, p_0)} \geq \frac{\frac{\partial V}{\partial z}(z_N, p_N)}{\frac{\partial V}{\partial z}(z_N, p_0)}. \tag{8}$$

This condition would be satisfied if an increase in the probability of loss would raise the marginal value of cash-on-hand. Indeed, this would imply that the left-hand side of the above equality would be larger than unity, whereas the right-hand side would be smaller than unity. We thus proved the following Lemma.

Lemma 1 *The uncertainty about the probability of loss increases (resp. reduces) the optimal initial level of prevention if the marginal value of wealth ($\partial V/\partial z$) is an increasing (resp. decreasing) function of the probability of loss.*

This result is intuitive. Suppose that the marginal value of wealth is increasing in the probability of loss. The occurrence of a loss in the first period raises the probability of loss in the second period. This raises the marginal value of wealth in a state where it is already large because of the loss itself. On the contrary, the absence of a loss in the first period reduces the probability of a loss in the second period. It also reduces the marginal value of wealth in a state where it is already relatively small because the agent is wealthy. In comparison to the case where the probability of a loss is certain, the learning process reduces the marginal value of wealth where it is low and it increases it where it is large. These effects are thus equivalent to an increase in the concavity of the value function with respect to wealth. In conclusion, this raises the optimal investment in prevention.

4 The effect of an increase in the probability of loss on the marginal value of wealth

We must thus examine the conditions under which an increase in the probability of loss raises the marginal value of wealth. We do this in two steps. First, we consider the pure static prevention problem (2). We derive the condition for $\partial v(s,p)/\partial s$ to be increasing in p. We then extend the analysis to take into account of the saving problem (3), by showing that this property is inherited by function V.

The following second Lemma relies on two indexes: the index of absolute risk aversion $A(c) = -u''(c)/u'(c)$ which measures the concavity of u, and the index of absolute prudence $P(c) = -u'''(c)/u''(c)$, which measures the degree of convexity of marginal utility. This latter concept has been introduced by Kimball (1990) to measure the strength of the precautionary saving motive in a lifecycle consumption model.

Lemma 2 *An increase in the expected probability of loss raises (resp. reduces) the marginal value of savings $\partial v/\partial s$ defined by program ( ) if and only if the ratio of absolute prudence to absolute risk aversion is smaller (resp. larger) than .*

Proof: The first-order condition to problem (2) is written as:

$$(k-1)pu'_L = (1-p)u'_N, \tag{9}$$

where $u_L^{(n)} = u^{(n)}(w_1 + Rs - \alpha^* - (L - k\alpha^*))$ and $u_N^{(n)} = u^{(n)}(w_1 + Rs - \alpha^*)$. Fully differentiating this condition with respect to p yields

$$\frac{\partial \alpha^*}{\partial p} = -\frac{(k-1)u'_L + u'_N}{(k-1)^2 pu''_L + (1-p)u''_N}. \tag{10}$$

The envelope theorem implies that

$$\frac{\partial v}{\partial s} = R\left[pu'_L + (1-p)u'_N\right].$$

By fully differentiating this condition with respect to p, we obtain that

$$\frac{1}{R}\frac{\partial^2 v}{\partial s \partial p} = u'_L - u'_N + \left[(k-1)pu''_L - (1-p)u''_N\right]\frac{\partial \alpha^*}{\partial p}.$$

If we replace $\partial \alpha^*/\partial p$ by its expression in (**??**), we obtain that the left-hand side of the above equality is positive if and only if

$$u'_L - u'_N \geq \frac{(k-1)pu''_L - (1-p)u''_N}{(k-1)^2 pu''_L + (1-p)u''_N}\left[(k-1)u'_L + u'_N\right].$$

After eliminating terms, this condition is equivalent to

$$\frac{u''_N}{(u'_N)^2} \leq \frac{u''_L}{(u'_L)^2}. \tag{11}$$

Because we assumed that $k \leq 1/p$, the first-order condition (9) implies that $u'_L \geq u'_N$. This means that $L - k\alpha^*$ is positive, namely, that the loss is not fully covered. It implies that consumption is larger in the no loss state than in the loss state. This implies that condition (11) is satisfied if and only if the function ϕ defined by $\phi(c) = u''(c)/(u'(c))^2$ is decreasing in c. This is equivalent to requiring that $u'''(c)(u'(c))^2 - 2u'(c)(u''(c))^2$ be nonpositive. This is the case if and only if P/A is uniformly smaller 2.■

Thus, when P/A is smaller than 2, an increase in the probability of loss raises the marginal value of savings. Of course, it also increases the optimal level of savings and the marginal value of cash-on-hand in the first period, as stated in the following Lemma.

