Reputation in the MarketPlace: Seller Heterogeneity and Trust in Pre-Trade Communication*

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Abstract. *A model of reputation is developed in which an agent of a less reliable type imitates the behavior of a more reliable type. Since agents of either type choose their action strategically, this model provides a theoretical foundation for the conventional model of reputation based on commitment types. Unlike the conventional model, reputation motives do not disappear even after the types are revealed. This model is applied to examine the extent to which reputation concerns with a rating system may discipline sellers in informing buyers on the benefits from trade. We then determine the impact of the possibility that sellers may restart as a new trader by obtaining a new identity..* (*JEL* Classification Codes: C73, D82, D83, L14)

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1 Introduction

It is widely recognized that reputational concerns in repeated interactions are one of the main forces that sustain the trust that is essential in effective transactions of goods and services whose quality is unknown to the buyers before purchase (experience goods). This problem is particularly acute in internet markets because, in addition to unobservability of the item for sale, the quality of delivery service is also subject to moral hazard. To help reputation mechanism work better, many trading websites adopt consumer rating systems.[1] However, the effectiveness of such rating systems is at best con-

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[1]See Dellarocas (2003 and 2005) and references therein for a discussion of reputation issues on Internet.

troversial according to the findings of recent studies.[2] In addition, there are practical issues that may undermine the rating systems as an effective reputation mechanism, for instance, a trader may restart with a new identity after damaged reputation. Given the rapid expansion of internet markets, it seems imperative for economic efficiency that these issues be analyzed from the strategic perspectives of economic agents.

In this article we provide some theoretical findings on the mechanism through which reputational concerns may resolve the moral hazard problem in online markets where different types of sellers coexist who are subject to varying degrees of moral hazard.

Reputation for sellers encompasses several dimensions. First it may reflect the beliefs on the ability of the seller to deliver a service of good quality. Second it may reflect the level of trust attached to various information that sellers may provide to the market. Reputation then evolves over time as consumers learn from the past record of the seller.

Our paper emphasizes that the two dimensions of reputation, ability and trust, are intrinsically related, by showing how trust emerges when there is adverse selection on ability. The idea is that consumers beliefs that sellers of high ability are trustworthy are self-enforcing. This being established, it will follow in return that communication helps mitigating the adverse selection problem and accelerates the learning process.

Specifically, in our model each seller randomly draws an item of either good or bad quality in each period and announces this quality as cheap talk. Each seller is of one of two private types, high or low ability: a high type seller draws a good quality item more frequently. Each item is traded at a price that is equal to the expected quality based on the seller's prevailing reputation and his announcement. The buyer learns the true quality and publicly reveals the truthfulness of the seller's announcement (rating), which updates the seller's reputation level accordingly.

Without communication, the model involves learning overtime through the observation of past quality, and prices reflect the evolution of beliefs on ability, but not the true quality of the item for sales. Allowing communication expands the set of equilibria with the possibility that some information on the item be credibly revealed by the seller.

Our main result is that in this model there is a *unique* equilibrium in which high type sellers always announce truthfully. In this equilibrium each and every truthful announcement increases the seller's reputation, which has the effect of increasing the price he receives in the next period if he claims his

[2] Various studies find marginal effect, e.g., Jin and Kato (2006); A recent study by Canals-Cerda (2008) report a significant effect based on eBay art auction data. See also the reference in Bajari and Hortacsu (2004).

item to be of a good quality. Low type sellers of all reputation levels falsely claim bad quality items to be good with a positive probability for short-term gain, after which their reputation vanishes. The probability of lying is a continuous but non-monotonic function of the prevailing reputation level of the low type seller.

Compared with the case with no communication, the equilibrium involves faster learning of the type as well as more information reflected in the price of the item. It thus mitigates the lemon problem substantially.

If sellers can start as a new seller by obtaining a new identity, then they would do so if their reputation level drops below the default level that new sellers start with. We show that this option increases cheating incentives by limiting the damage from abusing reputation and as a result, the probability that a low type seller lies is higher than when fresh restarts are infeasible, uniformly across all reputation levels.

If there are multiple trading places with separate records, but all prevent fresh restarts, sellers can switch places if their reputation is ruined in one trading place. This has a similar effect of reducing the effectiveness of reputation mechanism.

Our analysis makes several theoretical contributions to the reputation literature that, following Kreps-Wilson (1982) and Milgrom-Roberts (1982), has been developed along the adverse selection approach.[3] The approach has been particularly influential and theorizes the idea that a strategic/normal agent may refrain from a myopic selfish act so as to be perceived as one who is by nature incapable of untrustworthy behavior, which would bring future benefits by fostering cooperation. As such, this approach critically relies on the existence of commitment/crazy types but justification of such types has been only informal hitherto. We provide a theoretical foundation for such types: both types are strategic in our model yet the more reliable type always behaves trustworthy in equilibrium, which the less reliable type tries to imitate. Moreover the two types have the same preferences and differ only by the knowledge of a technological parameter. As far as we are aware, reputational behavior of this kind has not been formalized before.

This leads us to discuss another aspect of our reputational behavior. In the reputation equilibria supported by commitment types, agents lose reputational incentives entirely as soon as they are revealed to be of a strategic type. Even under imperfect monitoring, Cripps *et al.* (2004) show that reputational motives disappear in the long-run because agents' types get revealed eventually. In our model, agents' types get revealed within a finite time as well,[4] yet

[3] Mailath and Samuelson (2006) contains an extensive review of the literature.

[4] In most studies, the stage game payoff is independent of the reputation level in every period. Hence, the reputational behavior of normal/opportunistic type tend to be stationary under infinite-horizon perfect monitoring settings, exhibiting little dynamics of reputation

agents of the more reliable type maintain trustworthy behavior indefinitely. Since both types are strategic, this means that reputational motives do not dissipate in our model. It may be worth stressing that such behavior by the more reliable type is driven precisely by reputational concerns, because it is not viable without presence of the less reliable type.

Finally, this paper also makes a methodological innovation in establishing the existence and uniqueness of the equilibrium value function of the agent depending on the reputation level. In a model of financial experts who can manipulate the market by distorting information, Benabou and Laroque (1992) obtain existence and uniqueness by applying Blackwell's Theorem, which is not applicable to our model because our sellers may benefit by distorting information in only one direction, unlike financial market where information holders can benefit by distorting the market in either direction, which generates symmetry in the model. Consequently, we construct a mapping from the set of all monotonic functions to itself to which the Fan-Glicksberg Fixed Point Theorem can be applied to obtain existence of equilibrium value function. The uniqueness is separately obtained by analyzing the properties of the fixed point. In a recent independent work, Mathis, McAndrew and Rochet (2009) propose a constructive proof of existence in a related model of rating agencies with some benefits from the value at trade. Their proof relies on the fact that only positive claims can be verified which simplifies greatly the analysis.[5] In our model both positive and negative claims can be verified.

The next section describes the base model and defines equilibrium. Section 3 present some preliminary results. Then section 4analyzes the reputation mechanism and characterizes the unique reputation equilibrium in which the high type sellers always trade truthfully. Section 5 discusses its properties. Section 6 examines the case that sellers may restart as a new seller by obtaining a new identity and characterizes the stationary equilibrium. Some technical details are collected in Appendix.

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building and cashing in. One way of generating such dynamics is by introducing the stochastic importance level of each period (Sobel, 1985). Another way is to relate the payoff of opportunistic behavior to the reputation level in a natural manner, such as in our paper and Benabou and Laroque (1992). In these cases, a less reliable agent is bound to cheat given a chance if he had been lucky enough in the past to have built up his reputation above a certain threshold, thus revealing the type. The logic of Cripps *et al.* (2004) is somewhat different: A long enough history of an agent's past behavior reveals his type with a statistically overwhelming confidence and consequently, he would indulge in opportunistic actions because each such action makes only a negligible dent on his reputation.

[5] They also require a discontinuity in the payoff due an exogenous cost of loosing reputation, which is not present in our model.

2 Model

We consider a single market-place (or website) where sellers of different abilities interact with a large set of buyers. There are infinite periods $t = 1, 2, \cdots$, and a representative seller is either of high type ($\theta = h$) or low type ($\theta = \ell$) where $0 < \ell < h < 1$. The seller's type $\theta \in \{h, \ell\}$ is private information. The seller's perceived ability in each period t is captured by his *reputation* $\mu_t \in [0, 1]$, the posterior belief that the prospective buyers commonly attach to the seller being of a high type at the beginning of that period.

In each period t, a seller with reputation μ_t draws one item for sale of a random quality q_t which is good (g) with probability θ and bad (b) with probability $1 - \theta$ where $\theta \in \{h, \ell\}$ is the seller's type. We normalize as $g = 1$ or $b = 0$. Observing the quality of the item, the seller publicly makes a cheap talk announcement $m_t \in \{G, B\}$ about its quality, where the upper case of q is interpreted as announcing the quality to be $q \in \{g, b\}$.[6] We say that the agent lies if he announces B when $q_t = g$ or G when $q_t = b$, and tells the truth if he announces G when $q_t = g$ or B when $q_t = b$.[7]

The prospective buyers are myopic and try to maximize the expected quality minus the price paid. We assume a competitive demand side so that each item is traded at a price that is equal to the expected quality calculated, *a la* Bayes rule, based on μ_t and the seller's equilibrium strategy of announcing m_t. At the end of the trading period, the purchaser observes the true quality q_t and honestly reports it publicly. The seller's reputation is revised from μ_t to μ_{t+1} based on m_t and q_t, and the period $t+1$ starts. The seller's objective is to maximize the discounted sum of its revenue stream with discount factor $\delta \in (0, 1)$. At any date t, the full history of messages and items' quality of the seller is publicly known. The structure of this game, denoted by Γ, is common knowledge.

Our equilibrium concept is Perfect Bayesian equilibrium and we consider only stationary equilibria, i.e. equilibria such that the equilibrium strategies in each period depends only on the seller's reputation level of that period. Thus, the seller's strategy is represented by two functions $x^*(\mu, q)$ and $y^*(\mu, q)$ that denote, respectively, the probability that a seller of h-type and ℓ-type lies contingent on the prevailing reputation level $\mu \in [0, 1]$ and the quality $q \in \{g, b\}$ of the item drawn.

Given $x^*(\mu, q)$ and $y^*(\mu, q)$, define a "price/quality profile" $p^*_m(\mu)$ as the

[6] Alternatively, we may model that each seller posts a price p at which buyers either buy or not, and the purchaser of the item reports whether satisfied ($q \geq p$) or not ($q < p$). This produces the same equilibrium. We don't consider the possibility that the seller announces his type θ, although we conjecture that this would not change our results.

[7] Of course the labelling of the messages is somewhat arbitrary, but it will be unambiguous when we introduce the reputation equilibrium.

posterior probability that the item is of a good quality ($q = g$) when the seller with a reputation level μ announced $m \in \{G, B\}$, obtained by Bayes rule from the seller's strategy whenever possible. Being the expected quality, $p_m^*(\mu)$ is also the price at which the item will be traded.

A "transition rule" is a function $\pi_{mq}^*(\mu)$ that specifies the posterior probability of $\theta = h$ in the next period when in the current period $\Pr(\theta = h) = \mu$ and the seller sells an item of quality $q \in \{g, b\}$ after announcing $m \in \{G, B\}$. We require that $\pi_{mq}^*(\mu)$ be obtained by Bayes rule from the seller's strategy whenever possible.

Given $x^*(\mu, q)$, $y^*(\mu, q)$ and $\pi_{mq}^*(\mu)$ as above, we define the value function for $\theta \in \{h, \ell\}$, denoted by $V_\theta^*(\mu) : [0, 1] \to \mathbb{R}$, as the expected discounted sum of revenue stream of a seller of type θ and reputation μ.

Definition 1 *A collection* $(x^*, y^*, p_m^*, \pi_{mq}^*, V_\theta^*)$ *is a (Perfect Bayesian) equilibrium if the followings hold for each* $\theta = h, \ell$ *where* $z_h = x^*$ *and* $z_\ell = y^*$*:*

(i) $z_\theta(\mu, g) = 0$ *if* $p_G^*(\mu) + \delta V_t^*(\pi_{Gg}^*(\mu)) > p_B^*(\mu) + \delta V_t^*(\pi_{Bg}^*(\mu))$,
$z_\theta(\mu, g) = 1$ *if* $p_G^*(\mu) + \delta V_t^*(\pi_{Gg}^*(\mu)) < p_B^*(\mu) + \delta V_t^*(\pi_{Bg}^*(\mu))$;

(ii) $z_\theta(\mu, b) = 0$ *if* $p_B^*(\mu) + \delta V_t^*(\pi_{Bb}^*(\mu)) > p_G^*(\mu) + \delta V_t^*(\pi_{Gb}^*(\mu))$,
$z_\theta(\mu, b) = 1$ *if* $p_B^*(\mu) + \delta V_t^*(\pi_{Bb}^*(\mu)) < p_G^*(\mu) + \delta V_t^*(\pi_{Gb}^*(\mu))$;

(iii)
$$V_\theta^*(\mu) = \theta \left[(1 - z_\theta(\mu, g)) \left(p_G^*(\mu) + \delta V_\theta^*(\pi_{Gg}^*(\mu))\right) + z_\theta(\mu, b) \left(p_B^*(\mu) + \delta V_\theta^*(\pi_{Bg}^*(\mu))\right)\right]$$
$$+ (1 - \theta) \left[z_\theta(\mu, b) \left(p_G^*(\mu) + \delta V_\theta^*(\pi_{Gb}^*(\mu))\right) + (1 - z_\theta(\mu, b)) \left(p_B^*(\mu) + \delta V_\theta^*(\pi_{Bb}^*(\mu))\right)\right].$$

Before we turn to the characterization of the equilibria with adverse selection and cheap talk, let us discuss a few properties of our model.

3 Preliminary considerations

The term "reputation" in the economic literature encompasses several notions, two of them are present in our model. First, reputation may refer to the beliefs concerning the average quality provided by the seller to the market. In our model this corresponds to the beliefs μ_t on the type θ. Second, the notion of reputation may refer to the level of confidence that consumers have on the truthfulness of the announcement of the seller concerning the quality of the good. This notion thus refers more to trust than to beliefs on the type. As shown below however, the two concepts are closely related.

We use the term learning to refer to the fact that the mere observation of the history of quality q_t helps consumers improve their knowledge of the type of the seller, in a non-strategic manner.

3.1 The learning equilibrium

Suppose that there is no communication, say because the seller doesn't observe the quality of the good. Then in every period a seller's item is traded at a price equal to the expected quality

$$p_t = p^e(\mu_t) = \mu_t h + (1-\mu_t)\ell.$$

In this case the beliefs of the buyers over a seller's type evolves according the simple Bayes rule:

$$\begin{aligned}\mu_{t+1} &= \frac{\mu_t h}{\mu_t h + (1-\mu_t)\ell} > \mu_t \;\; if \; q_t = g \\ \mu_{t+1} &= \frac{\mu_t (1-h)}{\mu_t (1-h) + (1-\mu_t)(1-\ell)} < \mu_t \;\; if \; q_t = b.\end{aligned}$$

Beliefs and prices follow a martingale, so that the price increases or declines depending on whether the quality delivered last period is good or bad.

Notice that this equilibrium remains an equilibrium in the game γ described in Section 2, i.e., the so-called "babbling equilibrium." For instance such an equilibrium obtains when the seller always announces G and thus, the message m_t, containing no information content, is ignored. The beliefs and the price evolve as in the learning equilibrium above and it is clear that since announcement doesn't affect the continuation game, it is optimal for the seller to announce G.