Lemma 3 *An increase in the expected probability of loss raises (resp. reduces) the marginal value of cash-on-hand $\partial V/\partial z$ defined by program (3) and savings if and only if the ratio of absolute prudence to absolute risk aversion is smaller (resp. larger) than 2.*

Proof: Using the envelope theorem for program (3) yields

$$\frac{\partial V}{\partial z}(z,p) = u'(z-s^*) = \beta\frac{\partial v}{\partial s}(s^*,p).$$

Fully differentiating this system with respect to p, we obtain that

$$\frac{\partial s^*}{\partial p} = -\frac{\frac{\partial^2 v}{\partial s \partial p}(s^*,p)}{\beta^{-1}u''(z-s^*) + \frac{\partial^2 v}{\partial s^2}(s^*,p)},$$

and

$$\frac{\partial^2 V}{\partial z \partial p}(z,p) = -u''(z-s^*)\frac{\partial s^*}{\partial p}.$$

Because v is concave in s, we directly obtain that $\partial s^*/\partial p$ and $\partial^2 V/\partial z\partial p$ have the same sign than $\partial^2 v/\partial s\partial p$. Lemma 2 yields the result.■

There is an intuition for this result. Consider the special case $p = 1/k$ as our benchmark. In such a situation, the investment in prevention is actuarially fair, and it is optimal to fully prevent the risk. Suppose now that the probability of loss is reduced, thereby making the investment in prevention actuarially unfair. This reduction in the probability of loss increases the expected final consumption. Because the agent wants to smooth his consumption over time, this reduction in the probability reduces his willingness to save. The strength of this consumption smoothing effect is proportional to the index A of aversion to consumption fluctuation over time. But because the return on the preventive investment is actuarially unfair, it is optimal not to eliminate the risk completely. The presence of this future risk induces the representative agent to accumulate precautionary saving. The strength of this precautionary effect is proportional to P, as shown by Kimball (1990). Thus, a reduction in the probability of loss reduces savings if and only if the precautionary effect is dominated the consumption smoothing effect. This is the case when P/A is sufficiently small. From the above Lemma, this is in fact the case when P/A is smaller than 2. If the level of savings is reduced, the first period consumption is increased, and the marginal utility of consumption, together with the marginal value of cash-on-hand, is reduced. Thus, when P/A is smaller than 2, the optimal saving and the marginal value of cash-on-hand are positively correlated with the probability of loss.

Notice that the above results do not hold only for the case of a departure of p from the fair situation $p = 1/k$. They also hold for any marginal changes in the probability of loss. A marginal reduction in the probability of loss raises the expected final consumption, and it also raises the size of the optimal risk taken by the decision maker. The same consumption smoothing effect and the same precautionary effect work at the margin.

5 The main result and its discussion

We can combine our three lemmas to derive the following result, which is our main result.

Proposition 1 *The uncertainty about the probability of loss raises (resp. reduces) the efficient level of the preventive effort in the first period if the ratio of absolute prudence to absolute risk aversion is smaller (resp. larger) than unity.*

To sum up, when P/A is smaller than 2, the consumption smoothing effect dominates the precautionary effect. In such a situation, learning reduces the marginal value of wealth in the no loss state, and it raises the marginal value of wealth in the loss state. This induces the representative agent to raise the investment in the preventive activity. Notice that, as is well-known, log utility decision maker are optimally myopic. In this specific environment, this means that their optimal investment in prevention in the first period is not affected by the uncertainty surrounding the probability of loss.

Condition $P/A \leq 2$ is not new in the economics of uncertainty. Determining whether P/A is smaller than 2 or equivalently whether $1/u'(c)$ is convex in c appeared in different contexts. Drèze and Modigliani (1972) examined the reduction in the level of savings due to the introduction of complete insurance markets. They proved that the level of savings decreases in such a circumstance if and only if $d^2(U_1/U_2)/dc_2^2 \geq 0$ where U_j denotes partial derivatives of a non time-separable utility function $U(c_1, c_2)$. Assuming $U(c_1, c_2) = u(c_1) + v(c_2)$ this condition becomes $1/v'$ convex.

Condition $P/A \leq 2$ is also useful in industrial economics. For example, Gabel and Sinclair-Desgagné (1997) uses it in a problem of optimal audit in the principal-agent model. Finally, Debreu and Koopmans (1982) argue that a good measure of risk aversion should be the "concavity index" $-u''/(u')^2$. If we admit the Debreu and Koopmans index, decreasing absolute risk aversion is thus equivalent to $P/A \geq 2$. The Debreu-Koopmans index was recently used in political economy theory. For example, Alesina and Tabellini (1990) showed that the uncertainty in the identity of the median tomorrow generates a bias towards deficit if the concavity index is decreasing.

The more general condition $P/A \geq m$ is useful for various values of scalar m. For example, the case $m = 0$ corresponds to positive prudence, a condition that is standard since Leland (1968) to justify the precautionary motive to saving. The stronger condition corresponding to $m = 1$ is equivalent to decreasing absolute risk aversion. This condition is also very natural. However, decreasing absolute risk aversion ($P/A \geq 1$) may be compatible either with $P/A \leq 2$ or with $P/A \leq 2$. In particular, the condition $P/A \leq 2$ that is compatible with the precautionary principle means that absolute risk aversion can be decreasing, but not too much decreasing.

Because there does not exist any reliable estimation of the degree of absolute prudence, determining whether P/A is smaller or larger than 2 is thus an open question. Some light can be shed on this question by limiting the analysis to utility functions exhibiting constant relative risk aversion, namely those with $u'(c) = c^{-\gamma}$. In that case, relative risk aversion is constant and equal to γ. Moreover, $P/A = (\gamma + 1)/\gamma$. Therefore, when relative risk aversion is constant, P/A is smaller than 2 if and only if relative risk aversion is larger than unity. There exist compelling evidence that most people have a relative risk aversion larger than unity. For example, as is well-known from the so-called equity premium puzzle, it is hard to explain the observed asset prices over the century without relying to preferences exhibiting a degree of relative risk aversion smaller than 30. Gollier (2000a) presents a similar puzzle on the insurance demand: the standard insurance model cannot explain the very low level of deductible in insurance without making a similar assumption. An experienced reader in macroeconomics and in finance would observe that most researchers calibrate their model by using an index of relative risk aversion between 1 and 10. We conclude that there is a consensus on this interval in our profession, and that P/A should be smaller than 2, assuming that relative risk aversion be constant. According to Proposition

1, we should observe that agents invest more in prevention for risks whose probability of loss is more uncertain, ceteris paribus.

In this paper, we assume that the only source of information comes from observing the realization of the risk in the first period. As in Gollier, Jullien and Treich (2000), we could have considered the possibility to update our beliefs by observing other signals, as those coming from the improvement of scientific knowledge. However, it can be inferred from Gollier (2000b) that this form of learning has no effect on the optimal risk exposure in the first period when relative risk aversion is constant.

6 Extension to more than 2 periods

In this section, we show that our two-period model can be extended to any finite horizon T model without affecting our result.

Let $J_t(s, \widetilde{\pi})$ be the value of wealth at the beginning of period t. It is a function of wealth s available at that date, and of the current distribution $\widetilde{\pi}$ of the probability of loss. We have $J_{T+1}(s, \widetilde{\pi}) = u(w_T + Rs)$. Within period t, two different decisions are taken. At the end of the period, there is a consumption-saving decision which is written as

$$V_t(z, \widetilde{\pi}_{t+1}) = \max_s \quad u(z-s) + \beta J_{t+1}(s, \widetilde{\pi}_{t+1}) \tag{12}$$

for a given cash-on-hand z available after the realization of risk $\widetilde{x}_t$ during the period and the corresponding updated distribution $\widetilde{\pi}_{t+1}$ of the loss probability. Ex-ante, before the realization of $\widetilde{x}_t$, the representative agent determines the optimal level of prevention by solving the following problem:

$$J_{t-1}(s, \widetilde{\pi}_t) = \max_\alpha \quad [E\widetilde{\pi}_t]\, V_t(w_t + Rs - \alpha - (L - k\alpha), \widetilde{\pi}_{t+1} \mid \widetilde{\pi}_t, \widetilde{x}_t = L) \tag{13}$$

$$+(1 - [E\widetilde{\pi}_t])V_t(w_t + Rs - \alpha, \widetilde{\pi}_{t+1} \mid \widetilde{\pi}_t, \widetilde{x}_t = 0).$$