3.2 A single type

Now suppose that there is a single type, say type ℓ. (As we shall see below, this is different from saying that the buyers beliefs assign probability 1 on the type ℓ.) Due to risk-neutrality and a probability 1 to trade, our model has the feature that there is a zero value for the seller of transmitting information to the buyer in this case. The reason is that the ex-ante payoff is equal to the expected price which always coincides with the expected quality. The implication is that repeated interaction cannot help in fostering communication in this set-up.

To see this consider any equilibrium of our game when a seller's type is publicly "observed" to be ℓ. Because the price is the expected value conditional on the information available at date t, the ex-ante expected price must be equal to ℓ. Thus any equilibrium generates an expected payoff $\frac{\ell}{1-\delta}$.

Another consequence is that there cannot be any information transmitted through communication. To see this consider any period and suppose that the message is informative in this period. This occurs when the probability $y^*(b)$ of announcing $m = G$ when $q = b$ is not equal to the probability $1 - y^*(g)$ of announcing G when $q = g$. In the case the prices will differ for the two messages. But we have seen that the expected payoff from the next period on must be equal to $\frac{\ell}{1-\delta}$ and thus is independent of the message. Hence, the seller would announce with probability one the message that would generate the highest price, irrespective of q, which contradicts $y^*(b) \neq 1 - y^*(g)$.

Thus when the type of the seller is publicly observed, the unique equilibrium outcome is the no communication equilibrium outcome.

3.3 On communication in equilibrium

We say that an equilibrium involves communication if there is a positive probability that at some date the message conveys some information. In our model, there are two types of information that can be transmitted: information about the current level of quality g or b, and information about the type θ.

Before we turn to the equilibrium analysis it is worth noticing that the two types of information transmission are related in a non-trivial way. In our set-up there will be some information transmitted about the type θ if there is a positive probability that at some date

$$x^*(\mu_t, q_t) \neq y^*(\mu_t, q_t).$$

To see this, observe that if the strategy of the seller is independent of his type then it must be the case that the posterior μ_{t+1} depends only on the history of the realized quality $\mathbf{h}^t = (q_1, \cdots, q_t)$ and not on the history of message $\mathbf{m}^t = (m_1, ..., m_t)$, i.e., $\Pr(\theta \mid \mathbf{h}^t, \mathbf{m}^t) = \Pr(\theta \mid \mathbf{h}^t)$. The reason is that the distribution of $\mathbf{m}^t$ conditional on $\mathbf{h}^t$ and θ is independent of θ.

Similarly, there will be some information transmitted about the quality if there is a positive probability that at some date $p_G^*(\mu_t) \neq p_B^*(\mu_t)$ and both message can occur with positive probability.

We have seen above that communication about the quality of the item is not possible if there is a single type. This observation extends to the following property when there are multiple types, i.e., in a setting of adverse selection:

Property: Messages cannot convey information on the quality of the good unless they convey information on the type of the seller.

To see this suppose that no information is transmitted by the message on the type θ in equilibrium. Then we have seen that revised beliefs are only function of the history of quality $\mathbf{h}^t$. But this implies that at any date the expected future payoff of the seller is independent of his current announcement strategy. Thus, if $p_G^*(\mu_t) \neq p_B^*(\mu_t)$, then both types of sellers would choose to send with probability 1 the same message (that maximize the price), which would imply that the messages are uninformative, i.e., $p^*_G(\mu_t) \neq p^*_B(\mu_t)$, a contradiction.

Therefore, adverse selection and signalling about the type is a necessary ingredient for message to be a credible signal of quality in our environment.

4 Reputation equilibrium

Let us now turn to equilibrium analysis with reputation. As we wish to study whether adverse selection may help to induce truthful revelation of the quality of the good, we will focus on equilibria with the following properties:

(a) an h-type seller always tells the truth regardless of q so long as $\mu > 0$, i.e., $x^*(\mu, q) = 0$ for all $\mu > 0$; and

(b) the value function V^*_θ is non-decreasing for $\theta = h, \ell$.[8]

The first property captures the idea that beliefs about the types will generate trust in messages. The intuition behind this property is that building/maintaining reputation through truthful announcement of the quality is less costly for an h-type seller because he knows he will have more good draws than the type ℓ, whence he should announces the truth with a larger probability.

The property (b) states that a seller's expected profit increases with the market beliefs about his type. Notice that the expected quality of the good $\mu h + (1 - \mu)\ell$ is increasing with μ. This would be the payoff of the seller in a one-shot game or if $\delta = 0$. The property states that this monotonicity property extends to our dynamic setting, which seems natural.

We shall refer to this as a non-trivial reputation equilibrium:

Definition 2 *An equilibrium satisfying (a) and (b) is called a NTR-equilibrium.*

In characterizing the equilibrium, we proceed in three steps. First, we derive some useful properties of the equilibrium. Then, we use these properties to prove existence, continuity and uniqueness of the value function for the ℓ-type seller. Finally, we show that telling the truth is the optimal strategy for the h-type seller.

[8] We conjecture that condition (b) and Lemma 1 imply each other.

4.1 Some properties of NTR-equilibria

A first trivial remark is that it is not possible that the ℓ-type seller always tells the truth. Indeed a strategy $y^*(\mu, q) \equiv 0$ would yield a payoff $\ell/(1-\delta)$ and a price $p_G^*(\mu) = 1$. But then the seller would gain at least $1 + \delta\ell/(1-\delta) > \ell/(1-\delta)$ by reporting a bad quality item as good ($m = G$). Thus, full revelation of q by the seller is not possible. Nonetheless, intuition suggests that due to property (b) there is no incentive to misreport a good quality $q = g$, as this would reduce the current price without enhancing next period's reputation level. We first show that this is the case:

Lemma 1 *For any NTR-equilibrium,*

$$y^*(\mu, g) = 0 \quad \textit{and} \quad p_B^*(\mu) = 0 \quad \forall \mu \in (0, 1]. \tag{1}$$

Proof. First we claim that for any $\mu > 0$, $p_G^*(\mu) \geq p_B^*(\mu)$. Indeed suppose that $y^*(\mu, b) > 0$ then property (a) implies that $\pi_{Gb}^*(\mu) = 0 < \pi_{Bb}^*(\mu)$. Moreover $p_G^*(\mu) + \delta V_\ell^*(\pi_{Gb}^*(\mu)) \geq p_B^*(\mu) + \delta V_\ell^*(\pi_{Bb}^*(\mu))$ and property (b) implies that $p_G^*(\mu) \geq p_B^*(\mu)$. If $y^*(\mu, b) = 0$, on the other hand, $p_G^*(\mu) = 1 > p_B^*(\mu)$ by (a) and Bayes rule.

Now suppose $y^*(\mu, g) > 0$. Then (a) implies that $\pi_{Bg}^*(\mu) = 0 < \mu < \mu' = \pi_{Gg}^*(\mu)$. Moreover $p_G^*(\mu) + \delta V_\ell^*(\mu') \leq p_B^*(\mu) + \delta V_\ell^*(0)$, which implies, given (b) and $p_G^*(\mu) \geq p_B^*(\mu)$, that $p_G^*(\mu) = p_B^*(\mu)$ and $V_\ell^*(\mu') = V_\ell^*(0)$. Since $\mu < \mu'$,we also have $V_\ell^*(\mu) = V_\ell^*(0)$. But this then for $m \in \{G, B\}$, we have $V_\ell^*(\mu') = p_G^*(\mu) + \delta V_\ell^*(0) \geq p_m^*(\mu') + \delta V_\ell^*(0)$. Hence, $p_m^*(\mu') \leq p_G^*(\mu)$ should hold for $m = G, B$, which is an impossibility because some weighted average of $p_G^*(\mu')$ and $p_B^*(\mu')$ is the expected quality of an item drawn by a seller of reputation μ', hence must be strictly greater than $p_G^*(\mu) = p_B^*(\mu) = \mu h + (1-\mu)\ell$. Therefore, we have to conclude that $y^*(\mu, g) = 0$ for all $\mu > 0$.

Finally $y^*(\mu, g) = 0$ and (a) imply that $p_B^*(\mu) = 0$, since message $m = B$ occurs with positive probability but only when $q = b$. ■

Observe that in equilibrium, once a seller's reputation index falls to $\mu = 0$, he cannot increase his reputation level above 0, because Bayes rule dictates that $\pi_{mq}^*(0) = 0$ so long as an ℓ-seller with an item of quality q announces m with a positive probability according to the equilibrium strategy. Therefore, the seller whose reputation is 0 announces m that gives the highest $p_m^*(0)$ regardless of q, which implies that the equilibrium price is the same regardless of q, being equal to ℓ. This occurs when an ℓ-type seller's strategy when $\mu = 0$ is independent of q.

Since labeling of the messages is inconsequential due to the costless nature of cheap talk messages, for expositional ease we make the convention that an ℓ-seller announces G regardless of q when $\mu = 0$, i.e.,

$$y^*(0, g) = 0, \ y^*(0, b) = 1 \quad \text{and} \quad p_G^*(0) = \ell, \ p_B^*(0) = 0. \tag{2}$$

An immediate consequence of the fact that equilibrium messages don't convey any information when $\mu = 0$ is that the payoff when $\mu = 0$ is not affected by adverse selection. Indeed we have

Lemma 2 *For any NTR-equilibrium,* $V_\ell^*(0) = \frac{\ell}{1-\delta}$ *and* $V_h^*(1) = \frac{h}{1-\delta}$.

Proof. Consider the ℓ-type seller with $\mu = 0$. Then beliefs remain constant and the price is $p_G^*(0) = \ell$ at any point on the equilibrium path, which implies a discounted value $\ell/(1-\delta)$.

For the h-type seller with $\mu = 1$, again beliefs remain constant while the price is 1 with probability h and zero with probability $1-h$. The expected price at any date is then h which implies a discounted value $h/(1-\delta)$. ■

Having established above that $y^*(\mu, g) = 0$ and $p_B^*(\mu) = 0$ for all μ in any NTR-equilibrium, we now focus on the equilibrium values of $y^*(\mu, b)$. For notational ease, we use $y^*(\mu)$ as shorthand for $y^*(\mu, b)$ in the sequel. Thus an equilibrium, if it exists, is characterized by the probability that a ℓ-type seller announces G when $q = b$, the transition rule and the values functions. Notice that the h-type seller cannot announce truthfully for all beliefs unless the ℓ-type lies for some beliefs at least. Thus $y^*(\mu) \neq 0$ for some beliefs μ.

For each $\mu \in (0,1]$ and $y \in [0,1]$, we define

$$p_G(\mu, y) = \frac{\mu h + (1-\mu)\ell}{\mu h + (1-\mu)(\ell + (1-\ell)y)}. \tag{3}$$

Here, $p_G(\mu, y)$ is the expected quality of the product conditional on the seller having announced $m = G$. In equilibrium it is equal to the price so that

$$p_G^*(\mu) = p_G(\mu, y^*(\mu)) > 0,$$

whereas we have seen that when the seller announces B the price is $p_B^*(\mu) = 0$. The following observations are straightforward:

Property i: *For $y > 0$ and $\mu < 1$, $p_G(\mu, y)$ strictly increases in μ and strictly decreases in y, with $p_G(\mu, 0) = 1$ for all μ, $p_G(1, y) = 1$ for all y and $p_G(0, 1) = \ell$.*

Let us consider now beliefs. First notice that without loss of generality we can set $\pi_{Bg}^*(\mu) = 0$, as the seller never claim $m = B$ when $q = g$. Since V_θ^* is minimal at $\mu = 0$, this doesn't alter incentive compatibility conditions (i) in Definition 1.

Consider now the case where the seller reveals the true quality. For each $\mu \in (0,1]$ and $y \in [0,1]$, we define the following values whenever well-defined:

$$\pi_{Gg}(\mu) = \frac{\mu h}{\mu h + (1-\mu)\ell}, \quad (4)$$

$$\pi_{Bb}(\mu, y) = \frac{\mu(1-h)}{\mu(1-h) + (1-\mu)(1-\ell)(1-y)}. \quad (5)$$

Note that $\pi_{Gg}(\mu_t)$ is the posterior probability, μ_{t+1}, that $\theta = h$ when a seller of reputation μ_t announced $m_t = G$ and the purchaser reported $q_t = g$; and $\pi_{Bb}(\mu_t, y^*(\mu_t)) = \pi^*_{Bb}(\mu_t)$ is that when a seller of reputation μ_t announced $m_t = B$ and the purchaser reported $q_t = b$.

We then have

Property ii: *For $y < 1$ and $\mu < 1$, $\pi_{Gg}(\mu)$ and $\pi_{Bb}(\mu, y)$ strictly increases in μ, and $\pi_{Bb}(\mu, y)$ strictly increases in y, with $\pi_{Bb}(\mu, 1) = 1$ and $\pi_{Bb}(1, y) = 1$.*

This property implies that $V(\pi_{Bb}(\mu, y))$ (weakly) increases in y with $V(\pi_{Bb}(\mu, 1)) = V_\ell(1)$, for any non-decreasing function $V : [0, 1] \to R_+$.

When the seller lies and quality is b, posterior beliefs should be equal to $\pi^*_{Gb}(\mu) = 0$, except when $\mu = 1$ or $y^*(\mu) = 0$. However, for $\mu < 1$, there is no loss of generality in assuming that $\pi^*_{Gb}(\mu) = 0$ when $y^*(\mu) = 0$ since neither type lies when $q = b$ and $V^*_\theta(0)$ is the minimal value of the value function.

Determining $\pi^*_{Gb}(1)$ is a bit more delicate because it determines $V^*_\ell(1)$ and thus, the optimality of $y^*(\mu) = 1$ for $\mu < 1$, via determining the deviation value for an ℓ-type seller with reputation μ of announcing truthfully when $q = b$, which would induce belief $\pi_{Bb}(\mu, 1) = 1$.

The value of $\pi^*_{Gb}(1)$ plays a central role because a seller with "maximal reputation" $\mu = 1$ have the choice, upon drawing $q = b$, between maintaining its reputation with a zero current price ($p^*_B(1) = 0$) or losing its reputation with a current price of $p^*_G(1) = 1$. Losing reputation by lying then induces a drop in reputation, inducing a loss in future profits from $V^*_\theta(1)$ to $V^*_\theta(\pi^*_{Gb}(1))$. The the equilibrium behavior at $\mu = 1$ is characterized in the next lemma.

Lemma 3 *For any NTR-equilibrium,*

(i) *$y^*(1) = 1$ and $y^*(\mu)$ is continuous at $\mu = 1$;*

(ii) $\lim_{\mu \to 1} V^*_\ell(\mu) = \dfrac{1 - \delta(1 - \ell + \ell^2)}{(1-\delta)(1-\delta\ell)}$;

(iii) *if $y^*(\mu) < 1$ for some μ,* $V^*_\ell(1) = \dfrac{1 - \delta(1 - \ell + \ell^2)}{(1-\delta)(1-\delta\ell)}$ *and $V^*_\ell(\pi^*_{Gb}(1)) = V^*_\ell(0)$.*

Moreover, there exists a NTR-equilibrium with $\pi^*_{Gb}(1) = 0$ and the same announcement strategy $y^*(\mu)$ for all μ, and same value $V^*_\theta(\mu)$ for all $\mu < 1$.