We would be done if the condition $P/A \geq 2$ or $P/A \leq 2$ would be transmitted from J_{t+1} to V_t, and from V_t to J_{t-1}. We hereafter show that these inheritance properties hold. By using backward induction and Proposition 1, we will conclude that the initial level of prevention would be increased by the uncertainty surrounding the probability of loss.

Observe first that both the consumption-saving problem (12) and the prevention problem (13) can be written can be written as

$$h(w) = \max_{c(.)} \quad Eg(c(\widetilde{s}), \widetilde{s}) \tag{14}$$

$$s.t. \quad E\eta(\widetilde{s})c(\widetilde{s}) = w, \tag{15}$$

for some random variable $\widetilde{s}$, some function $g(.,.)$ that is concave with respect to its first argument and some positive function $\eta(.)$. For example, the consumption-saving problem corresponds to a binary distribution of $\widetilde{s} \sim (1, 1/(1+\beta); 2, \beta/(1+\beta))$, $g(c,1) = u(c)$, $g(c,2) = J_{t+1}(c, \widetilde{\pi}_{t+1})$, $\eta(1) = \beta + 1$, $\eta(2) = (\beta+1)/\beta$ and $z = w$. A similar exercise can be done for program (13). Thus the question is to determine whether function h in program inherits property $P_h/A_h = h'''h'/h''^2 \geq (\leq) 2$ from property

$$\frac{P_g}{A_g} = \frac{\frac{\partial^3 g}{\partial c^3}\frac{\partial g}{\partial c}}{\left[\frac{\partial^2 g}{\partial c^2}\right]} \geq (\leq) 2.$$

This property is proved in the next Proposition, whose technical proof is relegated to the Appendix. This result is extracted from Gollier (2001, Proposition 53, chapter 14).

Proposition 2 *Consider problem (14),(15) with a positive function η and a function $g(.,.)$ that is three time continuously differentiable and concave with respect to its first argument. Consider also any scalar m. P_h/A_h is uniformly smaller (resp. larger) than m whenever P_g/A_g is uniformly smaller (resp. larger) than m. It implies that the result presented in Proposition 1 holds when the time horizon has more than 2 periods.*

This result is related to the one by Caroll and Kimball (1996) who have shown that condition $P/A \geq m$ is a property that value functions inherits from the utility function in the standard consumption-saving problem, and for the standard portfolio problem. It is important to stress that the opposite property $P/A \leq m$ does not hold in general for the portfolio problem, when markets are incomplete. Our result states that this property is also inherited by the value function from the utility function when markets are complete, which is implicitly the case in our model with two states and two investment opportunities.

7 A numerical example with infinite horizon

It is not easy to describe the state contingent optimal prevention strategy in a finite horizon model, because this strategy depends upon the number of periods remaining. In this section, we estimate the effect of the uncertainty surrounding the probability of loss by assuming that the representative agent has an infinite horizon. The assumption that the time horizon is infinite is for the sake of simplicity, as it makes the optimal strategy independent of time. Our computations show that agents with a finite horizon follow almost the same strategy than if the horizon would be infinite if they have more than, say, 20 periods remaining. This is a standard "turnpike" property of dynamic strategies.

In an infinite horizon model, it is easier to consider a timing of the intra-period t by looking at the joint saving s_t and prevention α_{t+1} decisions that take place after the realization of risk $\widetilde{x}_t$. We therefore take the cash-on-hand z at that time as the state variable. This has the advantage to make the consumption in period t dependent of the risk in t only through the state variable z_t.