Proof. See Appendix. ■

The lemma states that without loss of generality we can set $\pi^*_{Gb}(1) = 0$. If the ℓ-seller announces truthfully with some probability, then this doesn't alter the equilibrium. In the case where the ℓ-seller always lies, there is some flexibility in setting the beliefs $\pi^*_{Gb}(1)$ that would affect the value $V^*_\ell(1)$ but not the value function $V^*_\ell(\mu)$ for $\mu < 1$. As an ℓ-seller never reaches the maximum reputation level $\mu = 1$ in equilibrium, this is irrelevant for the equilibrium path. The value $V^*_\ell(1)$ solves $V^*_\ell(1) = 1 + \delta(\ell V^*_\ell(1) + (1-\ell)V^*_\ell(0))$, yielding the value in the lemma.

Following the lemma we restrict to NTR-equilibria such that $\pi^*_{Gb}(1) = 0$. For such a NTR-equilibrium, we always have

$$V^*_\ell(1) = V^*_\ell(0) + \Delta$$

where

$$\Delta := \frac{1-\delta(1-\ell+\ell^2)}{(1-\delta)(1-\delta\ell)} - \frac{\ell}{1-\delta} = \frac{1-\ell}{1-\delta\ell} < 1. \tag{6}$$

4.2 Value and policy function for the ℓ-type

Let $\mathcal{F}$ denote the set of all non-decreasing functions $V : [0,1] \to \mathbb{R}$ such that $V(0) = \frac{\ell}{1-\delta}$ and $V(1) = \frac{\ell}{1-\delta} + \Delta$. For any NTR-equilibrium, in light of Lemma 3 and the conditions (i)-(iii) of Definition 1, $V^*_\ell \in \mathcal{F}$ and

$$\begin{aligned} V^*_\ell(\mu) &= \ell\left(p_G(\mu, y^*(\mu)) + \delta V^*_\ell(\pi_{Gg}(\mu))\right) \\ &\quad +(1-\ell)\left[y^*(\mu)\left(p_G(\mu, y^*(\mu)) + \delta V^*_\ell(0)\right) + (1-y^*(\mu))\delta V^*_\ell(\pi_{Bb}(\mu, y^*(\mu)))\right] \end{aligned} \tag{7}$$

where y^* satisfies

$$\begin{cases} y^*(\mu) = 0 & \text{if} \quad p_G(\mu, y^*(\mu)) < \delta\left(V^*_\ell(\pi_{Bb}(\mu, y^*(\mu))) - V^*_\ell(0)\right) \\ y^*(\mu) = 1 & \text{if} \quad p_G(\mu, y^*(\mu)) > \delta\left(V^*_\ell(\pi_{Bb}(\mu, y^*(\mu)))) - V^*_\ell(0)\right). \end{cases} \tag{8}$$

Note from (7) that V^*_ℓ is a fixed point of a mapping determined by the RHS of (7) via a "best response" function y^* that satisfies (8). We formally define this mapping on $\mathcal{F}$ with a view to applying a fixed point theorem.

For a given $V \in \mathcal{F}$, a "best-response" is a function $y_V : [0,1] \to [0,1]$ that satisfies

$$\begin{cases} y_V(\mu) = 0 & \text{if} \quad p_G(\mu, y_V(\mu)) < \delta\left(V(\pi_{Bb}(\mu, y_V(\mu))) - V(0)\right) \\ y_V(\mu) = 1 & \text{if} \quad p_G(\mu, y_V(\mu)) > \delta\left(V(\pi_{Bb}(\mu, y_V(\mu)))) - V(0)\right). \end{cases} \tag{9}$$

Lemma 4 *For any $V \in \mathcal{F}$, there exists a unique best-response y_V and $y_V(\mu) > 0$ for all $\mu \in [0,1]$.*

Proof. Properties i and ii imply that $p_G(\mu, y) - \delta\left(V(\pi_{Bb}(\mu, y)) - V(0)\right)$ is strictly increasing in y. Hence either it is always negative or always positive,, or there exists a unique value y_V where it is zero. Since $p_G(\mu, 0) = 1 > \delta(V(\pi_{Bb}(\mu, 0)) - V(0))$ for all μ by (6), $y_V(\mu) > 0$ on $[0, 1]$. ■

For $\mu = 0$, property ii implies that $\delta\left(V_\ell(\pi_{Bb}(0, y)) - V_\ell(0)\right) = 0 < p_G(0, y)$ for all $y < 1$. Hence $y_V(0) = 1$.

Let us define:

$$\bar{\mu} = \inf\left\{\mu \mid p_G(\mu, 1) > \delta\Delta\right\}.$$

Due to $\Delta < 1 = p_G(1, 1)$, $\bar{\mu}$ is smaller than 1. Then, $p_G(\mu, y) > \delta\left(V_\ell(\pi_{Bb}(\mu, y)) - V_\ell(0)\right)$ for all $y < 1$ when $\mu > \bar{\mu}$ which implies that $y_V(\mu) = 1$ if $\mu > \bar{\mu}$. The threshold $\bar{\mu}$ is positive if δ is large enough so that

$$\delta\Delta > \ell = p_G(0, 1) \quad \Longleftrightarrow \quad \delta > \delta_\ell = \frac{\ell}{1 - \ell + \ell^2}. \tag{10}$$

Then, for $\mu < \bar{\mu}$, at a best response $y \in (0, 1)$ we must have

$$\delta \lim_{y' \uparrow y}\left(V(\pi_{Bb}(\mu, y')) - V(0)\right) \leq p_G(\mu, y) \leq \delta \lim_{y' \downarrow y}\left(V(\pi_{Bb}(\mu, y')) - V(0)\right). \tag{11}$$

Thus, the unique best response function y_V is characterized as

$$y_V(\mu) = \begin{cases} 1 & \text{if} \quad \mu > \bar{\mu} \\ \text{the unique } y \text{ that satisfies (11)} & \text{if} \quad 0 < \mu \leq \bar{\mu} \\ 1 & \text{if} \quad \mu = 0. \end{cases} \tag{12}$$

To conclude the characterization of the "best-response":

Lemma 5 *For any $V \in \mathcal{F}$, $y_V(\mu)$ is continuous on $[0, 1]$, and $p_G(\mu, y_V(\mu))$ is nondecreasing with μ (as well as $\pi_{Bb}(\mu, y_V(\mu))$ for $\mu > 0$).*

Proof. For each $\mu \in (0, \bar{\mu}]$, by construction, $y_V(\mu)$ is intersection of the graph of $p_G(\mu, y)$ and the "connected" graph of $\delta(V(\pi_{Bb}(\mu, y)) - V(0))$, i.e., the graph is connected vertically at every discontinuity points by the shortest distance. Since both of the graphs are uniformly continuous as functions of μ, the intersection point changes continuously in μ, i.e., $y_V(\mu)$ is continuous on $\mu \in (0, \bar{\mu}]$. In addition, $y_V(\mu) \to 1$ as $\mu \to 0$ because $\pi_{Bb}(\mu, y) \to 0$ as $\mu \to 0$ for any $y < 1$. Since $y_V(\bar{\mu}) = 1$ by construction (using $y_V(0) = 1$ if $\bar{\mu} = 0$), it follows that $y_V(\mu)$ is continuous on $[0, 1]$.

For $\mu \geq \bar{\mu}$, we have $p_G(\mu, y_V(\mu)) = p_G(\mu, 1)$ which increases in μ by property i, while $\pi_{Bb}(\mu, 1) = 1$. For $0 < \mu \leq \bar{\mu}$, note that both of the graphs move upward as μ increases due to properties i and ii. Therefore the height of the intersection point also increases, i.e., $p_G(\mu, y_V(\mu))$ increases (weakly) in μ, as well as $\pi_{Bb}(\mu, y_V(\mu))$. ■

Finally, define a mapping $T : \mathcal{F} \to \mathcal{F}$ by

$$T(V)(\mu) := p_G(\mu, y_V(\mu)) + \delta\left(\ell V(\pi_{Gg}(\mu)) + (1-\ell)V(0)\right), \tag{13}$$

which is well-defined due to Lemma 6 below.

Lemma 6 $T(V_\ell) \in \mathcal{F}$.

Proof. From above, $p_G(\mu, y_V(\mu))$ increases (weakly) in μ, as well as $\pi_{Bb}(\mu, y_V(\mu))$. Since $\pi_{Gg}(\mu)$ increases in $\mu \in [0,1]$, this proves that $T(V)$, defined in (13), is non-decreasing in μ.

Next, since $y_V(0) = 1$ and $\pi_{Gg}(0) = 0$, we have

$$\begin{aligned} T(V)(0) &= p_G(0,1) + \delta\left(\ell V(0) + (1-\ell)V(0)\right) \\ &= \ell + \delta V(0) = \frac{\ell}{1-\delta} \end{aligned}$$

as desired. Finally, since $y_V(1) = \pi_{Gg}(1) = 1$ and $p_G(1,1) = 1$,

$$\begin{aligned} T(V)(1) &= 1 + \delta\left(\ell V(1) + (1-\ell)V(0)\right) \\ &= 1 + \delta\left(\ell\frac{1-\delta(1-\ell+\ell^2)}{(1-\delta)(1-\delta\ell)} - (1-\ell)\frac{\ell}{1-\delta}\right) \\ &= \frac{1-\delta(1-\ell+\ell^2)}{(1-\delta)(1-\delta\ell)}. \end{aligned}$$

■

The next result establishes that the value function of an equilibrium can be computed as a fixed point of the Bellman operator T.

Lemma 7 *For any NTR-equilibrium, $T(V_\ell^*) = V_\ell^*$ and $y^*(\mu) = y_{V_\ell^*}(\mu)$. If (10) does not hold, then $y^*(\mu) = 1$ for all $\mu \in [0,1]$, while if (10) holds $y^*(\mu) = 1$ for $\mu \geq \bar{\mu} > 0$.*

Proof. We start by observing that, for any non-decreasing V with $V(0) = \frac{\ell}{1-\delta}$,

$$\begin{aligned} V(\mu) = p_G(\mu,1) + \delta\big(\ell V(\pi_{Gg}(\mu)) + (1-\ell)V(0)\big) \quad \forall \mu \in (\bar{\mu},1) \neq \emptyset \\ \implies \lim_{\mu\to 1} V(\mu) - V(0) = \Delta \end{aligned} \tag{14}$$

because $p_G(\mu,1) \to 1$ and $\pi_{Gg}(\mu) \to 1$ as $\mu \to 1$.

Let $(x^*, y^*, p_m^*, \pi_{mq}^*, V_\theta^*)$ be an NTR-equilibrium. First consider the case that $\delta > \delta_\ell$ defined in (10). If $y^*(\mu) = 1$ for all $\mu < 1$, (14) would imply $\delta(V_\ell^*(\pi_{Bb}^*(\mu)) - V_\ell^*(\pi_{Gb}^*(\mu))) = \delta(V_\ell^*(1) - V_\ell^*(0)) \geq \delta\Delta > p_G(\mu,1)$ for sufficiently small $\mu > 0$, contradicting optimality condition (ii) of Definition

0. Hence, $y^*(\mu) \neq 1$ for some μ and thus, (7) and (8) are satisfied so that, as verified above when defining (12), $y^*(\mu) = 1$ for $\mu = 0$ and $\mu \in [\bar{\mu}, 1]$ and, in addition, $y^*(\mu) > 0$ and satisfies (**??**) for $\mu \in (0, \bar{\mu})$. Consequently, $y^*(.) \equiv y_{V_\ell^*}(.)$ and (7) is equivalent to (13) when $V = V_\ell^*$, which implies that $T(V_\ell^*) = V_\ell^*$.

Next, consider the case that (10) does not hold so that $y_{V_\ell^*}(.) \equiv 1$. Then, $\delta(V_\ell^*(\pi_{Bb}^*(\mu)) - V_\ell^*(\pi_{Gb}^*(\mu))) \leq \ell$ for all μ by Lemma 3 (iii), while $p_G^*(\mu) - p_B^*(\mu) > \ell$ for all $\mu > 0$ by Lemmas 1 and 4.1, hence $y^*(\mu) = y_{V_\ell^*}(\mu) = 1$ for all $\mu > 0$ by the condition (ii) of Definition 0. Since $y^*(0) = 1$ by Lemma 3 (i), we have again $y^*(.) \equiv y_{V_\ell^*}(.)$. ■

Thus, if (10) fails, all NTR-equilibria have a simple characterization: ℓ-sellers always lie upon drawing a bad quality product (and are honest otherwise).[9] Below we further our characterization of NTR-equilibria for the more interesting case that (10) holds, by examining the existence and properties of the fixed point of T.

Although $T : \mathcal{F} \to \mathcal{F}$ is well-defined, fixed point theorems may not be applied directly to T because $\mathcal{F}$ is not a compact set. This problem is resolved by showing that we can restrict attention to the set $\mathcal{F}^r$ of all right-continuous non-decreasing functions on [0,1], which is compact.[10] We will then show that T is continuous and apply Fan-Glicksberg Fixed Point Theorem[11] to T restricted to $\mathcal{F}^r$.

For this we will need two key results. The first result is continuity of the equilibrium value function.

Lemma 8 *If $T(V) = V$ then $V(\mu)$ is continuous and strictly increasing.*

Proof. To reach a contradiction, suppose that $T(V) = V$ yet V is not continuous, say at μ_1. Since $y_V(\mu) = 1$ for all $\mu \geq \bar{\mu}$ and $\pi_{Gg}(\mu)$ is continuous with $\pi_{Gg}(1) = 1$, (13) implies that

$$lim_{\mu \to 1} V(\mu) = \frac{1 - \delta(1 - \ell + \ell^2)}{(1 - \delta)(1 - \delta\ell)} = V(1).$$

Hence, we may assume that $\mu_1 < 1$. Then, since $p_G(\mu, y_V(\mu))$ is continuous in μ, (13) and dictates that $0 < \lim_{\mu \downarrow \mu_1} V(\mu) - \lim_{\mu \uparrow \mu_1} V(\mu) = \delta\ell \left(\lim_{\mu \downarrow \mu_2} V(\mu) - \lim_{\mu \uparrow \mu_2} V(\mu)\right)$ where $\mu_2 = \pi_{Gg}(\mu_1) > \mu_1$. Since $\mu_2 < 1$, applying analogous arguments repeatedly, we deduce that there must exist

[9] In this case the equilibrium may vary in $V_\ell^*(1)$ and $\pi_{Gb}(1,1)$ subject to $V_\ell^*(1) = 1 + \delta(\ell V_\ell^*(1) + (1 - \ell) V_\ell^*(\pi_{Gb}(1,1)))$, but they all generate the same equilibrium outcome.

[10] Note that $T(V)$ is right-continuous if V is because $y_V(\mu)$ is continuous in μ.

[11] This theorem states that an upper hemi-continuous convex valued correspondence from a nonempty compact convex subset of a convex Hausdorff topological vector space has a fixed point.

an infinite sequence μ_n such that both μ_n and $\lim_{\mu\downarrow\mu_n} V(\mu) - \lim_{\mu\uparrow\mu_n} V(\mu)$ increase in n, which is impossible because $V_\ell^*(1) - V_\ell^*(0)$ is bounded. Hence V is continuous.