We assume that the agent has a constant income w per period. We suppose that the probability p of loss can be either p_1 or $p_2 \leq p_1$. This implies that the beliefs in period t is completely characterized by π_t, the probability that p be equal to p_1. Using Bayes' rule, this probability will be increased to

$$\pi_{t+1}(\pi_t, \widetilde{x}_t = L) = \frac{\pi_t p_1}{\pi_t p_1 + (1-\pi_t)p_2}$$

in period $t+1$ if a loss is incurred in period t. On the contrary, it is reduced to

$$\pi_{t+1}(\pi_t, \widetilde{x}_t = 0) = \frac{\pi_t(1-p_1)}{\pi_t(1-p_1) + (1-\pi_t)(1-p_2)}$$

if no loss was observed in period t. Let z represent the cash-on-hand before consumption. The Bellman equation is written as

$$\begin{aligned} J(z,\pi) \;=\; \max_{s,\alpha} \quad & u(z-s) \\ & +\beta(\pi p_1 + (1-\pi)p_2) J(w + Rs - \alpha - (L - k\alpha), \pi_{t+1}(\pi, L)) \\ & +\beta(1 - \pi p_1 - (1-\pi)p_2) J(w + Rs - \alpha, \pi_{t+1}(\pi, 0)) \end{aligned}$$

where J is the value function depending upon the two state variables z and π. After some manipulations, we can rewrite the Bellman equation as

$$J(z,\pi) = \max_{C,A} \quad u(C) + \beta E\left[J(\overline{w} + R(z-C) + Ag(\widetilde{x}), \pi_{t+1}(\pi,\widetilde{x})) \mid \pi\right],$$

where C is the level of consumption, A is the non-prevented loss $L - k\alpha$, $\overline{w} = w - k^{-1}L$ is the per-period income net of the cost of eliminating the risk, and $g(L) = k^{-1} - 1$ and $g(0) = k^{-1}$.

We hereafter assume that the representative agent has a constant relative risk aversion γ which is larger than 1. Then, the value function and the policy functions are separable:

$$J(z,\pi) = j(\pi)\frac{\left(z + (\overline{w}/R)\right)^{1-\gamma}}{1-\gamma},$$

$$C(z,\pi) = c(\pi)\left(z + (\overline{w}/R)\right),$$

and

$$A(z,\pi) = a(\pi)\left(z + (\overline{w}/R)\right).$$

The level of consumption and the non-prevented loss are proportional to the level of wealth measured by the sum of cash-on-hand and the present value of the next net income. We hereafter focus on the policy functions where $(z + (\overline{w}/R))$ has been normalized to unity.

Functions $j(.)$, $c(.)$ and $a(.)$ must satisfy the following conditions:

$$c(\pi)^{-\gamma} = \beta R E\left[j(\pi_{t+1}(\pi,\widetilde{x}))(R(1-c(\pi)) + a(\pi)g(\widetilde{x}))^{-\gamma} \mid \pi\right],$$

$$E\left[j(\pi_{t+1}(\pi,\widetilde{x}))g(\widetilde{x})(R(1-c(\pi)) + a(\pi)g(\widetilde{x}))^{-\gamma} \mid \pi\right] = 0,$$

and

$$j(\pi) = (c(\pi))^{1-\gamma} + \beta E\left[j(\pi_{t+1}(\pi,\widetilde{x}))(R(1-c) + ag(\widetilde{x}))^{1-\gamma} \mid \pi\right],$$

for all $\pi \in [0,1]$.

There is no hope to solve this system analytically, except in the logarithmic case ($\gamma = 1$). In the following, we present two simulations in which relative risk aversion is set to 2, the rate of pure preference for the present is equal to 2% ($\beta = 0.98$) and the interest rate is fixed to 1% ($R = 1.01$). Moreover, we assume that $k = 2$: each dollar invested in prevention reduces the loss by 2 dollars. In Figure 2, we draw an heavy curve describing the optimal level of non-prevented loss when the probability of loss is either $p_1 = 10\%$ with probability π, or $p_2 = 1\%$ with probability $1-\pi$. We compare this policy to the one (thin curve) that would be optimal without any uncertainty on the probability of loss that would be set to $p = \pi p_1 + (1-\pi)p_2$. In accordance to our main result, the parameter uncertainty reduces the optimal level of the non-prevented loss, i.e., it raises the initial prevention effort. Consider for example a situation in which there is no uncertainty about the probability of loss, which would be set equal to $p = 0.055$. Compare this situation with the one in which the probability of loss is distributed as $(0.1, 0.5; 0.01, 0.5)$. Notice that the expected probability of loss is $p = 0.055$ in the two situations. In Figure 2, we see that the optimal size of the non-prevented loss is reduced by 7%, coming from 104% to 97% of current wealth.