Next, again to reach a contradiction, suppose V is not strictly increasing. Let μ' be the highest value such that $V(\mu)$ is constant on some interval $(\mu' - \varepsilon, \mu')$. Then $\mu' \leq \bar{\mu}$ by (3) and (12), and $\pi_{Gg}(\mu) > \mu'$ if $\mu \in (\mu' - \varepsilon, \mu')$ for sufficiently small $\varepsilon > 0$ by (4). But then for μ and μ'' such that $\mu' - \varepsilon < \mu < \mu'' < \mu'$, since $V_\ell(\pi_{Gg}(\mu)) < V_\ell(\pi_{Gg}(\mu''))$ and $p_G(\mu, y^*(\mu))$ is non-decreasing in μ as verified in the proof of Lemma 6, we would have

$$\begin{aligned} V_\ell^*(\mu') &= p_G(\mu, y^*(\mu')) + \delta\Big(\ell V_\ell(\pi_{Gg}(\mu')) + (1-\ell)V_\ell(0)\Big) \\ &> p_G(\mu, y^*(\mu)) + \delta\Big(\ell V_\ell(\pi_{Gg}(\mu)) + (1-\ell)V_\ell(0)\Big) = V_\ell^*(\mu), \end{aligned}$$

contradicting the supposition that $V_\ell^*(\mu)$ is constant on $(\mu' - \varepsilon, \mu')$. ■

The second result is a restriction on the evolution of beliefs in equilibrium.

Lemma 9 *If $T(V) = V$, then $\pi_{Bb}(\mu, y_V(\mu)) > \mu$ all $\mu > 0$.*

Proof. See Appendix. ■

The lemma implies that along any equilibrium path beliefs, and thus prices, increase until the point where the ℓ-type seller reveals his type by falsely claiming a high quality, at which point the price drops definitely to ℓ.

Theorem 1 *There exists a unique fixed point of T.*

Proof. See Appendix ■

As explained above, existence follows from the continuity of the operator T on the set $\mathcal{F}^r$. The result thus differs from Benabou-Laroque (1992) in that T may not be contraction mapping. More precisely, T is not nondecreasing in V so that Blackwell's Theorem cannot be applied. A second key difference is that Benabou and Laroque assume continuity while we don't restrict a priory to continuous functions.

Uniqueness result from the fact that the value function is uniquely defined for $\mu > \bar{\mu}$ and that beliefs are increasing along an equilibrium path with truthful announcement (Lemma 9). Indeed the latter implies that there is a unique way to "unravel" the value function from large to low μ. A similar idea is exploited in Mathis, McAndrew and Rochet (2009) to obtain a constructive proof of existence in a game of rating agencies.

4.3 Optimality for the h-seller and equilibrium

We have established that there is a unique pair of V_ℓ^* and $y^*(\mu)$ that is consistent with an equilibrium that satisfies the properties (a) and (b) introduced earlier. The price are then $p_G^*(\mu) = p_G(\mu, y^*(\mu))$ and $p_B^*(\mu) = 0$. We now establish the condition under which truthful announcing is an equilibrium strategy for an h-seller and thus, the NTR-equilibrium exists. In doing so, we set (w.l.o.g.) the off-equilibrium beliefs as

$$\pi_{Bg}^*(\mu) = 0 \leq \pi_{Gg}(\mu) \quad \forall \mu \in [0,1]$$

which ensures that beliefs always decrease after a false claim

For $\mu > 0$, let V_h^* be the value function of an h-seller calculated at the optimal strategy given y^* obtained from (12) for the fixed point V_ℓ^* of T and transition rules π_{mq}^* defined for y^*.

Recall that $x^*(0,q)$ has not been specified, yet. Thus $V_h^*(0)$ is to be determined. At $\mu = 0$, optimality of $x^*(0,g) = 0$ is obvious because $\pi_{Gg}(0) = 0 = \pi_{Bg}^*(0)$.

For the moment we restrict attention to the case where the h-seller lies with positive probability when $\mu = 0$ and $q = b$ so that

$$V_h^*(0) = \frac{\ell}{1-\delta} \geq h\ell + \delta\left(hV_h^*(0) + (1-h)V_h^*(\pi_{Bb}^*(0))\right) \tag{15}$$

We will later consider the possibility that $V_h^*(0)$ is larger, but this will not enlarge the set of parameters for existence. Notice that $x^*(0,b) = 1$ is an equilibrium strategy for an h-seller at $\mu = 0$ for posterior beliefs $\pi_{Bb}^*(0) = 0$. Thus condition (15) can always be satisfied.

Since $p_G(\mu, y) > \ell$ for all $\mu, y \in (0,1]$, an h-seller can warrant $\ell/(1-\delta)$ in any continuation subgame by always claiming $q = g$ and, therefore, $V_h^*(\mu) > V_h^*(0)$ for all $\mu > 0$.

Upon drawing $q = g$, an h-seller gets $p_G(\mu, y^*(\mu)) + \delta V_h^*(\pi_{Gg}(\mu))$ by reporting truthfully and $\delta V_h^*(0)$ by reporting untruthfully. Since $V_h^*(\pi_{Gg}(\mu)) \geq V_h^*(0)$, $x^*(\mu, g) = 0$ is optimal for all μ.

Let us now consider the strategy of the h-seller when $q = b$. Once $\mu = 1$ is reached, upon drawing $q = b$, an h-seller gets $\delta V_h^*(1)$ by reporting truthfully and $1 + \delta V_h^*(0)$ by reporting untruthfully by Lemma 3. Thus, it is optimal for an h-seller to report truthfully if and only if $\delta(V_h^*(1) - V_h^*(0)) \geq 1$ where $V_h^*(1) = h + \delta V_h^*(1)$, or equivalently,

$$h - \ell \geq \frac{1-\delta}{\delta} \quad \Longleftrightarrow \quad \delta \geq \delta_h = \frac{1}{h - \ell + 1}, \tag{16}$$

which we assume below. It is clear that no equilibrium exists that satisfies (a) and (b) if (16) fails.

For $\mu > 0$, we don't know the value of V_h^* but we can show that the value increases with the true type.

Lemma 10 *If $V_\ell^* = T\left(V_\ell^*\right)$, then $V_h^*(\mu) > V_\ell^*(\mu)$ for all $\mu > 0$.*

Proof. Let $V_h(\mu)$ be the value function from the following strategy of an h-seller: report $q = g$ truthfully and upon drawing $q = b$ for the first time report $m = G$ and get $V_h^*(0)$ in the continuation subgame. Then,

$$V_h(\mu) = \left[\sum_{t=0}^{\infty} h^t \delta^t p_G\left(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\right)\right] + \delta V_h^*(0)(1-h)\sum_{t=1}^{\infty} h^t\delta^t \quad (17)$$

where $\pi_{Gg}^t(\mu) = \pi_{Gg}(\pi_{Gg}^{t-1}(\mu))$ is defined in (38). Clearly $V_h^*(\mu) \geq V_h(\mu)$.

From equation 37 in the proof of lemma 9:

$$V_\ell^*(\mu) = \left[\sum_{t=0}^{\infty} \ell^t \delta^t p_G\left(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\right)\right] + \delta V_\ell^*(0)(1-\ell)\sum_{t=0}^{\infty} \ell^t\delta^t. \quad (18)$$

Substracting (18) from (17),

$$\begin{aligned} V_h(\mu) - V_\ell^*(\mu) \quad &= \quad \Big[\sum_{t=0}^{\infty}(h^t - \ell^t)\delta^t p_G\big(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big)\Big] \\ &\quad +\delta\Big(\frac{1-h}{1-\delta h} - \frac{1-\ell}{1-\delta\ell}\Big)V_\ell^*(0) \end{aligned}$$

Using $p_G\big(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big) > \ell$.

$$V_h(\mu) - V_\ell^*(\mu) > \frac{\delta(h-\ell)\ell}{(1-\delta h)(1-\delta\ell)} - \frac{\delta(1-\delta)(h-\ell)}{(1-\delta h)(1-\delta\ell)}V_\ell^*(0) = 0$$

■

The result expresses the fact that the h-seller obtains better draws of quality which implies that he would lose its reputation at longer delays if he were to imitate an ℓ-type.

Then, optimality of truthful announcement of q for the type-h seller follows. Upon drawing $q = b$ in any period with a prevailing posterior $\mu > 0$, an h-seller can guarantee himself at least $\delta V_h^*(\pi_{Bb}(\mu, y^*(\mu))$ by reporting truthfully in that period, but gets a payoff of $p_G(\mu, y^*(\mu)) + \delta V_h^*(0)$ by lying. From the previous lemma, announcing $m = B$ yields a higher payoff for the h-seller than for the ℓ-seller, while from (15), announcing $m = G$ would yield the same payoff for both types. Hence the h-seller announces $m = B$ with probability 1 if the ℓ-type does with positive probability. We then obtain

Theorem 2 *There exists a NTR-equilibrium of* Γ *if and only if* $\delta \geq \delta_h$. *The equilibrium outcome is unique.*

Proof. We know that at the unique fixed point of T the incentive compatibility conditions are satisfied for the ℓ-seller. Moreover, with $V_h^*(0) = \ell/(1-\delta)$ the optimal strategy for type h is to announce $m = q$ if $q = g$. It suffices to show that $x^*(\mu, b) = 0$ for $\mu > 0$.

For $\mu \in [\bar{\mu}, 1]$, this follows from (16) because $\pi_{Bb}(\mu, y^*(\mu)) = 1$ and $p_G(\mu, y^*(\mu)) \leq 1$.

For $\mu \in (0, \bar{\mu})$, observe from (11) and (12) that

$$\delta \left(V_\ell^* \left(\pi_{Bb}(\mu, y^*(\mu))\right) - V_\ell^*(0)\right) = p_G(\mu, y^*(\mu)). \tag{19}$$

From above $V_h^* \left(\pi_{Bb}(\mu, y^*(\mu))\right) > V_\ell^* \left(\pi_{Bb}(\mu, y^*(\mu))\right)$, while $V_\ell^*(0) = V_h^*(0)$, hence

$$\delta V_h^* \left(\pi_{Bb}(\mu, y^*(\mu))\right) - V_h^*(0) > p_G(\mu, y^*(\mu)). \tag{20}$$

This prove the optimality of $x^*(\mu, b) = 0$ for $\mu > 0$.

From what follows, NTR-equilibria may differ in $x^*(0, b)$ and $\pi_{Bb}^*(0)$ only. But since consistency requires that an h-seller starts with an initial reputation level $\mu > 0$ and an h-seller always tells the truth as per (a), specification of $x^*(0, b)$ is a part of off-equilibrium strategy. Therefore, the equilibrium outcome is unique.

The value function for the h-type is given by :

$$V_h^*(\mu) = \sum_{t=0}^{\infty} \sum_{\mathbf{h}^t \in H_g^t} \delta^t \rho(\mathbf{h}^t) p_G \left(\pi(\mathbf{h}^t, \mu), y^*(\pi(\mathbf{h}^t, \mu))\right) \tag{21}$$

where $H_g^t := \{g, b\}^{t-1} \times \{g\}$ is the set of all possible realizations of q for t periods with the requirement that $q = g$ in period t; $\rho(\mathbf{h}^t)$ is the ex ante probability that $\mathbf{h}^t \in H_g^t$ realizes; $\pi(\mathbf{h}^t, \mu)$ is the posterior belief at the beginning of period t calculated by Bayes rule from the prior belief μ along $\mathbf{h}^t$.

Observe that $V_h^*(\mu)$ is increasing in μ because $p_G(\mu, y^*(\mu))$, $\pi_{Gg}(\mu)$ and $\pi_{Bb}(\mu, y^*(\mu))$ all increase in μ as verified earlier. ■

To complete this characterization, we should point that because the incentive constraint of the h-seller is slack for $\mu > 0$, there is the possibility that the value $V_h^*(0)$ differs from $V_\ell^*(0)$. This is the case when $x^*(0, b) = 0$. For this to be true it must first be the case that $\pi_{Bb}^*(0)$ is large enough for a h-type to be willing to sacrifice a current profit ℓ to build a reputation at $\mu = \pi_{Bb}^*(0)$ but not so large that the ℓ-type would want to do so. A natural candidate would thus be the limit of $\pi_{Bb}^*(\mu)$ when μ goes to zero. Second, to extend the argument of the theorem on incentive compatibility of the h-type, we must preserve the property that the h-type has more incentive to build reputation: $V_h^*(\mu) - V_h^*(0) \geq V_\ell^*(\mu) - V_\ell^*(0)$. We show below that this is the case if h is large and the seller is patient enough.

Lemma 11 *If $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$ and δ is large enough, there exists an equilibrium such that $x^*(\mu, q) = 0$ for all $\mu \in [0,1]$ and $q \in \{g, b\}$.*

Proof. See Appendix. ■

5 Discussion of the equilibrium

Theorem 2 expresses that the equilibrium outcome is unique.

When $\delta_h \leq \delta \leq \delta_\ell$, the ℓ-type always announces $m = G$. In this case the dynamics of prices and beliefs is straightforward:

- The price p_t increases over time until a quality $q_t = b$ occurs, then
  - it definitely drops to ℓ if the type is ℓ
  - it is always equal to the quality, $p_t = q_t$ if the type is h.

When $\delta > \max(\delta_h, \delta_\ell)$ the ℓ-type randomizes between announcing the true quality B when $q = b$ and claiming a high price p_G with a message G and loosing reputation. Then the dynamics is more complex but we still have:

- The expected price $E(p_t \mid \mathbf{h}^t)$ increases until
  - either a quality $q_t = b$ occurs and the ℓ-type announce G, in which case the price definitely drops to ℓ;
  - or a quality $q_t = b$ occurs, the message is B and at least $\bar{t}$ periods has occurred, in which case the price is equal to the quality, $p_t = q_t$ for ever.

Let us compare this with the learning equilibrium. Two main differences are the price responses and the evolution of the beliefs.

First, notice that the information on the type is revealed much faster in the reputation equilibrium as stated in the next lemma.

Lemma 12 *In a NTR-equilibrium, the type of the seller is known in finite time with probability* 1.

Proof. If a message is invalidated at some date, the type is known to be low. As long as messages are truthful, beliefs increase so there is a maximal date $\bar{t}$ such that $\mu_t > \bar{\mu}$ for sure if $t > \bar{t}$. Then if there were no lie before $\bar{t}$, the type is discovered at the first date where the quality is $q_t = b$. But this occurs at some date with probability 1. ■

Thus communication helps mitigating the asymmetric information problem along two dimensions that are intrinsically related:

i) it helps credible communication of the true quality, thereby mitigating the lemon problem;

ii) it helps consumers learning the true type of the seller.

The key difference concerns the posterior beliefs following a quality $q = b$. In the learning equilibrium they are given by $\pi_{Bb}(\mu, 0) < \mu$. In the reputation equilibrium they take the value 0 if $m = G$ and $\pi^*_{Bb}(\mu) > \mu$ if $m = B$. The next graphics illustrates this under the assumptions that $h = 2/3$ and $\ell = 1/3$, and for an ad-hoc policy function $y^*(\mu) = \min(1 - \mu(3/4 - \mu), 1)$ (so that $y^* = 1$ for $\mu > 3/4$).[12]

Hence, in reputation equilibrium, posterior beliefs increases until they jump to 0 or 1, while in the learning model they randomly converges to 0 or 1.

Concerning prices, the price following $m = G$ is higher in the reputation equilibrium since the h-seller doesn't lie on quality, $p^*_G(\mu_t) > p^e(\mu_t)$, while it is lower when $m = B$. The next graphic illustrates the prices for an announcement G in the case of the NTR-equilibrium and of the learning equilibrium. It is assumed as above that $h = 2/3$ and $\ell = 1/3$, and that the policy function is $y^*(\mu) = \min(1 - \mu(3/4 - \mu), 1)$. The dashed line corresponds to the price in the learning equilibrium, $p^e(\mu) = \mu h + (1 - \mu)\ell$. The two prices coincide for $\mu = 0$ reflecting the fact that in this case the seller announces G for all q with probability 1. Then the price reflects positively the announce $m = G$ in the reputation equilibrium for two reasons: the h-seller announces truthfully the quality; the ℓ-seller announces a low quality with positive probability below $\bar{\mu}$.