Of course the effect of parameter uncertainty is increasing with the size of the uncertainty. In Figure 3, we draw the optimal policy when the probability of loss is either $p_1 = 0.5$ with probability π or $p_2 = 0.01$ with probability $1-\pi$. Suppose that $\pi = 0.5$. In such an uncertain environment, the optimal value of a equals 0.045. On the contrary, when there is no uncertainty about the probability of loss fixed at $p = 0.255$, the optimal a is equal to 0.504. This means that the uncertainty on p reduces the optimal retained loss by more than 90%. In fact, the optimal level of prevention with $a = 0.045$ would be optimal in the absence of parameter uncertainty only if the probability of loss would be fixed at $p = 0.48$. In other words, the parameter uncertainty should induce the representative agent to do as if the probability of loss would be set to $p = 0.48$, very close to the worst case scenario with $p_1 = 0.5$! This effect is increasing with the degree of risk aversion.

8 Conclusion

People should devote more preventive effort, not less effort, when the risk is more uncertain. The uncertainty surrounding the intensity of a risk should not be taken as an excuse to delay investing in prevention. We show that this intuitive guideline is indeed optimal for expected-utility maximizers as soon as the ratio of their absolute prudence to their absolute risk aversion is uniformly smaller than 2, as is the case when their relative risk aversion is constant and larger than unity. Because this latter assumption is sustained by

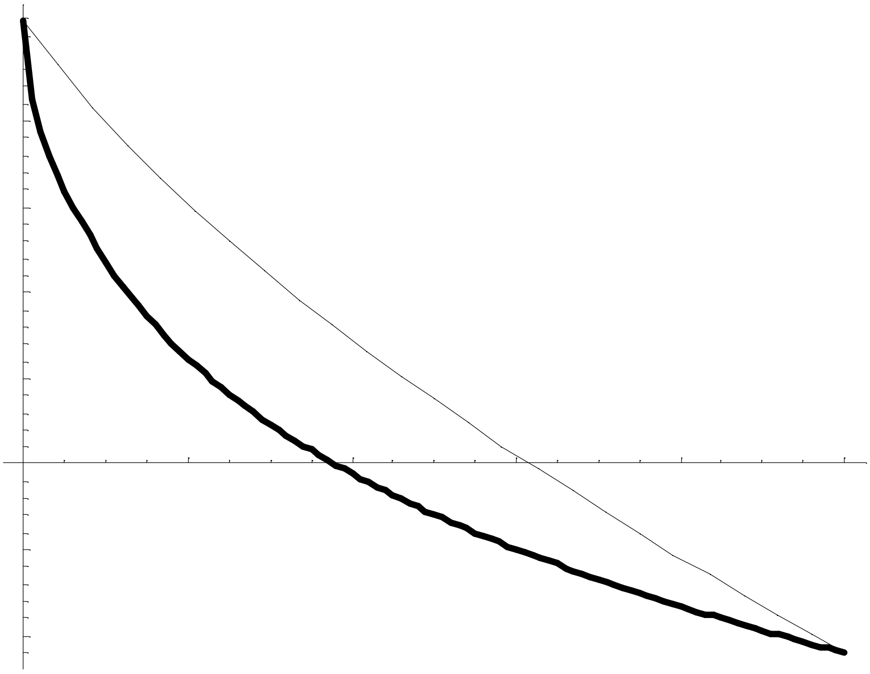

Figure 2: Optimal level of non-prevented loss when the probability of loss is distributed as $(0.1, \pi; 0.01, 1-\pi)$ (thick curve).