The dynamics differ also. To see that consider an equilibrium path and let τ be the date at which the type is revealed. In the reputation game the price when $q_t = g$ follows the beliefs and thus increases over time until date τ, while the price when $q_t = b$ is constant at zero. On the opposite the price in the learning equilibrium follows a martingale where the price of date t is independent of the realization of the quality at date t.

[12] This is just illustration as y^* is not optimal here.

6 Sellers can restart with a new identity

So far we have assumed that sellers stays for ever on the trading place and that memory is infinite. One of the issue in on-line market-place proposing reputation mechanism based on grading by consumers is that it may be difficult to keep track of the identity of the seller. When this is the case, a seller has the option at any date to erase his history by changing his identity and starts again as a new-comer.[13]

To address this issue, we need to put more structure on the dynamics behind the model because the incentive to change identity depends on the beliefs concerning new-comers, and these beliefs depend on equilibrium strategies.

Let us assume that there is a single platform on which trade can take place in an infinite horizon economy, that allows agents to keep track of the past records of sellers. There is a constant measure 1 of sellers on the platform in each period. Each seller dies with probability χ at the end of each period. These deaths are replaced by measure χ of new-born sellers at the beginning of the next period. Each new born seller is of h-type with probability μ_i.

If sellers cannot change identity, the model is the same as the one studied above, where the initial beliefs start at μ_i and the date t is interpreted as the age or seniority of the seller (the number of trading periods since he joined the platform).

Now suppose that a seller can start afresh with a new identity at any period. We wish to construct a reputation equilibrium similar to the NTR-equilibrium but accounting for this possibility.

A first remark is that in an NTR-equilibrium, the beliefs always increases when the message m is correct due to lemma 9. Although this would have to be verified, we can start with the presumption that this still holds. Then a seller will never change his identity unless he has announced $m = G$ while $q = b$, because at any date his future payoff is larger than the expected payoff of a new-born seller (because value functions are increasing). In particular h-sellers never restart.

However a ℓ-type may decide to restart if beliefs are too low.[14] In this case his reputation level is reset at the default level $\mu_0 \in (0, 1)$, at which point the value of continuation game is v_o for an ℓ-seller. To endogenously determine μ_0 and v_o we proceed in three steps.

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[13] The ability to do so depends on the technology used by the platforms. This is knwon to be an issue with Ebay for instance. This is less an issue when the platform controls the bank account coordinates or the social status of companies, as this would involves creating a new firm which is costly.

[14] Notice that for the same reason as above this will occur only after a lie $m = G$ for $q = B$. low.

First we treat v_o as a parameter representing an outside option value when a seller drops out of the market in question. We then determine the equilibrium for a given v_o which generates value functions $V_\theta^\dagger$.We then construct the μ_0 accounting for the equilibrium mass of new-born sellers of each types. We finally search for a fixed point where $v_o = V_\ell^\dagger(\mu_0)$.

6.1 Equilibrium with an outside option

Given an outside option value $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, consider a NTR-equilibrium of the modified game where there is no restart but the seller may opt out at the end of any period and obtain the value v_o in the subsequent period. Notice that the age doesn't convey additional information than in the game with no exit, because in equilibrium the seller only opts out when he has revealed it low type with a wrong message G.

Again a NTR-equilibrium is one that satisfies (a) and (b). As before[15] it is characterized by a probability $y^\dagger(\mu) > 0$ that a ℓ-seller announces $m = G$ when $q = b$ and value function $V_\theta^\dagger$. Then,

$$y^\dagger(0) = 1 \quad \text{and} \quad V_\ell^\dagger(0) = \ell + \delta v_o \in \left(\frac{\ell}{1-\delta}, v_o\right). \tag{22}$$

To see this, note that $V_\ell^\dagger(0) \geq p_G(0, y^\dagger(0)) + \delta v_o > \frac{\ell}{1-\delta}$. If $y^\dagger(0) < 1$, then $V_\ell^\dagger(0) = \ell p_G(0, y^\dagger(0)) + \delta(\ell V_\ell^\dagger(\pi_{Gg}(0)) + (1-\ell)(V^\dagger(\pi_{Bb}(0, y^\dagger(0))))) \leq \ell + \delta V_\ell^\dagger(0)$, which would contradict $V_\ell^\dagger(0) > \frac{\ell}{1-\delta}$. Hence, $y^\dagger(0) = 1$ and $V_\ell^\dagger(0) = \ell + \delta \max\{v_o, V_\ell^\dagger(0)\} = \ell + \delta v_o$.

In addition, an argument analogous to the proof of Lemma 3 establishes that (w.l.o.g. when $y^\dagger \neq 1$) :

$$\lim_{\mu \to 1} y^\dagger(\mu) = y^\dagger(1) = 1, \quad V_\ell^\dagger(1) = \frac{1 + \delta(1-\ell)v_o}{(1-\delta\ell)}, \text{ and } V_\ell^\dagger(\pi_{Gb}^\dagger(1)) = V_\ell^\dagger(0). \tag{23}$$

Let

$$\Delta_{v_o} := \delta \frac{1 - (1-\delta)v_o}{(1-\delta\ell)} = \delta(V_\ell^\dagger(1) - v_o). \tag{24}$$

Define $\mathcal{F}_{v_o}$ to be the set of all non-decreasing functions V on $[0, 1]$ such that $V(0) = \ell + \delta v_o$ and $V(1) = \frac{1+\delta(1-\ell)v_o}{(1-\delta\ell)}$. Define $y_V^\dagger(\mu)$ in the same manner as in (11) and (12) with $\bar{\mu}$ replaced by $\bar{\mu}^\dagger = \inf\{\mu \mid p_G(\mu, 1) > \delta\Delta_{v_o}\} < \bar{\mu}$. Define $T_{v_o} : \mathcal{F}_{v_o} \to \mathcal{F}_{v_o}$ as

$$T_{v_o}(V)(\mu) := p_G(\mu, y_V^\dagger(\mu)) + \delta\left(\ell \max\{v_0, V(\pi_{Gg}(\mu))\} + (1-\ell)v_0\right). \tag{25}$$

[15]The same argument shows that the ℓ-seller truthfully announces $q = g$.

It is straightforward to verify that $T_{v_o}(V_\ell) \in \mathcal{F}_{v_o}$.

Then, Lemmas 8, 14, 9 and Theorem 1 extend to T_{v_o}, establishing that, for any $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, there is a unique fixed point of T_{v_o} and it is continuous and strictly increasing.[16] We omit the proofs because they are analogous with straightforward changes due to the ℓ-seller opting to restart whenever his reputation level is so low that the continuation value falls short of v_o.[17]

To indicate their dependence on $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$ with compact notations, let $y^\dagger_{v_o}$ and $V^\dagger_{v_o}$ denote the strategy of the ℓ-seller and the fixed point of T_{v_o}, respectively. Now we characterize the stationary equilibrium.

Proposition 1 *For* $v_o \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$,

$$y^\dagger_{v_o}(\mu) > y^*(\mu)\ \forall \mu \in [\mu_0, \bar{\mu}) \quad \text{and} \quad y^\dagger_{v_o}(\mu) = y^*(\mu) = 1\ \forall \mu \in [\bar{\mu}, 1]. \tag{26}$$

Proof. In the Appendix. ∎

In the case where the value of the outside option is the same for both types, and thus is v_o for the h-seller, it is straightforward to verify that the existence theorem 2 applies provided that δ_h is replaced by the threshold δ_{v_o} solution of

$$\delta_{v_o}\left(\frac{h}{1-\delta_{v_o}} - v_o\right) = 1.$$

Thus an equilibrium exists for $\delta > \delta_{v_o}$. Then proposition 1 shows that at any level of beliefs there is more lying by the ℓ-type.

6.2 Equilibrium with restart

Let us extend the analysis to the case with restart. Then v_o and μ_0 are endogenous.

First we construct μ_0 for a given value of v_o. For the given v_o, we first find $y^\dagger_{v_o}$ as above. Then we need to measure the mass of ℓ-seller falsely announcing G while $q = b$.

For $q \in \{g, b\}$ let $\rho_\theta(q)$ be the ex ante probability that q realizes for a θ-seller where $\theta \in \{h, \ell\}$. Let $H^k := \{g, b\}^k$ and for any $\mathbf{h}^k = (q_1, \cdots, q_k) \in H^k$

[16] Since both types of sellers would restart if their reputation levels go below a threshold level, the conditions imposed in constructing $T^\dagger$ for μ below the threshold level are not necessary for equilibrium. Nonetheless, the fixed point of $T^\dagger$, restricted to μ above the threshold level satisfies all equilibrium conditions for ℓ-sellers. There may be other strategies of ℓ-seller that also meet equilibrium conditions, in conjunction with the honest behavior of h-sellers. Characterizing all such equilibria appears to be a daunting task, however, so our aim in this section is to investigate existence of stationary equilibrium.

[17] In the proof of lemma 9, $V^*_\ell(\mu) = \sum_{t=0}^{\infty} \delta^t \ell^t (p_G(\pi^t_{Gg}(\mu), y^*(\pi^t_{Gg}(\mu))) - \ell) + \frac{\delta v_o(1-\ell)+\ell}{1-\delta\ell}$, which implies that $V(\tilde{\mu}) - v_o < \sum_{t=0}^{\infty}(p_G(\pi^t_{Gg}(\tilde{\mu}), \hat{y}) - \ell)\delta^t \ell^t$ because $\frac{\delta v_o(1-\ell)+\ell}{1-\delta\ell} < v_o$.

let $\rho_\theta(\mathbf{h}^k)$ be the ex ante probability that $\mathbf{h}^k$ realizes for a θ-seller where $\theta \in \{h, \ell\}$.

Given $\mu_0 > 0$ and $\mathbf{h}^k = (q_1, \cdots, q_k) \in H^k$, let $\mathbf{h}^k_j = (q_1, \cdots, q_j)$ and $\pi(\mathbf{h}^k_j)$ be the posterior belief for a seller who has survived the history $\mathbf{h}^k_j$ without cheating, updated according to $y^\dagger_{v_o}$ conditional on the default belief μ_0:[18]

$$\pi(\mathbf{h}^k_1) = \frac{\mu_0 \rho_h(q_1)(1-\chi)}{\mu_0 \rho_h(q_1)(1-\chi) + (1-\mu_0)\rho_\ell(q_1)(1 - y^\dagger_{v_o}(\mu_0, q_1))(1-\chi)} \tag{27}$$

and recursively for $\mathbf{h}^k_j$,

$$\pi(\mathbf{h}^k_j) = \frac{\pi(\mathbf{h}^k_{j-1})\rho_h(q_j)}{\pi(\mathbf{h}^k_{j-1})\rho_h(q_j) + (1-\pi(\mathbf{h}^k_{j-1}))\rho_\ell(q_j)(1 - y^\dagger_{v_o}(\pi(\mathbf{h}^k_{j-1}), h_j))}. \tag{28}$$

Then, the probability that an ℓ-seller survives $\mathbf{h}^k$ without cheating is

$$\Pr(\mathbf{h}^k) = \prod_{j=1}^{k} \left[\rho_\ell(q_j)(1 - y^\dagger_{v_o}(\pi(\mathbf{h}^k_{j-1}), q_j))(1-\chi)\right] \tag{29}$$

where $\pi(\mathbf{h}^k_0) = \mu_0$. Consequently, the measure of nominally k-period old ℓ-sellers who restart in period $k+1$ for $k \geq 1$, is

$$\sum_{\mathbf{h}^k \in H^k} \chi_0 (1-\mu_0) \Pr(\mathbf{h}^k)(1-\ell) y^\dagger_{v_o}(\pi(\mathbf{h}^k), b)(1-\chi)$$

where χ_0 is a stationary state measure of all sellers who start in each period. This implies that the total measure of old ℓ-sellers who restart any period is $\chi_0(1-\mu_0)(1-\ell)(1-\chi)\Lambda(v_o)$ where:

$$\Lambda(v_o) := \sum_{k=1}^{\infty} \sum_{\mathbf{h}^k \in H^k} \Pr(\mathbf{h}^k) y^\dagger_{v_o}(\pi(\mathbf{h}^k), b).$$

Thus

$$\begin{aligned} \chi_0 &= \chi + \chi_0(1-\mu_0)(1-\ell)(1-\chi)\Lambda(v_o) \\ \implies & \quad \chi_0 = \frac{\chi}{1 - (1-\mu_0)(1-\ell)(1-\chi)\Lambda(v_o)}. \end{aligned}$$

Therefore, Bayes rule dictates that the following hold at a stationary state:

$$\mu_0 = \frac{\chi \mu_i}{\chi + \chi_0(1-\mu_0)(1-\ell)(1-\chi)\Lambda(v_o)} \tag{30}$$

[18] Recall $y^\dagger_{v_o}(\mu) = y^\dagger_{v_o}(\mu, b)$ and $y^\dagger_{v_o}(\mu, g) = 0$.

Solving for (30) allows to define the initial beliefs function $\mu_0^\dagger : (\frac{\ell}{1-\delta}, \frac{1}{1-\delta}) \to (0,1)$ as

$$\mu_0^\dagger(v_o) = \frac{\mu_i - \mu_i(1-\ell)(1-\chi)\Lambda(v_o)}{1 - \mu_i(1-\ell)(1-\chi)\Lambda(v_o)} < \mu_i \tag{31}$$

where the inequality follows from $0 < \Lambda(v_o) < 1$.

An equilibrium then consists of v_o, $y_{v_o}^\dagger$ and $V_{v_o}^\dagger$ such that $V_{v_o}^\dagger(\mu_0^\dagger(v_o)) = v_o$ which is shown to exist by continuity. Indeed we show in appendix

Lemma 13 *Let $\psi : (\frac{\ell}{1-\delta} + \frac{\eta}{2}, \frac{1}{1-\delta}) \to C_{[0,1]}$ be a mapping such that $\psi(v_o) = V_{v_o}^\dagger$ where $C_{[0,1]}$ is the set of all continuous functions on $[0,1]$. Then, ψ is continuous in v_o under the norm sup.*

Proof. See Appendix. ■

Once the existence of an equilibrium value function for the ℓ-type is established as well as initial beliefs, it just remains to show the h-seller finds optimal to announce the true quality. We then obtain

Theorem 3 *Assume that seller can change identity, then there exists a stationary NTR-reputation equilibrium for h and δ above some threshold.*

Proof. Note that, as $v_o \to \frac{\ell}{1-\delta}$, $\mu_0^\dagger(v_o)$ converges to a limit strictly greater than 0. Since the right derivative of $p_G(\mu, y_{v_o}^\dagger(\mu))$ with respect to μ is uniformly bounded away from 0 at $\mu = 0$, so is the right derivative of $V_{v_o}^\dagger(\mu)$ and consequently, $V_{v_o}^\dagger(\mu_0^\dagger(v_o)) > v_o$ for v_o sufficiently close to $\frac{\ell}{1-\delta}$.