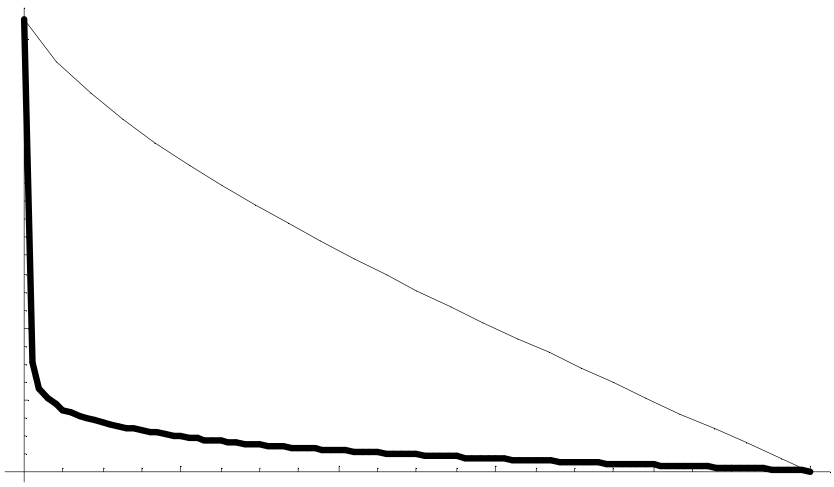

Figure 3: Optimal level of non-prevented loss when the probability of loss is distributed as $(0.5, \pi; 0.01, 1 - \pi)$ (thick curve).

the data, this paper provides a strong argument in favor of the precautionary principle.

Our model has its own limitations that should call for extensions. For example, we assumed that the size of the risk taken in the first period does not affect the quality of the signal. In reality, it is possible that the quality of the signal be increasing in the accepted size of the risk exposure. It could be interesting to examine the effect of this experimentation problem on the initial risk-taking attitude. A second potential extension would come from relaxing our assumption that the signal is immediately observed. In some cases, as for the mad cow disease or global warming, signals are observed a long time after risks have been undertaken. How do these delays affect the optimal dynamic risk management? Finally, we assumed in this paper that past decisions do not affect the current level of the risk. In many instances, there is a phenomenon of accumulation that takes place: the larger the risk accepted in the past, the larger the risk exposure today. For example, the risk of global warming in 100 years of now will depend upon the accumulation of greenhouse gas emissions. Gollier, Jullien and Treich (2000) examined a similar question, but without allowing for endogenous learning.

References

Alesina, A. and G. Tabellini, (1990), Voting on budget deficit, *American Economic Review*, 80, 37-49.

Brennan, M.J., (1998), The role of learning in dynamic portfolio decisions, *European Finance Review*, 1, 295-306.

Brennan, M.J., and Y. Xia, (1999), Assessing asset pricing anomalies, Discussion paper #19-99, The Anderson School at UCLA.

Briys, E. and H. Schlesinger, (1990), Risk aversion and the propensities for self-insurance and self-protection, *Southern Economic Journal*, 57, 458-67.

Caroll, C.D. and M. Kimball, (1996), On the concavity of the consumption function, *Econometrica*, 64, 981-92.

Debreu, G. and T. Koopmans, (1982) , Additively decomposed quasi-convex functions, *Mathematical Programming*, 24, 1-38.

Detemple, J.B., (1986), Asset pricing in an economy with incomplete information, *Journal of Finance*, 61, 383-392.

Drèze, J.H. and F. Modigliani, (1972) , Consumption decisions under uncertainty, *Journal of Economic Theory*, 5, 308-335.

Ehrlich, I., and G.S. Becker, (1972), Market insurance, self-insurance and self-protection, *Journal of Political Economy*, 80, 623-48.

Ellsberg, D., (1961), Risk, ambiguity, and the Savage axioms, *Quarterly Journal of Economics*, 75, 643-69.

Fox, C.R., and A. Tversky, (1995), Ambiguity aversion and comparative ignorance, *Quarterly Journal of Economics*, 110, 585-603.

Gabel, H.L. and B. Sinclair-Desgagné , (1997) , Environmental auditing in management systems and public policy, *Journal of Environmental Economics and Management*, 33, 331-46.

Gennotte, G., (1986), Optimal portfolio choice under incomplete information, *Journal of Finance*, 41, 733-749.

Gilboa, I. and D. Schmeidler, (1989), Maximin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18, 141, 153.

Godard, O., (1997), Social decision-making under conditions of scientific controversy, expertise and the Precautionary Principle, in *Integrating Scientific Expertise into Regulatory Decision-Making*, C. Joerges, K.-H. Ladeurand E. Vos editors, Nomos Verl.-Ges., Baden-Baden.

Gollier, C., (2000a), To insure or not to insure?: An insurance puzzle, mimeo, U. of Toulouse.

Gollier, C., (2000b), Optimal dynamic portfolio risk with first-order and second-order predictability, mimeo, U. of Toulouse.

Gollier, C., (2001), *The economics of risk and time*, MIT Press, forthcoming.

Gollier, C., (2001b), Should we beware of the precautionary principle?, *Economc Policy*, forthcoming.

Gollier C., Jullien B. and N. Treich, (2000), Scientific progress and irreversibility: An economic interpretation of the Precautionary Principle, *Journal of Public Economics*, 75, 229-53.

Gollier, C., and N. Treich, (2000), Decision-making under scientific progress: The economics of the Precautionary Principle, mimeo, U. of Toulouse.

Jullien, B., B. Salanié and F. Salanié, (1999), Should more risk-averse agents exert more effort?, *Geneva Papers on Risk and Insurance Theory*, 24, 19-28.

Kimball, M.S., (1990), Precautionary savings in the small and in the large, *Econometrica*, 58, 53-73.

Konrad, K., and S. Skaperdas, (1993), Self-insurance and self-protection: A nonexpected utility analysis, *Geneva Papers on Risk and Insurance Theory,* 18, 131-46.

Lee, K., (1998), Risk aversion and self-insurance-cum-protection, *Journal of Risk and Uncertainty*, 17, 139-50.

Leland, H.E., (1968), Saving and uncertainty: The precautionary demand for saving, *Quarterly Journal of Economics*, 465-473.

McCardle, K.F., and R.L. Winkler, (1992), Repeated gambles, learning, and risk aversion, *Management Science*, 38, 807-818.

Merton, R.C., (1973), An intertemporal capital asset pricing model, *Econometrica*, 41, 867-887.

Viscusi, K.W., and H. Chesson, (1999), Hopes and fears: the conflicting effects of risk ambiguity, *Decision Theory*, 47, 153-78.

Appendix: Proof of Proposition 2

Let $T_f(w) = -f'(w)/f''(w)$ denote the degree of absolute risk tolerance of function f, with f being function $h(.)$ or $g(., s)$. In the latter case, the derivatives are taken with respect to the first argument of the function. Notice that

$$T_f'(w) = -1 + \frac{P_f(w)}{A_f(w)}.$$

Therefore P_f/A_f is larger (smaller) than m if T_f' is larger (smaller) than $m-1$. Let $c(.) = \psi(., w)$ characterize the solution of program (14),(15). The first-order condition for this program is written as

$$g'(\psi(s, w), s) = \xi(w)\eta(s)$$

for all s, where $\xi(z)$ is the Lagrangian multiplier associated to constraint (15). Fully differentiating the first-order condition with respect to z yields

$$g''(\psi, s)\frac{\partial\psi}{\partial w} = \xi'(w)\eta(s).$$

Eliminating $\eta(s)$ from the last two equalties implies that

$$\frac{\partial\psi}{\partial w} = \frac{-\xi'(w)}{\xi(w)} T_g(\psi, s).$$

But notice that, by fully differentiating constraint (15) yields

$$1 = E\eta(\widetilde{s})\frac{\partial\psi(\widetilde{s}, w)}{\partial w} = \frac{-\xi'(w)}{\xi(w)} E\eta(\widetilde{s})T_g(\psi(\widetilde{s}, w), \widetilde{s}). \tag{16}$$

It implies that

$$\frac{-\xi(w)}{\xi'(w)} = E\eta(\widetilde{s})T_g(\psi(\widetilde{s}, w), \widetilde{s}).$$

But by the envelope theorem, $h'(w) = \xi(w)$ and, thus, $T_h(w) = -\xi(w)/\xi'(w)$. We conclude that

$$T_h(w) = E\eta(\widetilde{s})T_g(\psi(\widetilde{s}, w), \widetilde{s})$$

and

$$T_h'(w) = E\eta(\widetilde{s})\frac{\partial\psi(s, w)}{\partial w} T_g'(\psi(\widetilde{s}, w), \widetilde{s}).$$

Because of the first inequality in (16), we see that the above equality means that T'_h is a weighted average of $T'_g(\psi, s)$.

Suppose that P_g/A_g is uniformly smaller than m, then T'_g is uniformly smaller than $m-1$. Because T'_h is a weighted average of $T'_g(\psi(.,w),.)$, T'_h must be uniformly smaller than $m-1$, which in turn means that P_h/A_h must be smaller than m. ■