On the other hand, as $v_o \to \frac{1}{1-\delta}$, we have since $\mu_i < 1 : V_{v_o}^\dagger(\mu_0^\dagger(v_o)) \leq V_{v_o}^\dagger(\mu_i) < V_{v_o}^\dagger(1) \leq v_o$ for v_o sufficiently close to $\frac{1}{1-\delta}$. Moreover

Since $\mu_0^\dagger(v_o)$ is continuous in v_o from (31) and ψ is continuous, we must have $V_{v_o}^\dagger(\mu_0^\dagger(v_o)) = v_o$ for at least one $v_o \in (\frac{\ell}{1-\delta} + \eta, \frac{1}{1-\delta})$.

Let μ_0^s and v_o^s denote a pair of stationary default reputation level and value, i.e., $v_o^s = V_{v_o^s}^\dagger(\mu_0^s)$ and $\mu_0^s = \mu_0^\dagger(v_o^s)$. Note that to establish a stationary equilibrium, we still need to show that it is optimal for h-sellers to always report truthfully as long as $\mu \geq \mu_o^s$. Since the continuation value of h-seller after cheating is the equilibrium value of the default level μ_0^s, $V_h^\dagger(\mu_0^s)$, rather than $V_h^\dagger(0)$, the optimality condition of h-seller is more difficult to verify than when restarting is impossible. In fact, it has not been proved that for all stationary pair of μ_0^s and v_o^s, truthful reporting for all $\mu \geq \mu_0^s$ is optimal for h-sellers when ℓ-sellers report according to $y_{v_o^s}^\dagger(\mu)$ for $\mu \geq \mu_0^s$.

However, the proof of Lemma 11 relies on $V_\ell^*(0)$ being a constant, rather than $V_\ell^*(0) = \frac{\ell}{1-\delta}$ and consequently, applies analogously to $V_h^\dagger(\mu)$ defined as per (21) with y^* replaced by $y^\dagger_{v^s_o}$ for $\mu > \mu^s_o$. As a result, if $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$, for all sufficiently large $\delta < 0$ it constitutes an equilibrium for ℓ-sellers to report according to $y^\dagger_{v^s_o}(\mu)$ and h-sellers honestly for $\mu \geq \mu^s_0$ for any stationary pair μ^s_0 and v^s_o, provided that $\delta(V^\dagger_h(1) - V^\dagger_h(\mu^s_0)) \geq 1$.[19] It may be worth mentioning that this is a sufficient condition, so stationary equilibria in which h-sellers behave honestly may exist in a wider class of environments. ■

In any such equilibrium, the proof of Proposition 1 applies without change and, therefore, sellers' announcements are less reliable than when fresh restart with new identity is not possible.

However, this does not mean that untruthful announcements are more frequent in the market when restarts are possible than when not: ℓ-sellers who have lied once, rather than keep lying forever when $q = b$, would start afresh and announce according to $y^\dagger(\mu)$. In fact, when δ is close to 1 there will be more truthful announcements in the market when sellers are allowed to restart with a new identity.

Nevertheless, h-sellers tend to suffer more due to untrustworthy behavior of ℓ-sellers when restarts are possible, because such behavior by sellers who are "known" to be of ℓ-type (i.e., those with reputation level $\mu = 0$), which happens only when restarts are not allowed, does not affect h-sellers.

APPENDIX

Proof of Lemma 3:. (i) Suppose that $y^*(1) < 1$. Then, we would have:

$$V^*_\ell(1) = \ell\big(1 + \delta V^*_\ell(1)\big) + (1-\ell)\big(y^*(1)(1 + \delta V^*_\ell(\pi^*_{Gb}(1)) + (1 - y^*(1))\delta V^*_\ell(1)\big),$$

along with $1 + \delta V^*_\ell(\pi^*_{Gb}(1)) \leq \delta V^*_\ell(\pi^*_{Bb}(1)) = \delta V^*_\ell(1)$. Thus

$$\begin{aligned} V^*_\ell(1) \quad &\leq \quad \ell\big(1 + \delta V^*_\ell(1)\big) + (1-\ell)\delta V^*_\ell(1) = \ell + \delta V^*_\ell(1) \\ &\Longrightarrow \quad V^*_\ell(1) \leq V^*_\ell(0) \Longrightarrow V^*_\ell(1) = V^*_\ell(0). \end{aligned}$$

The value function V^*_ℓ would be constant which contradicts $1+\delta V^*_\ell(\pi^*_{Gb}(1)) \leq \delta V^*_\ell(\pi_{Bb}(1, y^*(1)))$. Hence, $y^*(1) = 1$.

Then

$$V^*_\ell(1) = 1 + \ell\delta V^*_\ell(1) + (1-\ell)\delta V^*_\ell(\pi^*_{Gb}(1))$$

and

$$1 + \delta V^*_\ell(\pi^*_{Gb}(1)) \leq \delta V^*_\ell(\pi_{Bb}(1, y^*(1))) = \delta V^*_\ell(1).$$

[19] The proof is omitted because it is the same as the proof of Lemma 11 with obvious changes, such as v^s_o and $\bar{\mu}^\dagger$ in place of $V^*_\ell(0)$ and $\bar{\mu}$, respectively.

Now suppose that $\lim_{\mu\to 1} y^*(\mu) \neq 1$. Then, the following holds for some $\eta > 0$: for any $\epsilon > 0$ there is $\mu_\epsilon < 1$ such that $1-\epsilon < \mu_\epsilon$ and $y^*(\mu_\epsilon) < 1-\eta$ and thus,

$$V_\ell^*(\mu_\epsilon) = \ell p_G(\mu_\epsilon, y^*(\mu_\epsilon)) + \delta\big(\ell V_\ell^*(\pi_{Gg}(\mu_\epsilon)) + (1-\ell)V_\ell^*(\pi_{Bb}(\mu_\epsilon, y^*(\mu_\epsilon)))\big). \quad (32)$$

Since $p_G(\mu_\epsilon, y^*(\mu_\epsilon)) \to 1$, $\mu_\epsilon \to 1$, $\pi_{Gg}(\mu_\epsilon) \to 1$, and $\pi_{Bb}(\mu_\epsilon, y^*(\mu_\epsilon)) \to 1$ as $\epsilon \to 0$, (32) would imply $\lim_{\mu\to 1} V_\ell^*(\mu) = \ell + \delta \lim_{\mu\to 1} V_\ell^*(\mu)$, i.e., $\lim_{\mu\to 1} V_\ell^*(\mu) = \frac{\ell}{1-\delta} = V_\ell^*(0)$. Then, $V_\ell^*(\mu) = V_\ell^*(0)$ for all μ which would imply that an ℓ-seller with low quality product would always lie as $p_G(\mu_\epsilon, y^*(\mu_\epsilon)) + \delta V_\ell^*(0) > \delta V_\ell^*(0)$. Hence $y^*(\mu) \equiv 1$, a contradiction. We conclude that $\lim_{\mu\to 1} y^*(\mu) = 1$.

(ii) Since for all sufficiently large $\mu < 1$, $y^*(\mu) > 0$ implies that $V_\ell^*(\mu) = p_G(\mu, y^*(\mu)) + \delta\left(\ell V_\ell^*(\pi_{Gg}(\mu)) + (1-\ell)V_\ell^*(0)\right)$, we get

$$\lim_{\mu\to 1} V_\ell^*(\mu) = \frac{1-\delta(1-\ell+\ell^2)}{(1-\delta)(1-\delta\ell)}.$$

(iii) Suppose that $y^*(\mu) \neq 1$ for some μ and that $\bar{v} = \frac{1-\delta(1-\ell+\ell^2)}{(1-\delta)(1-\delta\ell)} < V_\ell^*(1)$. Note that $\delta(\bar{v} - V_\ell^*(0)) < 1$ from (6).

If $\delta(V_\ell^*(1) - V_\ell^*(0)) > \ell$, then there is some $\mu \in (0,1)$ such that $\delta(V_\ell^*(1) - V_\ell^*(0)) > p_G(\mu, 1) > \delta(\bar{v} - V_\ell^*(0))$ so that $y^*(\mu)$ cannot be equal to 1 because $\delta(V_\ell^*(1) - V_\ell^*(0)) > p_G(\mu, 1)$, nor can it be less than 1 because $\delta(V_\ell^*(\pi_{Bb}(\mu, y)) - V_\ell^*(0)) < p_G(\mu, y)$ for all $y < 1$ by Lemma 4.1, which is impossible.

If $\delta(V_\ell^*(1) - V_\ell^*(0)) \leq \ell$, on the other hand, $y^*(\mu) = 1$ must hold for all $\mu \in (0,1]$ due to condition (ii) of Definition 0, because $p_G(\mu, y) > \ell \geq \delta(V_\ell^*(1) - V_\ell^*(0)) \geq \delta(V^*(\pi_{Bb}(\mu, y)) - V^*(\pi_{Gb}(\mu, y)))$ for any $y < 1$ by Lemma 4.1. Since we have restricted to $y^*(0) = 1$ above, we have encountered a contradiction to the supposition that $y^*(\mu) \neq 1$ for some μ.

Hence $\lim_{\mu\to 1} V_\ell^*(\mu) = V_\ell^*(1)$. This, together with $V_\ell^*(1) = 1 + \delta\left(\ell V_\ell^*(1) + (1-\ell)V_\ell^*(\pi_{Gb}^*(1)\right)$, implies $V_\ell^*(\pi_{Gb}^*(1)) = V_\ell^*(0)$.

Finally suppose we change $\pi_{Gb}^*(1)$ to zero. The value function $V_h^*(.)$ is unchanged, while the incentive compatibility condition for type h at 1 is still verified: $1 + \delta V_h^*(1) \geq \delta V_h^*(0)$ holds if it holds for $\pi_{Gb}^*(1) > 0$. For the ℓ-type seller, the value $V_\ell^*(1)$ is lower while $V_\ell^*(\mu)$ is unchanged for $\mu < 1$. But since $\mu_t = 1$ should never occur in equilibrium when the type is ℓ, all incentive compatibility are preserved. ■

Proof of Lemma 9: . To reach a contradiction, suppose that there exists μ such that $\pi_{Bb}(\mu, y_V(\mu)) < \mu$. Since y_V is continuous and $y_V(\mu) = 1$ for $\mu \geq \bar{\mu}$, there exists

$$\tilde{\mu} = \max\{\mu < 1 \mid \pi_{Bb}(\mu, y_V(\mu)) \leq \mu\} < \bar{\mu}. \quad (33)$$

Note that $\pi_{Bb}(\tilde{\mu}, y_V(\tilde{\mu})) = \tilde{\mu}$. Since

$$\pi_{Bb}(\mu, y) \geq \mu \iff y \geq \hat{y} := \frac{h-\ell}{1-\ell}, \quad (34)$$

it must be the case that $y_V(\tilde{\mu}) = \hat{y}$ and thus

$$p_G(\tilde{\mu}, \hat{y}) = \delta(V(\tilde{\mu}) - V(0)). \tag{35}$$

Expanding $V_\ell^*(\mu) = p_G(\mu, y^*(\mu)) + \delta\left(\ell V_\ell^*(\pi_{Gg}(\mu)) + (1-\ell)V_\ell^*(0)\right)$ by applying an analogous equation to $V_\ell^*(\pi_{Gg}(\mu))$ repeatedly, we get

$$\begin{aligned} V_\ell^*(\mu) &= \left[\sum_{t=0}^{\infty} \ell^t \delta^t p_G\left(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\right)\right] + \delta V_\ell^*(0)(1-\ell)\sum_{t=0}^{\infty} \ell^t \delta^t & (36)\\ &= \sum_{t=0}^{\infty} \delta^t \ell^t (p_G(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))) - \ell) + V_\ell^*(0). & (37) \end{aligned}$$

where $\pi_{Gg}^t(\mu) = \pi_{Gg}\left(\pi_{Gg}^{t-1}\right)$ is defined recursively so that

$$\pi_{Gg}^t(\mu) = \frac{\mu h^t}{\mu h^t + (1-\mu)\ell^t}. \tag{38}$$

Note from (38) that $\pi_{Gg}^t(\tilde{\mu}) > \tilde{\mu}$ for $t > 0$ and thus, $\pi_{Bb}(\pi_{Gg}^t(\tilde{\mu}), y_V(\pi_{Gg}^t(\tilde{\mu}))) > \pi_{Gg}^t(\tilde{\mu})$ by (33). Consequently, $y_V(\pi_{Gg}^t(\tilde{\mu})) > \hat{y}$ by (34). Therefore, since $p_G(\mu, y) \leq 1$ and decreases in y, (37) implies that

$$V(\tilde{\mu}) - V(0) < \sum_{t=0}^{\infty} (p_G(\pi_{Gg}^t(\tilde{\mu}), \hat{y}) - \ell)\delta^t \ell^t. \tag{39}$$

Since

$$p_G(\mu, \hat{y}) = \frac{\mu h + (1-\mu)\ell}{h} \tag{40}$$

from (3), we further deduce from (39) that

$$\begin{aligned} V(\tilde{\mu}) - V(0) &< p_G(\tilde{\mu}, \hat{y}) - \ell + \sum_{t=1}^{\infty} \left(\frac{\pi_{Gg}^t(\tilde{\mu})(h-\ell) + \ell(1-h)}{h}\right)\delta^t \ell^t \\ &< p_G(\tilde{\mu}, \hat{y}) - \ell + \sum_{t=1}^{\infty} \left(\frac{(h-\ell) + \ell(1-h)}{h}\right)\delta^t \ell^t \\ &= p_G(\tilde{\mu}, \hat{y}) - \ell + (1-\ell)\frac{\delta\ell}{1-\delta\ell} \\ &= p_G(\tilde{\mu}, \hat{y}) - \frac{(1-\delta)\ell}{1-\delta\ell} < p_G(\tilde{\mu}, \hat{y}) \end{aligned}$$

where the second inequality follows from $\pi_{Gg}^t(\tilde{\mu}) < 1$. Thus, we have reached a contradictory conclusion that (35) cannot hold at $\tilde{\mu}$. ∎ ∎

Proof of existence:. Define $\mathcal{F}^r$ as the set of all right-continuous non-decreasing functions on [0,1], endowed with the topology of the weak convergence. The set is compact and convex. ∎

Lemma 14 *$T(\mathcal{F}^r) = \mathcal{F}^r$ and T is continuous on $\mathcal{F}^r$.*

Proof. Consider a sequence V^n, $n = 1, 2, \cdots$, in $\mathcal{F}^r$ that converges to $V \in \mathcal{F}^r$ under the topology of the weak convergence. We show below that $T(V^n)(\mu)$ converges to $T(V)(\mu)$ at all continuity points of $T(V)$, which proves the lemma.

Let Ω be the set of all points where $V(\pi_{Gg}(\mu))$ is continuous. Since $\pi_{Gg}(\mu)$ is increasing, $[0,1]\backslash\Omega$ is countable. Since V is continuous at $\mu = \pi_{Gg}(\mu)$ if $\mu \in \Omega$ by continuity of π_{Gg}, it follows that $V^n(\pi_{Gg}(\mu))$ converges to $V(\pi_{Gg}(\mu))$ on Ω.

Next, Let $y_V(\mu)$ be as defined in (12) for V and $y_{V^n}(\mu)$ for V^n. Let Λ be the set of points where $V(\pi_{Bb}(\mu, y_V(\mu)))$ is continuous. Since $\pi_{Bb}(\mu, y_V(\mu))$ is non-decreasing on $(0,1]$ as verified in the proof of Lemma 6, $[0,1]\backslash\Lambda$ is countable. We now show that $y_{V^n}(\mu) \to y_V(\mu)$ for all $\mu \in \Lambda$.

Consider $\mu \in \Lambda$. That $y_{V^n}(\mu) \to y_V(\mu)$ is trivial from (12) if $\mu = 0$ or $\mu > \bar{\mu}$. Hence, suppose $0 < \mu \leq \bar{\mu}$ so that, denotiing V^n_- the left limit:

$$\delta\left(V^n_-(\pi_{Bb}(\mu, y_{V^n}(\mu))) - V(0)\right) \leq p_G(\mu, y_{V^n}(\mu)) \leq \delta\left(V(\pi_{Bb}(\mu, y_{V^n}(\mu))) - V(0)\right). \tag{41}$$

By taking a subsequence if necessary, we may assume that $y_{V^n}(\mu)$ converges to a limit y'. To reach a contradiction, suppose $y' \neq y_V(\mu)$. First, consider the case that $y' < y_V(\mu)$. Then, since $p_G(\mu, y)$ decreases with μ there exists $\varepsilon > 0$ such that

$$p_G(\mu, y_{V^n}(\mu)) > p_G(\mu, y_V(\mu)) + \varepsilon = \delta(V(\pi_{Bb}(\mu, y_V(\mu))) - V(0)) + \varepsilon$$

for sufficiently large n, where the equality follows because $\mu \in \Lambda$. From this we further deduce that

$$\begin{aligned} p_G(\mu, y_{V^n}(\mu)) &> \delta(V^n(\pi_{Bb}(\mu, y_V(\mu))) - V(0)) + \frac{\varepsilon}{2} \\ &> \delta(V^n(\pi_{Bb}(\mu, y_{V^n}(\mu))) - V(0)) + \frac{\varepsilon}{2} \end{aligned}$$

for sufficiently large n, where the first inequality follows because $V^n(\pi_{Bb}(\mu, y_V(\mu))) \to V(\pi_{Bb}(\mu, y_V(\mu)))$ for $\mu \in \Lambda$ and the second because $\pi_{Bb}(\mu, y)$ increases in y and $y_{V^n}(\mu) \to y' < y_V(\mu)$. However, this contradicts (41).

For the case $y' > y_V(\mu)$, we can apply the same reasoning using $V^n_-(\pi_{Bb}(\mu, y_{V^n}(\mu))) \geq V^n(\pi_{Bb}(\mu, y_V(\mu)))$ for n large leads to

$$p_G(\mu, y_{V^n}(\mu)) < \delta(V^n_-(\pi_{Bb}(\mu, y_{V^n}(\mu))) - V(0)) - \frac{\varepsilon}{2},$$

which again is a contradiction. We conclude that $y_{V^n}(\mu) \to y_V(\mu)$ for all $\mu \in \Lambda$.

Together with the earlier result that $V^n(\pi_{Gg}(\mu)) \to V(\pi_{Gg}(\mu))$ for all $\mu \in \Omega$, this establishes for all $\mu \in \Omega \cap \Lambda$ that

$$\begin{aligned} T(V^n)(\mu) &= p_G(\mu, y^{n*}(\mu)) + \delta\big(\ell V^n(\pi_{Gg}(\mu)) + (1-\ell)V(0)\big) \\ &\to p_G(\mu, y_V(\mu)) + \delta\big(\ell V(\pi_{Gg}(\mu)) + (1-\ell)V(0)\big) = T(V)(\mu) \end{aligned}$$

as $n \to \infty$. Finally, to verify this convergence at every continuity point of $T(V)(\mu)$, observe first that this convergence is trivial from (12) at $\mu = 0, 1$. For any other $\mu \notin \Omega \cap \Lambda$ at which $T(V)$ is continuous, one can find $\mu_1 \in \Omega \cap \Lambda \cap (0, \mu)$ arbitrarily close to μ and $\mu_2 \in \Omega \cap \Lambda \cap (\mu, 1)$ arbitrarily close to μ because $\Omega \cap \Lambda$ is dense in $[0, 1]$. Since $T(V^n)(\mu_1) \leq T(V^n)(\mu) \leq T(V^n)(\mu_2)$ and $T(V)(\mu_1) \leq T(V)(\mu) \leq T(V)(\mu_2)$, taking the limits we get

$$T(V)(\mu_1) \leq \liminf T(V^n)(\mu) \leq \limsup T(V^n)(\mu) \leq T(V)(\mu_2), \quad \text{and}$$

$$\sup_{\substack{\mu_1 \in \Omega\cap\Lambda \\ \mu_1 < \mu}} T(V)(\mu_1) = T(V)(\mu) = \inf_{\substack{\mu_2 \in \Omega\cap\Lambda \\ \mu_2 > \mu}} T(V)(\mu_2),$$

which imply, as desired, that $T(V^n)(\mu)$ converges to $T(V)(\mu)$ at every continuity point of $T(V)(\mu)$. ■
endproof. By Fan-Glicksberg Fixed Point Theorem (Fan, 1952; Glicksberg, 1952), T has a fixed point in $\mathcal{F}^r$. ■

Proof of uniqueness:. To reach a contradiction, suppose there are two fixed points V^1 and V^2. Notice that the level $\bar{\mu}$ is independent of V and

$$V^i(\mu) = p_G(\mu, 1) + \delta\left(\ell V(1) + (1-\ell)V(0)\right) \quad \forall \mu \geq \bar{\mu}. \tag{42}$$

Since $V^1(\mu) = V^2(\mu)$ for all $\mu \geq \bar{\mu}$, the following is well-defined:

$$\hat{\mu} := \min\{\mu \,|\, V^1(\mu') = V^2(\mu')\ \forall \mu' \geq \mu\} \in (0, \bar{\mu}]. \tag{43}$$

A "segment" for $i = 1, 2$, is a nonempty interval $I_i = [x, z] \subset [0, \bar{\mu}]$ such that $V^i(\mu) > V^j(\mu)$ for all $\mu \in (x, z)$ and $V^i(\mu) = V^j(\mu)$ for $\mu = x, z$, where $j \neq i$. A "region" for $i = 1, 2$, is a nonempty interval $R_i = [x, z] \subset [0, \bar{\mu}]$ such that $V^i(\mu) \geq V^j(\mu)$ for all $\mu \in I_i$ and there are $x', z' \in R_i$ such that $[x, x']$ and $[z', z]$ are segments for i. Let

$$p^i_G(\mu) := p_G(\mu, y_{V^i}(\mu)) \quad \text{and} \quad \pi^i_{Bb}(\mu) := \pi_{Bb}(\mu, y_{V^i}(\mu)) \quad \text{for} \quad i = 1, 2. \tag{44}$$

Recall that in the proof of Lemma 6, we have shown that both $p^i_G(\mu)$ and $\pi^i_{Bb}(\mu)$ weakly increase in μ. Since V^i strictly increases in μ by Lemma 8, the same reasoning establishes that

[A] $p^i_G(\mu)$ and $\pi^i_{Bb}(\mu)$ strictly increase in μ.

Next, we establish the following:

[B] If $V^1(\pi^i_{Bb}(\mu)) = V^2(\pi^i_{Bb}(\mu))$ for some $\mu > 0$ and some $i = 1, 2$, then $y^1(\mu) = y^2(\mu)$ and consequently, $p^1_G(\mu) = p^2_G(\mu)$ and $\pi^1_{Bb}(\mu) = \pi^2_{Bb}(\mu)$. If, in addition, $V^1(\pi_{Gg}(\mu)) = V^2(\pi_{Gg}(\mu))$ holds, then $V^1(\mu) = V^2(\mu)$.

Note that this observation is trivial for $\mu \geq \bar{\mu}$. Since

$$p^i_G(\mu) = \delta\big(V^i(\pi^i_{Bb}(\mu)) - V^i(0)\big) \quad \forall \mu \in (0, \bar{\mu}], \tag{45}$$

$V^1(\pi^i_{Bb}(\mu)) = V^2(\pi^i_{Bb}(\mu))$ implies $p^1_G(\mu) = p^2_G(\mu)$, which in turn implies $y^*_{V^1_\ell}(\mu) = y^*_{V^2_\ell}(\mu)$, from which the remaining claims of [B] follow.

We also establish the following:

[C] If $V^1(\pi^1_{Bb}(\mu)) > V^2(\pi^1_{Bb}(\mu))$ for some $\mu \in (0, \bar{\mu}]$, then $y^1(\mu) < y^2(\mu)$ and $p^1_G(\mu) > p^2_G(\mu)$.

Since $p_G(\mu, y)$ strictly decreases in y, $V^i(\mu)$ strictly increases in μ, and $p^1_G(\mu) = \delta\big(V^1(\pi^1_{Bb}(\mu)) - V^i(0)\big)$ for $\mu \in (0, \bar{\mu}]$, it follows that if $V^1(\pi^1_{Bb}(\mu)) > V^2(\pi^1_{Bb}(\mu))$ for some $\mu \in (0, \bar{\mu}]$ then the graphs of $p_G(\mu, y)$ and $\delta\big(V^2(\pi_{Bb}(\mu, y)) - V^2(0)\big)$ cross at $y^2(\mu) > y^1(\mu)$, hence $p^1_G(\mu) > p^2_G(\mu)$, proving [C].

Finally, since $\pi^i_{Bb}(\hat{\mu}) > \hat{\mu}$ by Lemma 9 and $\pi_{Gg}(\hat{\mu}) > \hat{\mu}$ by (4), due to continuity, there is $\mu' < \hat{\mu}$ such that $V^1(\mu') \neq V^2(\mu')$, $\pi^i_{Bb}(\mu') > \hat{\mu}$ and $\pi_{Gg}(\mu') > \hat{\mu}$. Then, $V^1(\pi^i_{Bb}(\mu')) = V^2(\pi^i_{Bb}(\mu'))$ by (43) and thus, $V^1(\mu') = V^2(\mu')$ by [B], a contradiction to the earlier assertion that $V^1(\mu') \neq V^2(\mu')$. This completes the proof. (Note that [C] is not used. ■

Proof of Lemma 11:. Let

$$V^o_h(\mu) := h \sum_{t=0}^{\infty} h^t \delta^t p_G\big(\pi^t_{Gg}(\mu), y^*(\pi^t_{Gg}(\mu))\big) \quad \forall \mu > 0$$

so that

$$V^*_h(\mu) = V^o_h(\mu) + (1-h)\delta \sum_{t=0}^{\infty} h^t \delta^t V^*_h\big(\pi_{Bb}(\pi^t_{Gg}(\mu), y^*(\pi^t_{Gg}(\mu))\big) \quad \forall \mu > 0. \tag{46}$$

In conjunction with (36),

$$V^o_h(\mu) - V^*_\ell(\mu) = \Big[\sum_{t=0}^{\infty} (h^{t+1} - \ell^t)\delta^t p_G\big(\pi^t_{Gg}(\mu), y^*(\pi^t_{Gg}(\mu))\big)\Big] - \delta V^*_\ell(0)\frac{1-\ell}{1-\delta\ell}$$

and thus,

$$\frac{dV_h^o(\mu)}{d\mu} - \frac{dV_\ell^*(\mu)}{d\mu} = \sum_{t=0}^{\infty}(h^{t+1} - \ell^t)\delta^t \frac{\partial p_G\big(\pi_{Gg}^t(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}^t(\mu)}{d\mu} \tag{47}$$

$$= \sum_{t=0}^{\infty}\delta^{2t}\Bigg[(h^{2t+1} - \ell^{2t})\frac{\partial p_G\big(\pi_{Gg}^{2t}(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}^{2t}(\mu)}{d\mu} + \delta(h^{2t+2} - \ell^{2t+1})\frac{\partial p_G\big(\pi_{Gg}^{2t+1}(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}^{2t+1}(\mu)}{d\mu}\Bigg]$$

$$> \sum_{t=0}^{\infty}\delta^{2t}\ell^{2t}\Bigg[(h-1)\frac{\partial p_G\big(\pi_{Gg}^{2t}(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}^{2t}(\mu)}{d\mu} + \delta(h^2 - \ell)\frac{\partial p_G\big(\pi_{Gg}^{2t+1}(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}(\pi_{Gg}^{2t}(\mu))}{d\mu}\frac{d\pi_{Gg}^{2t}(\mu)}{d\mu}\Bigg] \tag{48}$$

for $\mu \geq \bar{\mu}$ because $y^*(\mu) = 1$ for $\mu \geq \bar{\mu}$. By routine calculation, we get

$$(h-1)\frac{\partial p_G\big(\mu, 1\big)}{\partial \mu} + (h^2 - \ell)\frac{\partial p_G\big(\pi_{Gg}(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}(\mu)}{d\mu} = -\frac{h(1-h)(1-\ell)}{(1-(1-h)\mu)^2} + \frac{h^2(h^2-\ell)(1-\ell)\ell}{(\ell(1-\mu) + h^2\mu)^2}, \tag{49}$$

the derivative of which is

$$-2(1-\ell)\frac{h(1-h)^2}{(1-(1-h)\mu)^3} - \frac{\ell(h^3 - h\ell)^2}{(\ell(1-\mu) + h^2\mu)^3} < 0.$$

Since it is straightforwardly verified that (49) evaluated at $\mu = 1$ is positive if $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$, it further follow(49) is positive for all μ if $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$. This implies that (48) is positive for all $\mu \geq \bar{\mu}$ and consequently, from (46),

$$\frac{dV_h^*(\mu)}{d\mu} \geq \frac{dV_\ell^*(\mu)}{d\mu} \quad \forall \mu \geq \bar{\mu} \tag{50}$$

when $\delta < 1$ is sufficiently close to 1 if $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$.

Next, let $\mu_1 = \min\{\mu | \pi_{Gg}(\mu) \geq \bar{\mu}$ and $\pi_{Bb}(\mu, y^*(\mu)) \geq \bar{\mu}\}$ and consider $\mu \in [\mu_1, \bar{\mu}]$. Note that $\mu_1 < \bar{\mu}$ due to Lemma **??**. Since

$$\begin{aligned} V_h^*(\mu) &= hp_G(\mu, y^*(\mu)) + \delta\big(hV_h^*(\pi_{Gg}(\mu)) + (1-h)V_h^*(\pi_{Bb}(\mu, y^*(\mu)))\big) \text{ and} \\ V_\ell^*(\mu) &= p_G(\mu, y^*(\mu)) + \delta\big(\ell V_\ell^*(\pi_{Gg}(\mu)) + (1-\ell)V_\ell^*(0)\big), \end{aligned}$$

we deduce that $\frac{dV_h^*(\mu)}{d\mu} - \frac{dV_\ell^*(\mu)}{d\mu}$, which exists almost everywhere because both $V_h^*(\mu)$ and $V_\ell^*(\mu)$ are continuous and increasing, is equal to the derivative of

$$(1-h)\big(\delta V_h^*(\pi_{Bb}(\mu, y^*(\mu))) - p_G(\mu, y^*(\mu))\big) + \delta\big(hV_h^*(\pi_{Gg}(\mu)) - \ell V_\ell^*(\pi_{Gg}(\mu))\big),$$

which is positive due to (50) because $p_G(\mu, y^*(\mu)) = \delta\big(V^*_\ell(\pi_{Bb}(\mu, y^*(\mu))) - V^*_\ell(0)\big)$ for $\mu \le \bar{\mu}$. Repeated application of analogous argument establishes that $\frac{dV^*_h(\mu)}{d\mu} > \frac{dV^*_\ell(\mu)}{d\mu}$ for all $\mu > 0$ when $\delta < 1$ is sufficiently close to 1 if $h > \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}$. [See BJ-IUP-20090425.nb for calculation.]

Setting $\pi_{Bb}(0,1) = \lim_{\mu\to 0} \pi_{Bb}(\mu, y^*(\mu))$ and $V^*_h(0) = \lim_{\mu\to 0} V^*_h(\mu)$, this implies that h-seller prefers to tell the truth upon drawing $q = b$ whenever ℓ-seller is indifferent, i.e., when $\mu \in (0, \bar{\mu}]$. Then, $x^*(0, b) = 0$ is optimal by continuity of V^*_h, $p_G(\mu, y^*(\mu))$, $\pi_{Gg}(\mu)$, and $\pi_{Bb}(\mu, y^*(\mu))$. Finally, optimality of $x^*(\mu, g) = 0$ follows immediately from (**??**) as before. ■

Proof of Proposition 1:. Since $\delta\Delta_{v_o} < \delta\Delta < 1$, as before there is $\bar{\mu}^\dagger \in (0, \bar{\mu})$ such that $p_G(\bar{\mu}^\dagger, 1) = \delta(V^\dagger_\ell(\pi_{Bb}(\bar{\mu}^\dagger, 1)) - v_o)$ so that $p_G(\mu, y) > \delta(V^\dagger_\ell(\pi_{Bb}(\mu, y)) - v_o)$ for all $y \in [0,1]$ and, therefore, $y^\dagger_{v_o}(\mu) = 1$ for all $\mu \ge \bar{\mu}^\dagger$.

Note that $y^\dagger_{v_o}$ is continuous by construction (which is analogous to (12)) and $y^\dagger_{v_o}(\mu) \in (0,1)$ for $\mu < \bar{\mu}^\dagger$. To reach a contradiction, suppose $y^\dagger_{v_o}(\hat{\mu}) = y^*_{V^*_\ell}(\hat{\mu})$ for some $\hat{\mu} < \bar{\mu}^\dagger$ and $y^\dagger_{v_o}(\mu) > y^*_{V^*_\ell}(\mu)$ for all $\mu \in (\hat{\mu}, \bar{\mu})$. Then,

$$\begin{aligned}\delta\big(V^*_\ell(\pi_{Bb}(\hat{\mu}, y^*_{V^*_\ell}(\hat{\mu}))) - V^*_\ell(0)\big) &= p_G(\hat{\mu}, y^*_{V^*_\ell}(\hat{\mu})) \\ &= p_G(\hat{\mu}, y^\dagger_{v_o}(\hat{\mu})) = \delta\big(V^\dagger_\ell(\pi_{Bb}(\hat{\mu}, y^\dagger_{v_o}(\hat{\mu}))) - v_o\big)\end{aligned}$$

and thus,

$$V^*_\ell(\tilde{\mu}) - V^*_\ell(0) = V^\dagger_\ell(\tilde{\mu}) - v_o \quad \text{where} \quad \tilde{\mu} := \pi_{Bb}(\hat{\mu}, y^*_{V^*_\ell}(\hat{\mu})) > \hat{\mu} \tag{51}$$

and the inequality is from Lemma **??**. Furthermore, since

$$\begin{aligned} V^*_\ell(\tilde{\mu}) &= p_G(\tilde{\mu}, y^*_{V^*_\ell}(\tilde{\mu})) + \delta\big(\ell V^*_\ell(\pi_{Gg}(\tilde{\mu})) + (1-\ell)V^*_\ell(0)\big) \quad \text{and} \quad (52)\\ V^\dagger_\ell(\tilde{\mu}) &= p_G(\tilde{\mu}, y^\dagger_{v_o}(\tilde{\mu})) + \delta\big(\ell V^\dagger_\ell(\pi_{Gg}(\tilde{\mu})) + (1-\ell)v_o\big) \quad (53)\end{aligned}$$

while $p_G(\tilde{\mu}, y^*_{V^*_\ell}(\tilde{\mu})) \ge p_G(\tilde{\mu}, y^\dagger_{v_o}(\tilde{\mu}))$, (51)-(53) would imply

$$\delta\ell\big[\big(V^*_\ell(\pi_{Gg}(\tilde{\mu})) - V^*_\ell(0)\big) - \big(V^\dagger_\ell(\pi_{Gg}(\tilde{\mu})) - v_o\big)\big] \le (\delta-1)\big(v_o - V^*_\ell(0)\big) < 0. \tag{54}$$

Since $V^*_\ell(1) - V^*_\ell(0) = \Delta > \Delta_{v_o} = V^\dagger_\ell(1) - v_o$, there must exist $\mu' \in (\tilde{\mu}, 1)$ such that $V^*_\ell(\mu') - V^*_\ell(0) \le V^\dagger_\ell(\mu') - v_o$ and $V^*_\ell(\mu) - V^*_\ell(0) > V^\dagger_\ell(\mu') - v_o$ for all $\mu > \mu'$. However, since $p_G(\mu', y^*_{V^*_\ell}(\mu')) \ge p_G(\mu', y^\dagger_{v_o}(\mu'))$ and $\pi_{Gg}(\mu') > \mu'$, (52) and (53) evaluated at $\mu = \mu'$ imply that $V^*_\ell(\mu') - \delta V^*_\ell(0) > V^\dagger_\ell(\mu') - \delta v_o$ and consequently, $V^*_\ell(\mu') - V^*_\ell(0) > V^\dagger_\ell(\mu') - v_o$, contradicting the definition of μ'. [$V^\dagger_{v_o}(\mu) \ge V^*_\ell(\mu)$?] ■

Proof of Lemma 13:. Step 1. First, we prove that

Lemma 15 *For any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that $\forall v_o \in [\frac{\ell}{1-\delta}, \frac{1}{1-\delta}) \forall V_\ell \in \mathcal{F}_{v_o} \cap \mathcal{C}_{[0,1]}$ $\forall \mu$ and $\mu' \in (\varepsilon, \bar{\mu})$*

$$\frac{p_G(\mu', y^\dagger_{V_\ell}(\mu')) - p_G(\mu, y^\dagger_{V_\ell}(\mu))}{\mu' - \mu} \leq M_\varepsilon \tag{55}$$

■

Proof. Notice that we can find $k > 0$ such that $\frac{\partial p_G}{\partial \mu}$ is bounded above uniformly by k, and $\frac{\partial p_G}{\partial y}$ is bounded below uniformly by $-k$ Suppose that $\mu < \mu'$. Then if $y^\dagger_{V_\ell}(\mu') \geq y^\dagger_{V_\ell}(\mu)$: $\frac{p_G(\mu', y^\dagger_{V_\ell}(\mu')) - p_G(\mu, y^\dagger_{V_\ell}(\mu))}{\mu' - \mu} < k$ because p_G decreases in y.

Now suppose that $y^\dagger_{V_\ell}(\mu') < y^\dagger_{V_\ell}(\mu)$. Remind that $\pi_{Bb}\left(\mu, y^\dagger_{V_\ell}(\mu)\right)$ is non-decreasing. For $\mu > \varepsilon$ and $y \in [0,1]$, we can find k_ε such

$$\begin{aligned} \frac{\partial \pi_{Bb}(\mu, y)}{\partial \mu} &= \frac{(1-h)(1-\ell)(1-y)}{[\mu(1-h) + (1-\mu)(1-\ell)(1-y)]^2} < k_\varepsilon \\ \frac{\partial \pi_{Bb}(\mu, y)}{\partial y} &= \frac{(1-h)(1-\ell)(1-\mu)\mu}{[\mu(1-h) + (1-\mu)(1-\ell)(1-y)]^2} > \tilde{k}_\varepsilon \end{aligned}$$

Then

$$0 \leq \pi_{Bb}(\mu', y^\dagger_{V_\ell}(\mu')) - \pi_{Bb}(\mu, y^\dagger_{V_\ell}(\mu)) < k_\varepsilon(\mu' - \mu) + \tilde{k}_\varepsilon\left(y^\dagger_{V_\ell}(\mu') - y^\dagger_{V_\ell}(\mu)\right)$$

where we use $y^\dagger_{V_\ell}(\mu') < y^\dagger_{V_\ell}(\mu)$ and $\mu' > \mu$. Thus

$$y^\dagger_{V_\ell}(\mu') - y^\dagger_{V_\ell}(\mu) > -\frac{k_\varepsilon}{\tilde{k}_\varepsilon}(\mu' - \mu).$$

But then we have

$$\begin{aligned} p_G(\mu', y^\dagger_{V_\ell}(\mu')) - p_G(\mu, y^\dagger_{V_\ell}(\mu)) &< k(\mu' - \mu) - k\left(y^\dagger_{V_\ell}(\mu') - y^\dagger_{V_\ell}(\mu)\right) \\ &< \left(k + k\frac{k_\varepsilon}{\tilde{k}_\varepsilon}\right)(\mu' - \mu). \end{aligned}$$

We then set $M_\varepsilon = \left(k + k\frac{k_\varepsilon}{\tilde{k}_\varepsilon}\right)$. ■

Step 2. For the following notice that Next, we prove that

Lemma 16 *For any $\varepsilon > 0$, $\forall v_o \in [\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$*

$$\forall \mu > \varepsilon : D^+ V^\dagger_{v_o}(\mu) := \limsup_{\mu' \downarrow \mu} \frac{V^\dagger_{v_o}(\mu') - V^\dagger_{v_o}(\mu)}{\mu' - \mu} \leq \frac{M_\varepsilon}{1 - \delta h} \tag{56}$$

where $V^\dagger_{v_o}$ is the fixed point of T_{v_o} as noted earlier.

■

Proof. For v_o, there exists $\underline{\mu}$ defined by $V^{\dagger}_{v_o}(\pi_{Gg}(\underline{\mu})) = v_0$, such that

$$V^{\dagger}_{v_o}(\mu) = \begin{cases} p_G(\mu, y^{\dagger}_{v_o}(\mu)) + \delta v_o & \text{if} \quad \mu \leq \underline{\mu} \\ p_G(\mu, y^{\dagger}_{v_o}(\mu)) + \delta\left(\ell V^{\dagger}_{v_o}(\pi_{Gg}(\mu)) + (1-\ell)v_o\right) & \text{if} \quad \mu \geq \underline{\mu} \end{cases} \tag{57}$$

Since $\pi_{Gg}(\mu)$ is differentiable and $\frac{\ell}{h} \leq \frac{\partial \pi_{Gg}(\mu)}{\partial \mu} \leq \frac{h}{\ell}$, condition 55 implies that for $\mu > \varepsilon$

$$\begin{aligned} D^+V^{\dagger}_{v_o}(\mu) &\leq M_\varepsilon + \ell\delta \max_{\mu>\varepsilon}\left(D^+V^{\dagger}_{v_o}\right)\max_{\mu}\left(\frac{\partial \pi_{Gg}(\mu)}{\partial \mu}\right) \\ &\leq M_\varepsilon + h\delta \max_{\mu>\varepsilon}\left(D^+V^{\dagger}_{v_o}\right) \end{aligned}$$

Thus

$$D^+V^{\dagger}_{v_o}(\mu) \leq K = \frac{M_\varepsilon}{1-\delta h}$$

Step 3. Now, choose $v_o > \frac{\ell}{1-\delta}$. Notice that for a sufficiently small $\eta > 0$, and in particular smaller than $v_o - \frac{\ell}{1-\delta}$, the operator T_{v_o} can be extended to $\mathcal{F}^{\eta}_{v_o} \cap \mathcal{C}_{[0,1]}$ where $\mathcal{F}^{\eta}_{v_o} := \cup_{v_o-\eta\leq v\leq v_o+\eta}\mathcal{F}_v$.

Lemma 17 *The operator*

$$T_{v_o} : \mathcal{F}^{\eta}_{v_o} \cap \mathcal{C}_{[0,1]} \to \mathcal{C}_{[0,1]} \text{ is continuous in sup norm for every at } v_o, \tag{58}$$

■

Proof. Observe that if $\max_{\mu\in[0,1]} |V'_\ell(\mu) - V_\ell(\mu)| < \epsilon$ then

$$\max_{\mu\in[0,1]} |T_{v_o}(V'_\ell)(\mu) - T_{v_o}(V_\ell)(\mu)| < \epsilon + \delta\epsilon$$

follows from (25) because $|p_G(\mu, y^{\dagger}_{V'_\ell}(\mu)) - p_G(\mu, y^{\dagger}_{V_\ell}(\mu))| < \epsilon$ due to (**??**).

Step 4. Again fix v_o and η small. Observe from (25) that for $v = v_o + \kappa$ where $\kappa < \eta$, for all $V \in \mathcal{F}^{\eta}_{v_o} \cap \mathcal{C}_{[0,1]}$, $p_G(\mu, y^{\dagger}_V(\mu))$ changes by at most $\delta\kappa$ when v_o changes to v. Thus

$$T_{v_o}(V)(\mu) - 2|\delta\kappa| \leq T^{\dagger}_v(V)(\mu) \leq T_{v_o}(V)(\mu) + 2|\delta\kappa| \quad \forall \mu \in [0,1].$$

Then

$$T_{v_o}(V^{\dagger}_v)(\mu) - 2|\delta\kappa| \leq V^{\dagger}_v(\mu) = T^{\dagger}_v(V^{\dagger}_v)(\mu) \leq T_{v_o}(V^{\dagger}_v)(\mu) + 2|\delta\kappa|. \tag{59}$$

Since the closure of the set of $\mathcal{F}^{\eta}_{v_o} \cap \mathcal{C}_{[0,1]}$ is compact and contains $V^{\dagger}_v$ for all κ near 0, there is limit point of $V^{\dagger}_v$ as $\kappa \to 0$, say $\lim V^{\dagger}_v$.

We claim that for $v_0 - \eta > \frac{\ell}{1-\delta}$, the limit of $V^{\dagger}_v$ is a $V^{\dagger}_{v_o}$. For this we need the following lemma

Lemma 18 *Fix $v_o - \eta > \frac{\ell}{1-\delta}$. There exists $\varepsilon > 0$ such for all $v > v_o - \eta$ and $\mu < 2\varepsilon : V_v^\dagger(\mu) = p_G(\mu, 1) + \delta v$.*

■

Proof. TO BE DONE

Now take the set $[0, 2\varepsilon]$, $\max\left|V_v^\dagger(\mu) - V_{v_0}^\dagger(\mu)\right| = |v - v_0|$ converges to zero.

Consider now the set $[\varepsilon, 1]$, then there exists some K_ε such that $V_v^\dagger(\mu)$ is K_ε-Lipschitz on $[\varepsilon, 1]$.Then from Arzelà-Ascoli Theorem, the set K_ε-Lipschitz is compact under the norm sup. Assume that $V_v^\dagger$ converges to some $\lim V_v^\dagger$ (or extract a subsequence), then $\lim V_v^\dagger$ is K_ε-Lipschitz, and thus continuous on $[\varepsilon, 1]$.

Combining $[0, 2\varepsilon]$and $[\varepsilon, 1]$, we see that $V_v^\dagger$ converges to a continuous function $\lim V_v^\dagger$.

But then $T_{v_o}(V_v^\dagger) \pm 2|\delta\kappa|$ converges uniformly to $T_{v_o}(\lim V_v^\dagger)$. Thus from (59)

$$T_{v_o}(\lim V_v^\dagger) \leq \lim V_v^\dagger \leq T_{v_o}(\lim V_v^\dagger).$$

Thus $\lim V_v^\dagger = V_{v_o}^\dagger$. This proves continuity of ψ at v_o. ■

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